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Dispersive Approach to Power-Behaved Contributions in QCD Hard Processes[1]

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Abstract

We consider power-behaved contributions to hard processes in QCD arising from non-perturbative effects at low scales which can be described by introducing the notion of an infrared-finite effective coupling. Our method is based on a dispersive treatment which embodies running coupling effects in all orders. The resulting power behaviour is consistent with expectations based on the operator product expansion, but our approach is more widely applicable. The dispersively-generated power contributions to different observables are given by (log-)moment integrals of a universal low-scale effective coupling, with process-dependent powers and coefficients. We analyse a wide variety of quark-dominated processes and observables, and show how the power contributions are specified in lowest order by the behaviour of one-loop Feynman diagrams containing a gluon of small virtual mass. We discuss both collinear safe observables (such as the e^+e^- total cross section and τ hadronic width, DIS sum rules, e^+e^- event shape variables and the Drell-Yan K-factor) and collinear divergent quantities (such as DIS structure functions, e^+e^- fragmentation functions and the Drell-Yan cross section).

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1 Introduction

Power-behaved contributions to hard collision observables are by now widely recognized both as a serious difficulty in improving the precision of tests of perturbative QCD and as a way to explore non-perturbative effects. Such contributions are manifest as power-varying discrepancies between fixed-order perturbative predictions and experiment. For some quantities [1–4], the leading power contributions are of order Λ/Q where Q is the relevant energy scale and $\Lambda \sim 300$ MeV is the characteristic scale of QCD. In this case, even at scales $Q \sim M_Z \gg \Lambda$, power-behaved terms can be comparable to $\mathcal{O}(\alpha_s^2)$ perturbative contributions. In other processes [5–8], such as deep inelastic lepton scattering (DIS) and Drell-Yan lepton pair production, contributions of order Λ^2/Q^2 are important because Q is not so large. At even lower scales, as in the hadronic decay of the τ lepton, a stronger suppression of power contributions is needed in order for a perturbative analysis to be applicable [9–13].

Such contributions are usually regarded as "power-suppressed corrections" to the perturbative prediction, in which context they might appear negligible in comparison with perturbative terms of any order. From this viewpoint the former could not be considered before summing up the entire perturbative series, which in itself is ambiguous. In practice, however, power contributions can easily be distinguished since they are rapidly varying in contrast to the slowly (logarithmically) varying fixed-order perturbative contributions. Thus we can rather regard them as *power enhanced*, in the sense that they grow more rapidly as the hard process scale Q decreases. In view of their phenomenological relevance, it is important to study any possible source of power-behaved terms which might be a general feature of non-perturbative QCD.

So far, there is no rigorous systematic method for calculating power contributions in QCD. In some cases the operator product expansion (OPE) provides a framework for the classification of possible power-behaved terms, but it gives little indication of their coefficients [14,15]. General studies of the divergence of the perturbation series in high orders have led to the concept of renormalons [16], which are singularities in the complex plane of the Borel variable conjugate to the running coupling. In particular, infrared renormalons lie on the integration contour of the Borel variable. In regularizing these singularities one generates power-behaved contributions whose coefficients are ambiguous. Formal mathematical manipulations alone cannot resolve the infrared renormalon problem, which is of a physical nature, namely the notion of the running coupling in the small k^2 regime.

In this context, it is natural to consider the possibility that a finite running coupling might be definable at low scales, at least as an effective measure of the strength of interaction at large distances. One may hope in this way to take into account the main effects of confinement at a sufficiently inclusive level. This idea can only be meaningful if the effective coupling so introduced is found to be universal in some approximation. Its specific form, $\alpha_{\rm eff}(k^2)$, at small k^2 then determines the power contributions. Thus the experimental analysis of these contributions will constrain the form of $\alpha_{\rm eff}(k^2)$ at small scales. The theorists' task is to analyse the power contributions which are generated by the non-perturbative regime of the running coupling. To this end, one should study perturbative Feynman diagrams for hard process observables, in which the scale of the running coupling is not restricted to the hard region, but also runs into the non-perturbative domain.

A technique for studying power contributions within the language of perturbative QCD is to introduce a small gluon mass [1,7,12,17], which may be used as a "trigger" for long-distance contributions. In the present paper, we attempt to put this approach on a somewhat clearer theoretical basis. We work with a gluon field of zero mass, and introduce a dispersive representation of the running coupling $\alpha_s(k^2)$, which is assumed to remain finite all the way down to $k^2 = 0$. In this way

$\alpha_s(k^2)$ is expressed in terms of an effective coupling $\alpha_{\rm eff}(\mu^2)$ depending on the dispersion variable μ. In this representation μ plays a role similar to a gluon mass as far as the phase space and Feynman denominators are concerned. The region of small values of μ is responsible for power contributions. Since the running coupling is by assumption universal, these contributions can be related for different processes. We study the corrections to a wide range of quark-dominated processes due to a single off-shell gluon with time-like or space-like virtuality. The results should provide the means to test experimentally the concept of a universal infrared-finite coupling as an effective measure of strong interactions at large distances, as reflected in sufficiently inclusive hard process observables. Our discussion is mainly in the context of Abelian gauge theory, although we argue that its application to the non-Abelian case may be justified for leading power contributions.

In characterizing the effective coupling at low momentum scale we assume, in agreement with the standard ITEP-OPE approach [15], that non-perturbative physics does not affect the high-momentum region, that is, the propagation of quarks and gluons at short distances (soft confinement). Within our method, this implies that the non-perturbative "effective coupling modification", $\delta\alpha_{\rm eff}(\mu^2)$, which is responsible for power corrections in hard distributions, is essentially restricted to the small μ^2 region and must not directly introduce power corrections into α_s itself (notice again that μ^2 is a dispersive variable and we are working in Minkowski space). To satisfy this condition, $\delta\alpha_{\rm eff}(\mu^2)$ must have vanishing moments with respect to μ^2, at least for the first few (integer) powers of μ^2. If this were not the case, the predictions of the operator product expansion would be invalidated.

We start in Section 2 by discussing the extent to which a running coupling might be definable beyond perturbation theory. We introduce a dispersive representation of α_s in terms of a spectral density function $\rho_s(\mu^2)$ and then in terms of the effective coupling $\alpha_{\rm eff}(\mu^2)$. We recall how this representation leads to a natural choice of the scale of α_s in deep inelastic scattering, and we define the corresponding 'physical scheme' which we adopt for the QCD coupling in the perturbative region. Next we define the effective coupling $\alpha_{\rm eff}(\mu^2)$, related to the spectral density function, which will be a central quantity in our treatment of power contributions.

In Section 3 we present the method by which we compute leading power contributions from integrals involving the "effective coupling modification" $\delta\alpha_{\rm eff}$. We show that in quark-dominated processes, ie those involving no gluons at the Born level, the leading effect of the running coupling is determined by the behaviour of $\alpha_{\rm eff}(\mu^2)$ and a process-dependent *characteristic function* $\mathcal{F}$. This is a function of the ratio $\epsilon = \mu^2/Q^2$, together with any relevant dimensionless ratios of hard scales $\{x\}$, given by the sum of all one-loop graphs computed with a non-zero gluon mass μ. The power contributions that we seek to estimate are associated with the non-analytic terms in the small-ϵ expansion of this function. Each such term implies a power correction proportional to an integral of $\delta\alpha_{\rm eff}(\mu^2)$ times a corresponding non-analytic weight function. Thus the task becomes to compute $\mathcal{F}(\epsilon,\{x\})$ for various processes and to see whether the predicted power contributions are consistent with experiment for some choice of the behaviour of the "effective coupling modification" $\delta\alpha_{\rm eff}$ at low scales.

In Section 4 we present results on the characteristic functions for a variety of processes and discuss their behaviour at small ϵ and the associated power contributions. In many cases we find strong enhancement factors, either logarithms of Q^2 or singular functions of the auxiliary variables $\{x\}$, which lead us to hope that these contributions may be the dominant non-perturbative effects, at least in some important regions of phase space. In the case of Λ/Q contributions, no other possible sources of such terms are known, and therefore our results should provide useful semi-quantitative estimates of their relative magnitudes in different observables.

We do not undertake any detailed phenomenological analyses in the present paper, but confine ourselves to providing the theoretical results on which future analyses could be based. We end with a summary of results and a discussion of the limitations of our approach and its possible further extensions and applications.

2 Running coupling

In hard processes initiated by a quark, the inclusive quantities at a given order in perturbation theory receive contributions from an extra gluon both from virtual corrections to the amplitude at the previous order ($k^2 < 0$) and from a new production channel ($k^2 \geq 0$). We discuss here how one can associate a running coupling with this gluon. For large virtuality $|k^2|$ it is clear how to identify the running coupling and the proper scale for its argument using perturbative methods. For small momenta, $k^2 \to 0$, the very language of quarks and gluon becomes scarcely applicable, and the notion of the running QCD coupling becomes more elusive. This situation is opposite to the case of the Abelian theory, in which the physical fine structure coupling is defined at vanishing photon momentum, $\alpha(0)$, and appears, for example, in the Thomson cross section.

We explore here the hypothesis that in a non-Abelian theory the notion of the running coupling can also be extended to the region of vanishing momenta, at least in some effective sense. In doing so we seek to use the language and methods of perturbative QCD to probe some non-perturbative phenomena, in particular the leading power contributions to hard cross sections. In order to formulate this hypothesis more precisely, we assume that in QCD the running coupling can be represented by a dispersion relation which is inspired by the Abelian theory. Therefore we first recall the case of QED and then describe the extension to QCD.

2.1 Space-like gluon

Abelian case. Consider first the exchange of a photon with negative virtuality $-k^2$ (hereafter we always define $k^2 > 0$), which results in the appearance of the running coupling $\alpha(k^2)$. In an Abelian theory, due to the simple form of the Ward identities (*i.e.* the cancellation of the fermion wave function and vertex renormalization constants), the running coupling can be reconstructed using dispersion techniques (see [18]). The dispersion relation for the running coupling reads

$$\begin{aligned}\alpha(k^2) \equiv \alpha(0)\cdot \mathcal{Z}_3(-k^2) &= -\int_0^\infty \frac{d\mu^2}{\mu^2+k^2}\,\rho(\mu^2) \\ &= \alpha(0) + k^2\int_0^\infty \frac{d\mu^2}{\mu^2}\,\frac{\rho(\mu^2)}{\mu^2+k^2}\,,\end{aligned} \tag{2.1}$$

with $\mathcal{Z}_3$ the photon wave function with the normalization $\mathcal{Z}_3(0) = 1$. The spectral function $\rho(\mu^2)$ (positive in QED) has support only on the positive real axis and is obtained by considering all discontinuities associated with the virtual photon. It is given by the discontinuity of $\mathcal{Z}_3$ at positive virtuality,

$$\rho(\mu^2) \equiv \frac{\alpha(0)}{2\pi i}\left[\,\mathcal{Z}_3(\mu^2 - i\epsilon) - \mathcal{Z}_3(\mu^2 + i\epsilon)\right] \;=\; -\frac{1}{2\pi i}\,\mathrm{Disc}\,\left\{\alpha(-\mu^2)\right\}. \tag{2.2}$$

The dispersive representation (2.1) has the following non-perturbative implication: the coupling $\alpha(k^2)$ is regular in the full k^2-complex plane with a cut on the negative real axis. Therefore this representation gives a coupling that is free from the spurious QED Landau singularity and deviates from the standard perturbative e.m. coupling $\alpha^{\mathrm{PT}}(k^2)$ by a negligible amount $\mathcal{O}\left(k^2/\Lambda^2\right)$ ($\Lambda \sim 10^{30-40}$GeV).

Non-Abelian case. Consider now the exchange of a gluon with negative virtuality $-k^2$, which should result in the appearance of the running coupling $\alpha_s(k^2)$. In the QCD case one again seeks to identify the Feynman diagrams responsible for the running coupling with the one-gluon reducible graphs, which include gluon and quark self-energy insertions together with vertex corrections. Such a class of diagrams does not constitute a gauge-invariant set. Their gauge-dependent part is cancelled by corresponding contributions from the one-gluon irreducible graphs (*e.g.*, two-gluon exchange). Therefore one can argue that these gauge-dependent contributions from the one-gluon reducible set of graphs remain finite at $k^2 \to 0$, while the gauge-invariant part behaves as $1/k^2$, contributing to the renormalization of the pole of the gluon propagator. Since we are going to concentrate on the leading behaviour at small k^2, we shall not analyse the contributions that are regular for $k^2 \to 0$. We emphasize again that in non-Abelian theories the very notion of one gluon exchange becomes elusive at the level of relative k^2 corrections, at which two gluons can mimic the one-gluon exchange, the two processes being comparable and related by gauge invariance.

Inspired by the Abelian theory, we shall assume that in QCD the running coupling $\alpha_s(k^2)$ still satisfies a (formal) dispersion relation of the form (2.1):

$$\alpha_s(k^2) \equiv \alpha_s(0) \cdot \mathcal{Z}(-k^2) = -\int_0^\infty \frac{d\mu^2}{\mu^2+k^2}\,\rho_s(\mu^2)\,, \qquad \rho_s(\mu^2) = -\frac{1}{2\pi i}\,\mathrm{Disc}\left\{\alpha_s(-\mu^2)\right\}\,. \tag{2.3}$$

We note that the assumption of such a dispersion relation (with causal support) implies the absence of the perturbative Landau pole at $k^2 = \Lambda^2$, the QCD scale, but it also involves such ill-defined quantities as the coupling $\alpha_s(0)$ and the spectral density ρ_s at small scales μ^2. However, as we shall see shortly, $\alpha_s(0)$ will not appear explicitly in physical observables, while the contribution of the spectral density in the small μ^2 region will be suppressed by powers of μ^2.

Since in QCD the cancellation between the vertex and fermion propagator corrections due to Ward identities ($\mathcal{Z}_\Gamma \mathcal{Z}_q = 1$) does not hold, the quantity $\mathcal{Z}$ in (2.3) is not simply the gluon propagator correction. As is well known, both the non-Abelian part of the vertex renormalization correction $\mathcal{Z}_\Gamma^{(\mathrm{na})}$ and the gluon propagator factor $\mathcal{Z}_g$ contribute (in a gauge-dependent way) in forming the running coupling. Therefore in the case of QCD the quantity $\mathcal{Z}$ in (2.3) is now given by the product

$$\mathcal{Z} \;=\; \mathcal{Z}_\Gamma^{(\mathrm{na})} \cdot \mathcal{Z}_g \cdot \mathcal{Z}_\Gamma^{(\mathrm{na})}\,. \tag{2.4}$$

This expression shows that the function $\rho_s(\mu^2)$ is not literally a spectral density (although we shall continue to call it by that name) since it is not necessarily positive. In QED the function $\rho(\mu^2)$ is given by the discontinuity of the photon propagator, so that it is positive for any value of μ^2 as required by unitarity. This implies that $\alpha(k^2)$ increases as k^2 increases. In QCD instead $\alpha_s(k^2)$ decreases with increasing k^2 and so $\rho_s(\mu^2)$ is negative, at least in the perturbative domain. This is not in conflict with unitarity since, according to (2.4), the function $\rho_s(\mu^2)$ is given by the discontinuity of the gluon propagator correction ($\mathcal{Z}_g$) together with that of the non-Abelian vertex corrections ($\mathcal{Z}_\Gamma^{(\mathrm{na})}$). For example, in a physical gauge (axial, planar; see [19]) the first is determined by the total gluon splitting probability,

$$\mathrm{Disc}\,\mathcal{Z}_g \propto \int_0^1 dz \left\{ 2N_c \left(\frac{1}{z} + \frac{1}{1-z} - 2 + z(1-z)\right) + \sum_1^{n_f} \left(z^2 + (1-z)^2\right)\right\}$$

and gives an infrared-divergent contribution, which is positive as required by unitarity. The second (vertex corrections) gives a (divergent) *negative* contribution,

$$2\,\mathrm{Disc}\,\mathcal{Z}_\Gamma^{(\mathrm{na})} \;\propto\; -4N_c \int_0^1 \frac{dz}{z}\,,$$

so that the total discontinuity is finite and gauge-independent but not positive definite:

$$\mathrm{Disc}\left(\mathcal{Z}_\Gamma^{(\mathrm{na})}\mathcal{Z}_g\mathcal{Z}_\Gamma^{(\mathrm{na})}\right) \propto \int_0^1 dz \left\{2N_c\left(-2+z(1-z)\right) + \sum_1^{n_f}\left(z^2+(1-z)^2\right)\right\} = -\frac{11}{3}N_c + \frac{2}{3}n_f = -\beta_0\,.$$

2.2 Time-like gluon

We consider now the case of a gluon with zero ($k^2=0$) or positive virtuality ($k^2>0$) corresponding to the contribution of new production channels. In inclusive quantities the integration over k^2 has to be performed up to some kinematic limit $k^2_{\max} \sim Q^2$, the relevant hard scale. From this additional gluon correction one reconstructs the running coupling, provided one is able to factorize the following combination

$$\alpha_s(0)\,\delta(k^2) \;+\; \frac{\rho_s(k^2)}{k^2} \;=\; \frac{\alpha_s(0)}{\pi}\,\mathrm{Im}\left\{\frac{\mathcal{Z}(k^2)}{-k^2-i\varepsilon}\right\}. \tag{2.5}$$

The first contribution is due the on-shell gluon while the second is due to the sum over all final states generated by a positive-virtuality gluon. In order to factorize this combination, we need to consider observables which are sufficiently inclusive that all new production channels contribute to the same value of the observable.

Operationally our approach is similar to the so-called "naive non-Abelianization" procedure based on resummation of fermion loop diagrams [12,20]. The usual motivation for concentrating on fermion loop contributions is to avoid problems with gauge invariance. However we believe that the argument given earlier, that gauge-dependent terms are less singular at $k^2 \to 0$, justifies the application of our method to the *leading* power corrections in QCD.

In the next subsection we describe how the running coupling emerges in inclusive quantities of this type. The dispersive representation (2.5) allows one to identify the proper scale for the argument of the running coupling, and to define a 'physical' scheme for the definition of the QCD scale Λ.

2.3 Argument of the coupling

For the sake of illustration we consider the non-singlet quark distribution $F(Q^2,x)$ in DIS [18]. Here p and q are the momenta of the incoming quark and of the hard photon probe respectively, so that $Q^2=-q^2$ and $x=Q^2/2(pq)$ is the Bjorken variable.

The evolution equation for F is obtained by considering the emission of an additional gluon of momentum k. For simplicity we discuss only the real emission contribution; virtual corrections can be included in a straightforward way. One has to take into account both the on-shell contribution at $k^2=0$, with coupling $\alpha_s(0)$, and the contribution from the continuum involving the discontinuity through multi-parton states, with strength given by the spectral density $\rho_s(k^2)$. Introducing a minimum transverse momentum Q_0 for the emitted partons, the non-singlet quark distribution $F(Q^2,Q_0^2,x)$ is obtained by convoluting the combination (2.5) with the matrix element squared $\mathcal{F}(Q^2,k^2,k_\perp^2,x,z)$ for emission of an additional off-shell gluon of virtuality k^2, transverse momentum $k_\perp^2$ and fraction of momentum $(1-z)=2k\cdot(q+xp)\,x/Q^2$. We have

$$F(Q^2,Q_0^2,x) \;=\int_x^1\frac{dz}{z}\int_{Q_0^2}^{Q^2}dk_\perp^2\int_0^{Q^2}dk^2\,\mathcal{F}(Q^2,k^2,k_\perp^2,x,z)\left(\alpha_s(0)\delta(k^2)+\frac{\rho_s(k^2)}{k^2}\right), \tag{2.6}$$

where the integrations over $k_\perp^2$ and the timelike gluon virtuality k^2 actually run up to the kinematical limit, $W^2=Q^2(1-x)/x\sim Q^2$.

To see how the running coupling emerges in the *anomalous dimension* it suffices to consider the quasi-collinear region $k_\perp^2 \ll Q^2$. In the collinear limit the function $\mathcal{F}$ can again be expressed in terms of the distribution F itself as follows

$$\mathcal{F}(Q^2, k^2, k_\perp^2, x, z) \simeq F(Q^2, k_\perp^2, \frac{x}{z}) \frac{C_F}{2\pi} \left(\frac{P(z)}{k_\perp^2 + zk^2} - \frac{(1-z)\, zk^2}{(k_\perp^2 + zk^2)^2} \right) . \tag{2.7}$$

The first term in the brackets, involving the standard $q \to q$ splitting function $P(z) = (1+z^2)/(1-z)$, is related to the $k^2 = 0$ limit of the scattering matrix element. The second (subleading) contribution, non-singular for $k^2 \to 0$, originates from the k^2-corrections to the matrix element.

In order to show how the integral over the virtual boson mass k^2 in (2.6) reproduces an effective coupling running with $k_\perp^2$ as scale, it suffices to perform two steps, namely to extend the k^2 integration to infinity and then to apply the dispersion relation. The extension of the integration region is possible since the k^2 integral is rapidly convergent above $k^2 \sim k_\perp^2/z \lesssim k_\perp^2/x$ which, in the collinear approximation, is much smaller than the kinematical boundary $k^2 \leq W^2 \sim Q^2$.

Given this simplification, one can use the dispersion relation (2.5) to obtain for the k^2 integrals in (2.6)

$$\int_0^\infty dk^2 \left(\frac{P(z)}{k_\perp^2 + zk^2} - \frac{(1-z)\, zk^2}{(k_\perp^2 + zk^2)^2} \right) \left(\alpha_s(0)\delta(k^2) + \frac{\rho_s(k^2)}{k^2} \right) = P(z) \frac{\alpha_s(k_\perp^2/z)}{k_\perp^2} - (1-z) \frac{d\alpha_s(k_\perp^2/z)}{dk_\perp^2} , \tag{2.8}$$

and the evolution equation becomes

$$F(Q^2, Q_0^2, x) = \int_x^1 \frac{dz}{z} \int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} F(Q^2, k_\perp^2, \frac{x}{z}) \frac{C_F}{2\pi} \left\{ P(z)\, \alpha_s(k_\perp^2/z) - (1-z) \frac{d\alpha_s(k_\perp^2/z)}{d\ln k_\perp^2} \right\} . \tag{2.9}$$

Thus, by using the spectral function and the dispersion relation (2.5), we have reconstructed the running coupling associated with an extra gluon with positive virtuality. Moreover, as in [19,21,22], we have shown that the dispersion relation allows one to identify the physical scale for the argument of the coupling.

It is clear that in this way one has included some important contributions from higher orders proportional to the first beta function coefficient β_0. In particular in (2.9) one finds the following contributions:

1. The last term in the curly bracket of (2.9) can be written as

$$-(1-z) \frac{d\alpha_s(k_\perp^2/z)}{d\ln k_\perp^2} = -(1-z) \frac{\beta_0}{4\pi} \alpha_s^2(k_\perp^2) + \dots , \qquad \left(\beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f \right) ;$$

2. Expanding the argument of the coupling in the main term, we obtain

$$P(z)\, \alpha_s(k_\perp^2/z) = P(z)\, \alpha_s(k_\perp^2) + P(z) \ln z \left(\frac{\beta_0}{4\pi} \right) \alpha_s^2(k_\perp^2) + \dots$$

The above terms of order α_s^2 are indeed present in the two-loop contribution to the anomalous dimension [23].

QCD scale. Two comments are in order concerning the "physical scale" of the coupling.
1) Rewriting the expression in the curly brackets in (2.9) as

$$P(z)\,\alpha_s \;-\; (1-z)\,\frac{d\,\alpha_s}{d\ln k_\perp^2} \;=\; \left\{\frac{2z}{1-z}\cdot\alpha_s \;+\; (1-z)\cdot\left[\alpha_s - \frac{\beta_0}{4\pi}\,\alpha_s^2\right]\right\}, \tag{2.10}$$

one observes that beyond the first loop the coupling of the two pieces of the gluon radiation probability $P(z) = 2z/(1-z) + (1-z)$ effectively acquire different arguments. This is because these two contributions to the splitting function are physically different. The first term comes from the universal "soft" bremsstrahlung distribution ($d\omega/\omega$), which is independent of the nature of the incoming parton and corresponds to the gluon polarization in the scattering plane (longitudinal polarization), while the second one is due to "hard" gluons ($\omega\, d\omega$), depends on initial parton spin and consists equally of longitudinal and transverse polarization.
2) The universal nature of *soft* gluon bremsstrahlung may be used to define a "physical" QCD coupling beyond the first loop. To do so one has to analyse the higher-order terms of the anomalous dimension and to absorb the singular contributions $\propto (1-z)^{-1}$ systematically into a redefinition of the QCD scale Λ, which determines the intensity of radiation of relatively soft gluons.

Such a physical scheme is related to the popular $\overline{\mathrm{MS}}$ scheme as follows. In the $\overline{\mathrm{MS}}$ scheme the intensity of soft bremsstrahlung is given by an infinite series in a formally defined parameter $\alpha_{\overline{\mathrm{MS}}}$. In particular, both the first and the second loop $\overline{\mathrm{MS}}$ splitting functions $P(z)$ and $P^{(2)}_{\overline{\mathrm{MS}}}(z)$ contain the "soft singularity" $(1-z)^{-1}$. Redefining the coupling in such a way that the new two-loop splitting function $P^{(2)}(z)$ does not contain the soft singularity (the so-called MC-scheme of [24]) one arrives, in second order, at the relation

$$\alpha_{\overline{\mathrm{MS}}}P(x) + \alpha^2_{\overline{\mathrm{MS}}}\,P^{(2)}_{\overline{\mathrm{MS}}}(x) \;=\; \alpha_s\,P(x) + \alpha_s^2\,P^{(2)}(x) \;+\; \mathcal{O}\left(\alpha_s^3\right)\;; \tag{2.11a}$$

$$\alpha_s = \alpha_{\overline{\mathrm{MS}}}\left(1 + \frac{\alpha_{\overline{\mathrm{MS}}}}{2\pi}\,\mathcal{K}\right), \qquad\qquad \Lambda = \Lambda_{\overline{\mathrm{MS}}}\exp(\mathcal{K}/\beta_0)\,, \tag{2.11b}$$

with

$$\mathcal{K} \;=\; C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{10}{9}n_f T_R\,, \tag{2.11c}$$

which is equal to 4.0 ± 0.5 for $n_f = 4 \mp 1$.

In this scheme the second loop contribution $P^{(2)}(z)$ is at least one power of $(1-z)$ down with respect to the singular part of the main bremsstrahlung spectrum, $P(z)$. Moreover, as advocated also in Ref. [25], $P^{(2)}(z)$ no longer contains any n_f-dependence, this ill-defined quantity being completely absorbed, to this order, into the momentum dependence of the coupling (for more details see [26]).

Because of the classical character of soft bremsstrahlung, such a definition of the coupling proves to be universal with respect to the nature of the source of radiation. In particular the two-loop *gluon* splitting function also becomes infrared regular in the physical scheme described above, *i.e.*, $P^{(2)}_{g\to gg}(z)/P_{g\to gg}(z) \to 0$ with $z \to 1$. For this reason we shall use this physical scheme for the QCD coupling.

2.4 Effective coupling

The spectral density ρ_s may be related to the running coupling by the following formal transformation:

$$\begin{aligned}\alpha_s(k^2) &= -\int_0^\infty \frac{d\mu^2}{\mu^2+k^2}\rho_s(\mu^2) = -\int_0^\infty \frac{dv}{1+v}\rho_s(vk^2) \\ &= -\int_0^\infty \frac{dv}{1+v}\exp\left\{\ln v \frac{d}{d\ln k^2}\right\}\cdot\rho_s(k^2) \\ &= \frac{\pi}{\sin\left(\pi\frac{d}{d\ln k^2}\right)}\rho_s(k^2)\,.\end{aligned} \tag{2.12}$$

In terms of the differential operator $\mathcal{P}$,

$$\mathcal{P} \equiv \frac{d}{d\ln\mu^2}\,, \tag{2.13}$$

one may thus write the inverse relation as

$$\rho_s(\mu^2) = \frac{1}{\pi}\sin(\pi\mathcal{P})\,\alpha_s(\mu^2)\ . \tag{2.14}$$

Introducing the *effective coupling* $\alpha_{\rm eff}(\mu^2)$ defined by

$$\rho_s(\mu^2) = \frac{d}{d\ln\mu^2}\alpha_{\rm eff}(\mu^2)\,, \tag{2.15}$$

we have

$$\alpha_s(k^2) = k^2\int_0^\infty \frac{d\mu^2}{(\mu^2+k^2)^2}\alpha_{\rm eff}(\mu^2) \tag{2.16}$$

with inverse

$$\alpha_{\rm eff}(\mu^2) \equiv \frac{\sin(\pi\mathcal{P})}{\pi\mathcal{P}}\,\alpha_s(\mu^2)\,. \tag{2.17}$$

It follows from Eq. (2.17) that in the perturbative domain $\alpha_s \ll 1$, the standard and effective couplings are approximately the same:

$$\alpha_{\rm eff}(\mu^2) = \alpha_s(\mu^2) - \frac{\pi^2}{6}\frac{d^2\alpha_s}{d\ln^2\mu^2} + \ldots \;=\; \alpha_s \,+\, \mathcal{O}\left(\alpha_s^3\right)\ . \tag{2.18}$$

In what follows we shall look upon $\alpha_{\rm eff}$ itself as an effective measure of QCD interaction strength which extends the physical perturbative coupling down to the non-perturbative domain.

3 Dispersive method

Consider a dimensionless inclusive quantity $F(Q^2,\{x\})$ for a hard process involving only quarks at the Born level. Here Q^2 is the hard scale and $\{x\}$ stands for any further relevant dimensionless parameters, *e.g.*, Bjorken x or its conjugate moment variable N (DIS structure functions), particle energy fraction (inclusive e^+e^- annihilation spectra), the ratio M^2/s (the Drell-Yan process), or a jet shape variable. The one-loop prediction for $F(Q^2,\{x\})$, obtained from the squared amplitudes involving one additional gluon, can be improved by using a dispersive method based on the representation of the running coupling in (2.3) for the space-like case and (2.5) for the time-like case. In this way one takes into account relevant higher-order perturbative effects and is also able to study power contributions arising from the non-perturbative behaviour of the coupling at low scales.

3.1 Basic equation

To describe the method we concentrate first on collinear safe quantities. The generalization to include collinear singular observables will be presented shortly. We have to consider the two cases in which the additional gluon contributes to real production channels and to a virtual correction.

Consider first the case in which the additional gluon contributes to new production channels. As discussed in Sect. 2.2, one has to take into account both the on-shell contribution at $k^2 = 0$, with coupling $\alpha_s(0)$, and the contribution from the continuum involving the discontinuity through multi-parton states (from gluon branching), with strength given by the spectral density $\rho_s(k^2)$. For all these contributions at fixed k^2 one would like to factorize the combination (2.5). This is possible only if the observable is fully inclusive with respect to gluon branching, as in the case of total cross sections, DIS structure functions, etc. For less inclusive quantities, such as jet shape observables, gluon branching may give a different contribution [27]. The non-factorizable contribution is higher-order in $\alpha_{\rm eff}$. However, since $\alpha_{\rm eff}$ enters at a low scale, such terms could still be comparable to the factorizable first-order contribution $\propto \alpha_{\rm eff}$. Therefore our method can only give quantitative predictions for less inclusive quantities if $\alpha_{\rm eff}$ at low scales is sufficiently small.

We denote by $\mathcal{F}^{\rm real}(Q^2,\{x\},k^2)$ the squared amplitude for the emission of a gluon with timelike virtuality $k^2 \geq 0$, apart from the coupling. $\mathcal{F}^{\rm real}(Q^2,\{x\},k^2)$ is defined to vanish outside the real phase space $0 < k^2 < k^2_{\rm max}(Q^2,\{x\})$. Then the real emission contribution to the inclusive quantity $F(Q^2,\{x\})$ is given by the sum of the on-shell and continuum contributions (see Eq. (2.5)):

$$F^{\rm real}(Q^2,\{x\}) = \alpha_s(0)\mathcal{F}^{\rm real}(0) + \int_0^\infty \frac{dk^2}{k^2}\rho_s(k^2)\cdot\mathcal{F}^{\rm real}(k^2)\,. \tag{3.1}$$

Consider now the case in which the additional gluon contributes to virtual corrections. This contribution is present when the observable under consideration does not vanish in the Born approximation. In this contribution the virtuality of the gluon is spacelike, $-k^2 < 0$, and the coupling is given by $\alpha_s(k^2)$. The contribution can then be written as

$$F^{\rm virt}(Q^2,\{x\}) = \int_0^\infty \frac{dk^2}{k^2}\alpha_s(k^2)\mathcal{M}(Q^2,\{x\},-k^2)\,,$$

with $\mathcal{M}$ the integrand for the Feynman diagram containing the (massless) virtual gluon, apart from the coupling. We now use the dispersive representation (2.3) for the running coupling to write

$$\begin{aligned} F^{\rm virt} &= \alpha_s(0)\int_0^\infty \frac{dk^2}{k^2}\mathcal{M}(-k^2) + \int_0^\infty \frac{d\mu^2}{\mu^2}\rho_s(\mu^2)\int_0^\infty \frac{dk^2}{k^2+\mu^2}\mathcal{M}(-k^2) \\ &= \alpha_s(0)\mathcal{F}^{\rm virt}(0) + \int_0^\infty \frac{d\mu^2}{\mu^2}\rho_s(\mu^2)\cdot\mathcal{F}^{\rm virt}(\mu^2)\,, \end{aligned} \tag{3.2}$$

where

$$\mathcal{F}^{\rm virt}(Q^2,\{x\},\mu^2) \equiv \int_0^\infty \frac{dk^2}{k^2+\mu^2}\mathcal{M}(Q^2,\{x\},-k^2) \tag{3.3}$$

is the one-loop virtual correction that would be produced by a gluon with a finite mass μ^2.

The result thus has the same structure both for the real (3.1) and for the virtual (3.2) contributions. Combining them to form

$$\mathcal{F}(Q^2,\{x\},\mu^2) \equiv \mathcal{F}^{\rm real}(Q^2,\{x\},\mu^2) + \mathcal{F}^{\rm virt}(Q^2,\{x\},\mu^2)\,,$$

which we shall refer to as the *characteristic function*, the improved one-loop formula reads (omitting the external variables Q^2 and $\{x\}$)

$$F \;=\; \alpha_s(0)\mathcal{F}(0) + \int_0^\infty \frac{d\mu^2}{\mu^2}\rho_s(\mu^2)\cdot\mathcal{F}(\mu^2) \;=\; \int_0^\infty \frac{d\mu^2}{\mu^2}\rho_s(\mu^2)\cdot\left[\mathcal{F}(\mu^2)-\mathcal{F}(0)\right], \tag{3.4}$$

where we have made use of the formal relation (2.3) to eliminate $\alpha_s(0)$. By introducing the effective coupling $\alpha_{\rm eff}(\mu^2)$ (2.17) using Eq. (2.15) and integrating by parts, we can write

$$F(Q^2,\{x\}) \;=\; \int_0^\infty \frac{d\mu^2}{\mu^2}\,\alpha_{\rm eff}(\mu^2)\cdot\dot{\mathcal{F}}(Q^2,\{x\};\mu^2)\,, \qquad\qquad \dot{\mathcal{F}} \equiv -\frac{\partial\mathcal{F}}{\partial\ln\mu^2}\,. \tag{3.5}$$

The relation (3.4) or (3.5) is our basic equation for studying both the perturbative part and non-perturbative contribution to F. For a collinear safe observable $\dot{\mathcal{F}}(\mu^2)$ vanishes for $\mu^2\to 0$ and $\mu^2\to\infty$, and the integral is dominated by the region $\mu^2\sim Q^2$. One observes that the integrand contains two well separated scales: Q^2, the scale of the characteristic function $\mathcal{F}(\mu^2)$, and Λ^2, the QCD parameter, which is the scale of the effective coupling $\alpha_{\rm eff}(\mu^2)$ or $\rho_s(\mu^2)$. The perturbative contribution is related to the scale Q^2, while the power-behaved contribution is related to Λ^2.

Let us stress that although in our derivation we have introduced the quantity $\alpha_s(0)$, which may make little sense in the context of QCD, the final results (3.4) or (3.5) involve only the spectral density $\rho_s(\mu^2)$ or the effective coupling $\alpha_{\rm eff}(\mu^2)$ convoluted with a function that vanishes at $\mu^2=0$. Therefore the basic equation may make sense even if $\alpha_s(0)$ does not exist.

In what follows we shall look upon $\alpha_{\rm eff}$ defined in (2.17) as an effective measure of QCD interaction strength which extends the physical perturbative coupling down to the non-perturbative domain. While the form of $\alpha_{\rm eff}(k^2)$ at large k^2 is well known, we treat $\alpha_{\rm eff}$ at small scales as a phenomenological function whose behaviour is to be determined from experiment.

The characteristic function $\mathcal{F}$ is process-dependent and is obtained in practice by computing the relevant one-loop graphs with a non-zero gluon mass μ. Note that we do *not* intend to imply that the gluon has in any sense a real effective mass, but only that the dispersive representation can be expressed in this way.

In the above derivation it was implied that the characteristic function is well-defined at $\mu^2=0$. This is true for collinear safe quantities such as the total e^+e^- annihilation cross section, DIS sum rules, the Drell-Yan K factor, jet shape variables, etc. On the other hand for collinear singular quantities, such as DIS structure functions, the Drell-Yan cross section, and inclusive e^+e^- parton fragmentation functions, we have

$$\mathcal{F}(Q^2,\{x\};\mu^2) = P(\{x\})\cdot\ln\frac{Q^2}{\mu^2} \;+\; \mathcal{F}^{\rm reg}(Q^2,\{x\};\mu^2)\,, \tag{3.6}$$

where $\mathcal{F}^{\rm reg}(Q^2,\{x\};0)$ is finite. We cannot use the representations (3.4) for the collinear singular distribution itself, but rather for the scaling violation rate $Q^2\partial F/\partial Q^2$. In this case the function $\dot{\mathcal{F}}$ is involved, which has a finite $\mu^2\to 0$ limit. Then Eq. (3.5) becomes

$$Q^2\frac{\partial}{\partial Q^2}F(Q^2,\{x\}) \;=\; \int_0^\infty \frac{d\mu^2}{\mu^2}\,\alpha_{\rm eff}(\mu^2)\cdot\ddot{\mathcal{F}}(Q^2,\{x\};\mu^2)\,, \tag{3.7}$$

where, from (3.6) we have

$$\ddot{\mathcal{F}}(Q^2,\{x\};\mu^2) \;=\; \left(\epsilon\frac{\partial}{\partial\epsilon}\right)^2\mathcal{F}^{\rm reg}(\epsilon,\{x\})\,, \tag{3.8}$$

and we have used the fact that $\dot{\mathcal{F}}$ is a function of $\epsilon = \mu^2/Q^2$. The function $\ddot{\mathcal{F}}$ vanishes as $\epsilon \to 0$ and $\epsilon \to \infty$, and therefore the integral in (3.7) is now well defined. For simplicity, we continue the discussion in terms of $\mathcal{F}$ and $\dot{\mathcal{F}}$, with the understanding that one should substitute $\dot{\mathcal{F}}$ and $\ddot{\mathcal{F}}$ when studying the scaling violation in a collinear singular quantity.

We now discuss the general behaviour of the characteristic function. Since $\mathcal{F}$ depends only on dimensionless ratios, we write

$$\mathcal{F}(Q^2, \{x\}; \mu^2) = \mathcal{F}(\epsilon, \{x\}) , \qquad \dot{\mathcal{F}} \equiv -\epsilon \frac{\partial}{\partial \epsilon} \mathcal{F}(\epsilon, \{x\}) , \qquad \epsilon \equiv \frac{\mu^2}{Q^2} . \tag{3.9}$$

We describe the form of the characteristic function $\mathcal{F}(\epsilon, \{x\})$ for a collinear safe quantity, and in particular its behaviour for large and small ϵ. For a collinear singular quantity, the same behaviour is obtained for its regular part $\mathcal{F}^{\rm reg}(\epsilon, \{x\})$.

Large ϵ. At large ϵ the characteristic function either vanishes identically because of the phase space boundary (in the absence of virtual corrections to this order, as is the case for jet shape variables), or decreases as a negative power of ϵ as a consequence of the renormalizability of the theory. It follows that the logarithmic derivative $\dot{\mathcal{F}}$ also vanishes at infinity, so that the integral in Eq. (3.5) is well-defined.

Small ϵ. The behaviour at small ϵ is crucial for the analysis of non-perturbative power contributions. By definition of a collinear safe quantity, the behaviour of $\dot{\mathcal{F}}$ near $\epsilon = 0$ is of the form

$$\dot{\mathcal{F}}(\epsilon) \; = \; \epsilon^p \, \left[f(\ln \epsilon) + \epsilon \, g(\ln \epsilon) + \mathcal{O}\left(\epsilon^2\right) \right] , \tag{3.10}$$

with $p > 0$, where f and g are polynomials of degree not higher than 2:

$$f(\ln \epsilon) = f_2 \ln^2 \epsilon + f_1 \ln \epsilon + f_0 , \tag{3.11}$$

and similarly for $g(\ln \epsilon)$.

As an example, we show in Figs. 1 and 2 the behaviour of $\mathcal{R} \equiv 2\pi \mathcal{F}/C_F$ and its derivative $\dot{\mathcal{R}}$ for the e^+e^- total annihilation cross section. In this case we have $\dot{\mathcal{R}} \sim 3\epsilon^2 + 2\epsilon^3 \ln \epsilon - 3\epsilon^3$ at small ϵ, as shown by the dashed curve in Fig. 2, so $p = 2$, $f(\ln \epsilon) = 3$ and $g(\ln \epsilon) = 2 \ln \epsilon - 3$.

We show finally how the standard one-loop result is recovered from Eq. (3.5). Since $\dot{\mathcal{F}}$ vanishes both for $\epsilon \to 0$ and for $\epsilon \to \infty$, the main contribution to the integral comes from $\epsilon \sim 1$, that is $\mu^2 \sim Q^2$. Therefore one can extract the leading contribution by taking the value of the effective coupling at some characteristic scale

$$\bar{Q}^2 \equiv c(\{x\}) \cdot Q^2$$

outside the integration, to obtain

$$F(Q^2, \{x\}) = \alpha_{\rm eff}(\bar{Q}^2) \cdot \mathcal{F}(Q^2, \{x\}; 0) \; + \; \int_0^\infty \frac{d\mu^2}{\mu^2} \left[\alpha_{\rm eff}(\mu^2) - \alpha_{\rm eff}(\bar{Q}^2) \right] \cdot \dot{\mathcal{F}}(Q^2, \{x\}; \mu^2) . \tag{3.12}$$

The remaining integral contributes both to higher-order perturbative terms and to non-perturbative power contributions. Roughly speaking, the former come from the region $\mu^2 \sim \bar{Q}^2 \sim Q^2$ and the latter from $\mu^2 \sim \Lambda^2 \ll Q^2$. The small-$\mu^2$ region is outside perturbative control but, to the extent that $\alpha_{\rm eff}$ may be defined universally and obtained from experimental data, its contribution is determined by the behaviour of $\mathcal{F}$ at small values of ϵ. In the remainder of this Section, we consider various approaches to the evaluation of the integral in (3.12).

3.2 Renormalons

For large Q^2 the main contribution to the integral in (3.12) still comes from the region $\mu^2 \sim Q^2$. One might take this as sufficient motivation for evaluating this contribution by expanding $\alpha_{\rm eff}(\mu^2)$ around $\mu^2 = Q^2$ (for simplicity we consider here $\bar{Q}^2 = Q^2$). In this region the effective coupling $\alpha_{\rm eff}(\mu^2)$ can be approximated by its one-loop (or two-loop) perturbative expression $\alpha_s^{\rm PT}(\mu^2)$

$$\alpha_{\rm eff}(\mu^2) \simeq \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)} \equiv \alpha_s^{\rm PT}(\mu^2)\,, \qquad \mu^2 \gg \Lambda^2\,. \tag{3.13}$$

Making the replacement

$$\alpha_{\rm eff}(\mu^2) \Rightarrow \alpha_s^{\rm PT}(\mu^2) \;=\; \alpha_s^{\rm PT}(Q^2) \,+\, \alpha_s^{\rm PT}(Q^2)\sum_{k=1}^{\infty}\left(\frac{\beta_0\alpha_s^{\rm PT}(Q^2)}{4\pi}\;\ln\frac{Q^2}{\mu^2}\right)^k\,, \tag{3.14}$$

one finds

$$\begin{aligned} F(Q^2) - \alpha_{\rm eff}(Q^2)\mathcal{F}(0) \Rightarrow & \alpha_s^{\rm PT}(Q^2)\sum_{k=1}^{\infty}\left(\frac{\beta_0\alpha_s^{\rm PT}(Q^2)}{4\pi}\right)^k\; C_k\,, \\ C_k = & \int_0^{\infty}\frac{d\epsilon}{\epsilon}\left(\ln\frac{1}{\epsilon}\right)^k\;\dot{\mathcal{F}}(\epsilon)\,, \end{aligned} \tag{3.15}$$

where the simplified notation $\mathcal{F}(Q^2,\{x\},\mu^2) = \mathcal{F}(\epsilon)$ has been used. The deficiency of the perturbative expansion becomes apparent from the fact that the resulting coefficients C_k exhibit factorial growth [16]. This is associated with both the ultraviolet and the infrared integration regions in ϵ.

The ultraviolet contribution to C_k is estimated by integrating (3.15) over $\epsilon > 1$. If $\dot{\mathcal{F}}(\epsilon)$ vanishes like ϵ^{-q} at large ϵ, one finds for large k

$$C_k^{\rm UV} \sim \int_1^{\infty}\frac{d\epsilon}{\epsilon}\left(\ln\frac{1}{\epsilon}\right)^k\;\epsilon^{-q}\;\sim\;(-1)^k\,k!\,. \tag{3.16}$$

This corresponds to an ultraviolet renormalon. Such an alternating series can be evaluated by Borel summation. This is because in the ultraviolet integration region of (3.12) the replacement $\alpha_{\rm eff}(\mu^2) \to \alpha_s^{\rm PT}(\mu^2)$ is a reliable approximation and the contribution of this region can in fact be evaluated explicitly without any expansion.

The infrared contribution to C_k is estimated by integrating over the region $\epsilon < 1$. Using the small ϵ behaviour in (3.10) one finds

$$C_k^{\rm IR} = \int_0^{1}\frac{d\epsilon}{\epsilon}\left(\ln\frac{1}{\epsilon}\right)^k\;\epsilon^p\,f(\ln\epsilon)\;\sim\;k! \tag{3.17}$$

One again finds a factorial behaviour (an infrared renormalon). In this case however the coefficients are *non-alternating* and therefore the series is not Borel-summable. Attempts to ascribe meaning to such a series by brute force tend to give rise to unphysical *complex* contributions at the level of Q^{-2p} terms. This is generally interpreted as an intrinsic uncertainty in the summation of the perturbative series[3]. It is important to recognize that formal mathematical manipulations alone cannot resolve this problem, which is of a physical nature. One requires genuinely new physical input, namely a power-behaved 'confinement' contribution, to obtain a sensible answer. In this paper we advocate the hypothesis that such an input, for sufficiently inclusive Minkowskian observables, may be embodied in the form of $\alpha_{\rm eff}(\mu^2)$ at small μ^2.

[3]In fact, infrared renormalons are a purely perturbative phenomenon and have no direct relation to the presence of the Landau singularity in the running coupling; for a detailed discussion see [29].

3.3 Soft confinement and power corrections

The standard operator product expansion (OPE) approach by Shifman, Vainshtein and Zakharov [15] quantifies confinement effects in terms of additive contributions to Euclidean quantities and supplies the fuel for the impressive machinery of the ITEP sum rules (for a review see [28]).

The OPE approach is based on the hypothesis that the entire effect of confinement in vacuum correlators of currents may be embodied into "condensates" (vacuum expectation values of gauge- and Lorentz-invariant colourless operators built from gluon and quark fields). The basic ITEP idea was to separate the long- and short-distance contributions and to treat them on different bases. In the Euclidean region such a separation is straightforward: for $k^2 = k_0^2 + |\vec{k}|^2 > \lambda^2 \sim 1\ \mathrm{GeV}^2$ one employs perturbation theory — in particular, one uses the purely perturbative expression α_s^{PT} for the coupling in Feynman diagrams for a given correlator. The complementary region $k^2 < \lambda^2$, on the other hand, is treated phenomenologically. Upon integration, the latter region gives a power-behaved contribution of the order of $(\lambda^2/Q^2)^p$.

As far as gluon propagation is concerned, within the logic of the present paper one may write equivalently

$$\alpha_s(k^2) \;=\; \alpha_s^{\mathrm{PT}}(k^2) + \delta\alpha_s(k^2)\,, \tag{3.18}$$

with $\delta\alpha_s$ a modification in the effective interaction strength at small momentum scales responsible for non-perturbative effects. In agreement with the ITEP point of view, while this contribution generates power corrections to the hard distributions, it should not modify the behaviour of the running coupling at large momentum scales.

Let us stress that the separation (3.18) does not mean that the *perturbative* contribution as such is free from power-behaved terms. This very question is almost meaningless: one would have to keep under control an infinite series of logarithmic high-order terms prior to addressing the problem of perturbatively-generated power corrections. As we have mentioned above, due to the "infrared renomalon problem" the perturbative series is apparently not summable to such a level of accuracy. At the same time, the separation prescription (3.18) cures the problem operationally: as long as perturbative integrals are cut off in the infrared, resummation of the infinite series triggered by the running of α_s^{PT} is harmless. To obtain a physically sensible (and reasonably accurate) answer it suffices to calculate a few terms of the perturbative expansion and then to add power-behaved terms as a contribution of essentially different (non-perturbative, confinement) origin.

There is a strong point of the OPE ideology, which one may refer to as the "*soft* confinement" scenario. The OPE prescription implies that the propagation of quarks and gluons with *large* (Euclidean) momenta remains unaffected by non-perturbative physics, even at the level of power-suppressed terms. Formally, one might argue that the propagators of coloured fields have no gauge-invariant meaning. Nonetheless a power-suppressed variation, say a $1 + \lambda^2/k^2$ correction to the gluon propagator in the *ultraviolet* region $k^2 \gg \lambda^2$, would inevitably introduce an additional "*non-perturbative*" contribution. This would have no relation to the region of small momentum flow in the corresponding Feynman diagrams for the current correlators, which region is responsible for the formation of condensates. As a result, the original separation idea would not be valid. Thus the term $\delta\alpha_s(k^2)$ in (3.18) should decay faster than some (sufficiently large negative) power $(k^2)^{-p_{\max}}$ for viability of the notion of a condensate of dimension $2p \leq 2p_{\max}$.

Physically the OPE prescription corresponds to a picture of *smooth* non-perturbative large-scale vacuum fields with a typical size $\sim \lambda^{-1}$. If such fields were non-singular at short distances, the gluon propagator (and thus the running coupling) would be subject to exponentially small corrections only. An instanton–anti-instanton gas as a representative model for non-trivial vacuum fields sets an upper bound $p_{\max} < \beta_0 \sim 9$, above which small-size instantons start to disturb the propagation of quarks and gluons.

Following the discussion in the previous sections, we are now in a position to implement a similar logic for quantifying confinement contributions to Minkowskian observables.

The effective coupling $\alpha_{\rm eff}(\mu^2)$ appearing in (3.4) and (3.5) cannot be defined perturbatively below $\mu^2 \sim \lambda^2 \sim 1$ GeV^2, where the very language of quarks and gluons is scarcely applicable. In this region we expect an "effective coupling modification" $\delta\alpha_{\rm eff}$ which generates the non-perturbative interaction strength $\delta\alpha_s$ in (3.18) via the dispersion relation (2.16), *i.e.*

$$\delta\alpha_s(k^2) = k^2 \int_0^\infty \frac{d\mu^2}{(\mu^2+k^2)^2}\, \delta\alpha_{\rm eff}(\mu^2)\,. \tag{3.19}$$

At first sight, one might feel free to choose an arbitrary form for $\delta\alpha_{\rm eff}$ at small scales to model confinement effects. However, on inspecting (3.19) one observes that, generally speaking, a finite modification of the effective coupling at low scales will affect the ultraviolet behaviour of the coupling in the Euclidean region by an amount proportional to $1/k^2$.

As discussed above, such a modification would ruin the basis of the OPE approach. One has therefore to require that at least the first $p_{\rm max}$ integer moments of this coupling vanish. A similar constraint has been discussed in [30].

Consider, for example, the non-perturbative gluon condensate which contributes to the Adler D-function (see, e.g., [16]). To first order in α_s it is given by the integral

$$\frac{2\pi^2}{3}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle_{\rm NP} = \frac{3C_F}{2\pi}\int_0^{U^2} dk^2\, k^2\, \delta\alpha_s(k^2)\,, \tag{3.20}$$

where U^2 is the ultraviolet cutoff. Substituting the representation (3.19) for $\delta\alpha_s$ in terms of the function $\delta\alpha_{\rm eff}$ and performing the integration over k^2, one obtains

$$\frac{2\pi^2}{3}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle_{\rm NP} = \frac{3C_F}{2\pi}\int_0^{\infty} d\mu^2\, \delta\alpha_{\rm eff}(\mu^2)\left[U^2 - 2\mu^2 \ln\frac{U^2}{\mu^2} + \mu^2 \right].$$

Convergence of the integral in (3.20) implies

$$\int_0^\infty \frac{d\mu^2}{\mu^2}\, \left(\mu^2\right)^p \delta\alpha_{\rm eff}(\mu^2) \;=\; 0\,; \quad p = 1, 2\,, \tag{3.21}$$

that is, the vanishing of the first two moments of $\delta\alpha_{\rm eff}(\mu^2)$. The result now reads

$$\frac{\pi^2}{9}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle_{\rm NP} \;=\; \frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\, \mu^4 \ln\mu^2\, \delta\alpha_{\rm eff}(\mu^2)\,. \tag{3.22}$$

Notice this (log-)moment integral is independent of the scale of the logarithm.

Inspecting higher moments (operators of higher dimension), one finds it necessary to impose on $\delta\alpha_{\rm eff}(\mu^2)$ the set of restrictions (3.21) with $p = 1, 2, \ldots p_{\rm max}$, to respect the criterion of "soft confinement" as seen through Euclidean eyes. Applied to a Minkowskian observable F, this means that only those terms in the small-μ^2 behaviour of the characteristic function $\dot{\mathcal{F}}(\mu^2)$ that are *non-analytic* in μ^2 will contribute to $F^{\rm NP}$. Among them are the terms with non-integer p and/or those with $\log\mu^2$-enhanced asymptotic behaviour.

Eq.(3.22) gives an example of a "measurement" of one of the log-moments of $\delta\alpha_{\rm eff}$. In the rest of the present paper we study how this and other non-zero moments of $\delta\alpha_{\rm eff}$ enter into different

Minkowskian observables. According to Eq. (3.5), the corresponding contribution to a generic collinear safe observable F will be of the form

$$F^{\rm NP}(Q^2,\{x\}) = \int_0^\infty \frac{d\mu^2}{\mu^2}\,\delta\alpha_{\rm eff}(\mu^2)\dot{\mathcal{F}}(Q^2,\{x\};\mu^2)\,. \tag{3.23}$$

The leading non-zero contribution to the integral will come from the first non-analytic term in the small-ϵ expansion of $\mathcal{F}(Q^2,\{x\};\mu^2) = \mathcal{F}(\epsilon,\{x\})$, Eq. (3.10). If p is not an integer, the first term is non-analytic and the leading contribution is proportional to $Q^{-2p}\ln^q Q$ where $f_q \ln^q\epsilon$ is the largest non-vanishing term in Eq. (3.11). If p is an integer and f_1 or f_2 is non-vanishing, the leading contribution is proportional to $Q^{-2p}\ln^{q-1} Q$, because the first singular contribution in μ^2 will be of the form $\mu^{2p}\ln\mu$. If p is an integer and both f_1 and f_2 are zero, the contribution will be proportional to $Q^{-2(p+1)}\ln^{q-1} Q$, where q is now specified by the largest non-vanishing term of $g(\epsilon)$.

We shall call power-behaved contributions of this type *dispersively-generated* power corrections. An important question to be addressed is: how much of the full power correction to a given observable is due to these dispersively-generated terms?

For quantities like DIS structure functions and the Drell-Yan K-factor, dispersively-generated $1/Q^2$ power contributions arise which are given by the first (log- and log^2-enhanced) moment of $\delta\alpha_{\rm eff}(\mu^2)$. The origin of these contributions may be traced back to a *universal* $1/Q^2$-suppressed non-perturbative correction to the relevant *hard cross sections* (coefficient functions). At the same time, from general OPE considerations one expects $1/Q^2$ corrections proportional to the hadronic matrix elements of the relevant twist-4 operators, which depend on the target hadron. Since the dispersively-generated terms have a definite, calculable dependence on the hard process kinematics, it may be possible to identify kinematic regions in which they are dominant.

In the case of e^+e^- -annihilation, which is a hard process free from initial-state hadrons, the situation might be simpler. In particular, for the practically important case of *linear* ($1/Q$) corrections to event shapes, dispersively-generated terms could describe the entire contributions to different quantities in a universal way. Such a hypothesis seems plausible not only because there is no competition from the conventional OPE contributions, but also because it follows naturally from the picture of soft (local in phase space) hadronization, which one invokes to explain the observed similarity (duality) between the calculable distributions of partons and the measured distributions of hadrons from QCD jets [31].

4 Applications

In this section we apply the dispersive method of the previous section to various hard processes dominated by quarks. Recall that the object of central importance is the characteristic function $\mathcal{F}(\epsilon)$ for the emission of a gluon with mass-squared $\mu^2 = \epsilon Q^2$ at the hard scale Q^2. For power corrections, the relevant contribution is given by the leading non-analytic term in the small-ϵ behaviour of the logarithmic derivative $\dot{\mathcal{F}}$, which is of the general form (3.10). The coefficient of the resulting contribution is then determined by the function $f(\ln\epsilon)$ (or by the next-to-leading function $g(\ln\epsilon)$ if the leading term is analytic). Since f or g is a polynomial in $\ln(\mu^2/Q^2)$ of degree not more than two, we introduce for future use the moment integrals

$$A_{2p} \;=\; \frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\,\mu^{2p}\;\delta\alpha_{\rm eff}(\mu^2) \tag{4.1}$$

and their derivatives

$$A'_{2p} = \frac{d}{dp}A_{2p}\;,\qquad A''_{2p} = \frac{d^2}{dp^2}A_{2p}\;, \tag{4.2}$$

which have respectively an extra factor of $\ln \mu^2$ and $\ln^2 \mu^2$ in the integrand of Eq. (4.1). Then all dispersively-generated power corrections proportional to Q^{-2p} can be represented as linear combinations of A_{2p}, A'_{2p} and A''_{2p}. Since the integrand must be non-analytic in μ^2, the unprimed moments A_{2p} can only contribute when $p \leq p_{\max}$ is not an integer, while the primed (log) moments can contribute for any p. The quantity on the right-hand side of Eq.(3.22), for example, is denoted by A'_4, corresponding to a $1/Q^4$ correction to the Adler D-function.

In general the characteristic function $\mathcal{F}(\epsilon)$ has both real and virtual contributions. Since the latter are universal we discuss them first. Their form depends on whether the momentum transfer in the hard process is space-like or time-like.

Consider the one-gluon virtual correction to a hard process (*e.g.* DIS) with a space-like momentum transfer $-q^2 = Q^2 > 0$. The total correction to the renormalized hard interaction vertex is of the form $C_F\alpha_s\mathcal{V}_s/2\pi$ where

$$\begin{aligned}
\mathcal{V}_s(\epsilon) &= -2\int_0^1 dz(1-z)^2\int_0^\infty \frac{dk^2\,Q^2}{(k^2+\mu^2)(k^2+zQ^2)} = -2\int_0^1 dz\frac{(1-z)^2}{z-\epsilon}\ln\frac{z}{\epsilon} \\
&= 2(1-\epsilon)^2\left[\mathrm{Li}_2(1-1/\epsilon)-\frac{\pi^2}{6}\right]-\frac{7}{2}-(3-2\epsilon)\ln\epsilon+2\epsilon \qquad (4.3a)\\
&= 2(1-\epsilon)^2\left[\mathrm{Li}_2(\epsilon)+\ln\epsilon\ln(1-\epsilon)-\frac{1}{2}\ln^2\epsilon-\frac{\pi^2}{3}\right]-\frac{7}{2}-(3-2\epsilon)\ln\epsilon+2\epsilon\,,
\end{aligned}$$

where

$$\mathrm{Li}_2(u) \equiv -\int_0^u \frac{dt}{t}\ln(1-t)\,.$$

For a time-like process ($q^2 = Q^2 > 0$, as in e^+e^- annihilation and the Drell-Yan process), the virtual correction is $C_F\alpha_s\mathcal{V}_t/2\pi$ where

$$\begin{aligned}
\mathcal{V}_t(\epsilon) &= \mathrm{Re}\ \mathcal{V}_s(-\epsilon) = -2\int_0^1 dz\,\frac{(1-z)^2}{z+\epsilon}\,\ln\frac{z}{\epsilon} \\
&= 2(1+\epsilon)^2\left[\mathrm{Li}_2(-\epsilon)+\ln\epsilon\ln(1+\epsilon)-\frac{1}{2}\ln^2\epsilon+\frac{\pi^2}{6}\right]-\frac{7}{2}-(3+2\epsilon)\ln\epsilon-2\epsilon\,.
\end{aligned} \qquad (4.3b)$$

Next we combine these with the real contributions for various hard processes and analyse the resulting power corrections. We consider first collinear safe and then collinear singular processes.

4.1 Total e^+e^- annihilation cross section

The most straightforward application of the method is to calculate the power correction to the e^+e^- annihilation cross section $R(Q^2)$, given to first order by (we suppress the standard parton model normalization factor $R^0 = N_c\sum_f e_f^2$)

$$R^{(1)} = 1 + \frac{3C_F}{4\pi}\,\alpha_s(Q^2) + \dots\,. \qquad (4.4)$$

In our approach, this result originates from perturbative evaluation of the expression

$$R \;=\; 1 + \frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\alpha_{\mathrm{eff}}(\mu^2)\cdot\dot{\mathcal{R}}(Q^2,\mu^2)\,, \qquad \dot{\mathcal{R}} \equiv -\frac{d}{d\ln\mu^2}\mathcal{R}(Q^2,\mu^2)\ , \qquad (4.5)$$

where $\mathcal{R}$ is the characteristic function obtained from diagrams involving one gluon with mass μ. This function has both a real and a virtual contribution (see Fig. 1). The latter is simply given by the function $\mathcal{V}_t(\epsilon)$ in (4.3b). To obtain the real contribution we consider the emission of a quark, an antiquark and a gluon of momenta p, $\bar{p}$ and k respectively ($k^2 = \mu^2$, $p^2 = \bar{p}^2 = 0$). The matrix element assumes a simple form in terms of the scaled quark and antiquark energies,

$$x = 2pQ/Q^2 \,, \quad \bar{x} = 2\bar{p}Q/Q^2 \,,$$

which satisfy the phase-space constraints

$$(1-x)(1-\bar{x}) \geq \epsilon \,, \quad x + \bar{x} \leq 1 - \epsilon \,.$$

The matrix element squared is then of the form $C_F \alpha_s \mathcal{M}_{ee}/2\pi$, where

$$\mathcal{M}_{ee}(x, \bar{x}, \epsilon) = \frac{(x+\epsilon)^2 + (\bar{x}+\epsilon)^2}{(1-x)(1-\bar{x})} - \frac{\epsilon}{(1-x)^2} - \frac{\epsilon}{(1-\bar{x})^2} \,. \tag{4.6}$$

The real contribution to the characteristic function $\mathcal{R}$ is thus

$$\begin{aligned} \mathcal{R}^{(r)}(\epsilon) &= \int_{ph.sp.} dx\, d\bar{x} \mathcal{M}_{ee}(x, \bar{x}, \epsilon) \\ &= -2(1+\epsilon)^2 \left[2\mathrm{Li}_2(-\epsilon) + 2\ln\epsilon \ln(1+\epsilon) - \frac{1}{2}\ln^2\epsilon + \frac{\pi^2}{6} \right] + 5 + (3 + 4\epsilon + 3\epsilon^2)\ln\epsilon - 5\epsilon^2 \,, \end{aligned} \tag{4.7}$$

with $\epsilon \leq 1$. At the edge of phase space ($\epsilon \to 1$) the distribution vanishes rapidly, as

$$\mathcal{R}^{(r)}(\epsilon) = \frac{1}{10}(1-\epsilon)^5 \left\{1 + \mathcal{O}\left(1-\epsilon\right)\right\} \,. \tag{4.8}$$

Finally the complete characteristic function is

$$\begin{aligned} &\text{for} \quad \epsilon > 1 \,, \quad \mathcal{R}(\epsilon) = \mathcal{V}_t(\epsilon) \,, \\ &\text{for} \quad \epsilon < 1 \,, \quad \mathcal{R}(\epsilon) = \mathcal{R}^{(r)}(\epsilon) + \mathcal{V}_t(\epsilon) \\ &\qquad = -2(1+\epsilon)^2 \left[\mathrm{Li}_2(-\epsilon) + \ln\epsilon \ln(1+\epsilon) \right] + \frac{3}{2} + (2 + 3\epsilon)\epsilon \ln\epsilon - 2\epsilon - 5\epsilon^2 \,. \end{aligned} \tag{4.9}$$

The function $\mathcal{R}(\epsilon)$ is plotted in Fig. 1 together with its logarithmic derivative $\dot{\mathcal{R}}$ in Fig. 2. The behaviour for small and large ϵ is as follows:
For $\epsilon \ll 1$

$$\mathcal{R} = \frac{3}{2}(1-\epsilon^2) - \frac{2\epsilon^3}{3}\left(\ln\epsilon - \frac{11}{6}\right) + \mathcal{O}\left(\epsilon^4 \ln\epsilon\right) \,. \tag{4.10a}$$

For $\epsilon \to \infty$

$$\mathcal{R} = \frac{2}{3\epsilon}\left(\ln\epsilon + \frac{11}{6}\right) + \mathcal{O}\left(\frac{\ln\epsilon}{\epsilon^2}\right) \,. \tag{4.10b}$$

These two limiting forms are in fact related by the following symmetry of the e^+e^- characteristic function.

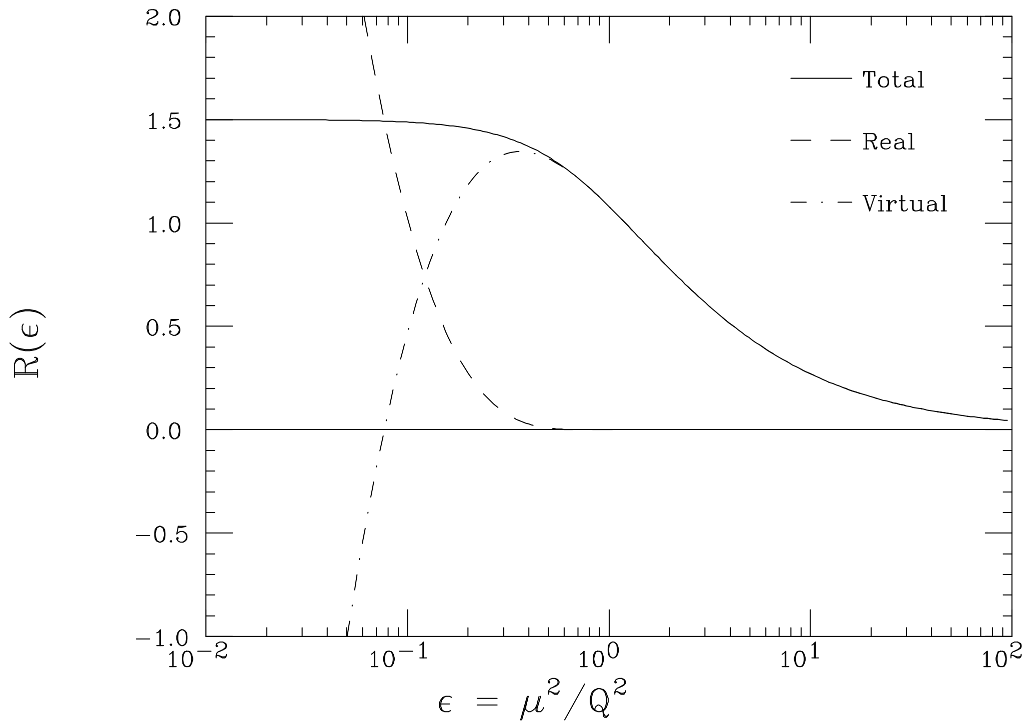

Figure 1: Characteristic function for e^+e^- total cross section, $\mathcal{R}$.

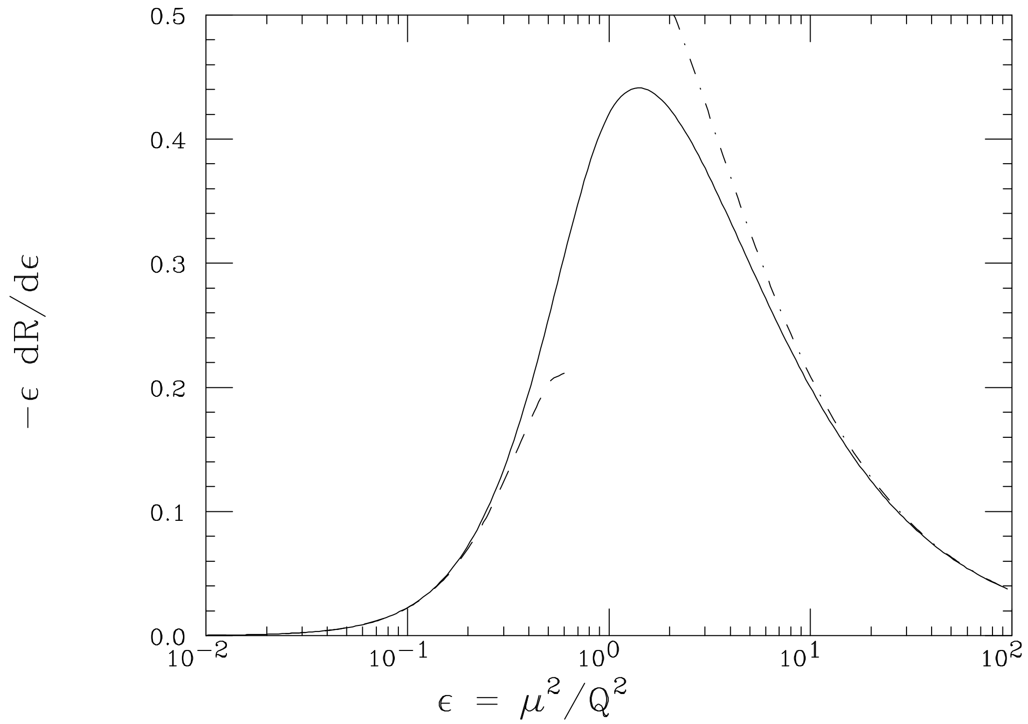

Figure 2: Derivative of characteristic function for e^+e^- total cross section, $\dot{\mathcal{R}}$. Dashed and dot-dashed curves show the limiting behaviour at small and large ϵ, respectively.

Symmetry. The function $\mathcal{R}$ given in (4.9) satisfies the following inversion symmetry

$$\frac{1}{\epsilon}\left[\mathcal{R}(\epsilon)-\frac{3}{2}\right] \;=\; \epsilon\left[\mathcal{R}\left(\frac{1}{\epsilon}\right)-\frac{3}{2}\right]. \tag{4.11}$$

The behaviour for $\epsilon \to 0$ follows from this symmetry and the fact that for $\epsilon \to \infty$ the characteristic function $\mathcal{R}(\epsilon) = \mathcal{V}_t(\epsilon)$ vanishes in the way specified in (4.10b).

The symmetry (4.11) also explains the fact that in the sum of real and virtual contributions one finds that not only the terms singular as $\mu^2 \to 0$,

$$\ln^2\mu^2\,, \qquad \ln\mu^2\,,$$

cancel, as required by the Bloch-Nordsieck theorem, but that the cancellation extends [7,17] also to the following finite terms

$$\mu^2\ln^2\mu\,, \quad \mu^2\ln\mu\,, \quad \mu^2\,, \quad \mu^4\ln^2\mu\,, \quad \mu^4\ln\mu\,.$$

Perturbative evaluation of $R(Q^2)$. Now we are in a position to evaluate R according to (4.5). Since $\dot{\mathcal{R}}$ vanishes both for small and large ϵ,

$$\dot{\mathcal{R}} = 3\epsilon^2 + 2\epsilon^3\ln\epsilon + \ldots\,, \quad \epsilon \to 0\,, \qquad \dot{\mathcal{R}} = -\frac{2}{3\epsilon}\ln\epsilon + \ldots \quad \epsilon\to\infty\,, \tag{4.12}$$

the main contribution to (4.9) comes from $\epsilon \sim 1$, that is $\mu^2 \sim Q^2$ (see Fig. 2). Therefore the leading perturbative contribution to $R^{\mathrm{PT}}(Q^2)$ is obtained by taking the value of the coupling constant $\alpha_{\mathrm{eff}}(\bar{Q}^2)$ outside the integration and one finds

$$R^{\mathrm{PT}}(Q^2) \;=\; 1 + \frac{C_F}{2\pi}\alpha_{\mathrm{eff}}(\bar{Q}^2)\mathcal{R}(0) \;+\; \mathcal{O}\left(\alpha_s^2(\bar{Q}^2)\right)\,, \tag{4.13}$$

where $\mathcal{R}(0) = 3/2$ and $\bar{Q}^2 \sim Q^2$ should be chosen in the vicinity of the peak, so that $\alpha_{\mathrm{eff}}(\bar{Q}^2) \approx \alpha_s(Q^2)$ and one obtains the result (4.4).

Leading power correction. As indicated in Eq. (4.12), the first non-analytic term in the expansion of $\dot{\mathcal{R}}$ at small ϵ is of order $\epsilon^3\ln\epsilon$. Thus the leading power-behaved contribution is given in terms of the moment integral (4.1) for the effective coupling as

$$R^{\mathrm{NP}} \simeq 2\frac{A_6'}{Q^6}\,. \tag{4.14}$$

The absence of a $1/Q^2$ contribution follows from the lack of any suitable dimension-two operators, in the massless quark limit. In principle one might have expected that a $1/Q^4$ contribution could be present, due to the gluon condensate $\langle\alpha_s G^2\rangle$. As we have already remarked, such a contribution is indeed expected in the Adler D-function, but to first order in $\delta\alpha_{\mathrm{eff}}$ it does not appear in R itself, which is related to the discontinuity of the D-function [16]. In order for a $1/Q^4$ term to appear in R one would need a $\ln Q^2/Q^4$ term in D. At the same time, a leading power correction of the form (4.14) is consistent with the OPE, since the D-function does acquire a log-enhanced contribution from operators of dimension six (see [12,13]).

Hadronic width of the τ lepton. The τ decay width [9–13] is closely related to the e^+e^- annihilation cross section. It is normally expressed in terms of the quantity R_τ, which for massless quarks is

$$R_\tau = 2\int_0^{m_\tau^2} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right) R(s)\,. \tag{4.15}$$

Thus we can write

$$R_\tau \;=\; 1+\frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\alpha_{\rm eff}(\mu^2)\cdot\dot{\mathcal{R}}_\tau(m_\tau^2,\mu^2) \tag{4.16}$$

where

$$\dot{\mathcal{R}}_\tau(m_\tau^2,\mu^2) = 2\int_0^{m_\tau^2} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)\dot{\mathcal{R}}(s,\mu^2)\,. \tag{4.17}$$

Defining $y=\mu^2/m_\tau^2$ and $\epsilon=\mu^2/s$, we can write this in the form

$$\dot{\mathcal{R}}_\tau(y) = 2y\int_y^\infty \frac{d\epsilon}{\epsilon^2}\left(1-\frac{y}{\epsilon}\right)^2\left(1+2\frac{y}{\epsilon}\right)\dot{\mathcal{R}}(\epsilon)\,. \tag{4.18}$$

The form of $\dot{\mathcal{R}}_\tau(y)$ is shown by the solid curve in Fig. 3. At large y the behaviour is

$$\dot{\mathcal{R}}_\tau(y) = \frac{1}{5y}\left(\ln y+\frac{107}{60}\right)+\mathcal{O}\left(\frac{\ln y}{y^2}\right)\;, \tag{4.19}$$

shown by the dot-dashed curve. At small y

$$\dot{\mathcal{R}}_\tau(y) = 8\,(4-3\zeta(3))\,y-18y^2+\left(6\ln^2 y-12\ln y+\frac{32}{3}\right)y^3\;+\;\ldots\,. \tag{4.20}$$

(dashed).

The fact that the non-analyticity of the expansion (4.20) starts at order y^3 is again in accord with the absence of a dimension-four contribution in the operator product expansion [12,15]. In this case the $\ln^2 y$ factor indicates a logarithmic enhancement of the power-behaved correction to R_τ, which takes the form

$$R_\tau^{\rm NP}\;\simeq -6\,(2A_6'\ln m_\tau^2+2A_6'-A_6'')/m_\tau^6\,. \tag{4.21}$$

4.2 DIS structure function F_2

We next consider the deep inelastic scattering process in which a quark, with momentum p, is probed by a hard spacelike photon of momentum q ($q^2=-Q^2$, $x=Q^2/2pq$). Recall from Sect. 3 that in the case of collinear singular quantities, such as DIS structure functions (or their moments) and e^+e^- fragmentation functions, the characteristic function $\dot{\mathcal{F}}(Q^2,\{x\};\mu^2)$ has a finite Q^2-independent collinear limit, $\dot{\mathcal{F}}(Q^2,\{x\};0)\equiv P(x)$. For small ϵ (inside kinematical limits),

$$\dot{\mathcal{F}}(\epsilon) = P(x)+\dot{\mathcal{F}}_{reg}(\epsilon)\,, \tag{4.22}$$

where $\dot{\mathcal{F}}_{reg}(\epsilon)$ vanishes as a power of ϵ at $\epsilon\to 0$, as does $\dot{\mathcal{F}}$ for the collinear safe case, see (3.10). As a consequence, the function $\ddot{\mathcal{F}}$ vanishes as a power for both $\epsilon\to 0$ and $\epsilon\to\infty$. This quantity determines the scaling violation according to Eq. (3.7). More precisely, defining the moments of the non-singlet part of the structure function F_2 as

$$F_2(Q^2,N) = \int_0^1 x^{N-1}F_2(Q^2,x)\,dx\;, \tag{4.23}$$

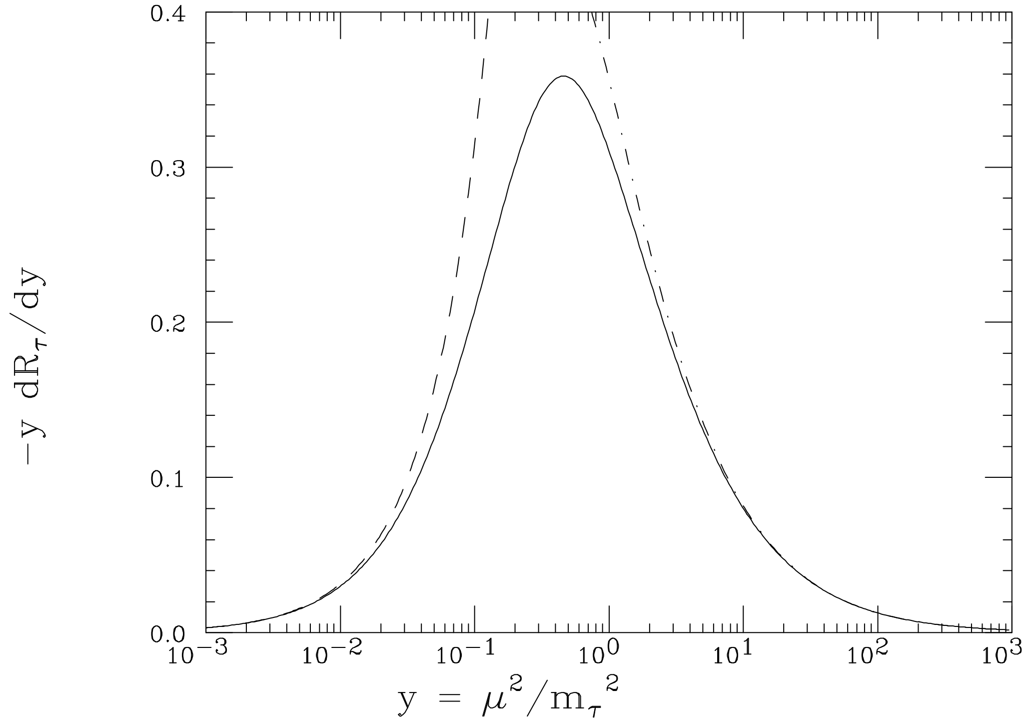

Figure 3: Derivative of characteristic function for τ decay, $\dot{\mathcal{R}}_\tau$. Dashed and dot-dashed curves show the limiting behaviour at small and large y, respectively.

the scaling violation is described by

$$\Gamma_N(Q^2) \equiv Q^2 \frac{\partial}{\partial Q^2} \ln F_2(Q^2, N) \;=\; \frac{C_F}{2\pi} \int_0^\infty \frac{d\mu^2}{\mu^2}\, \alpha_{\rm eff}(\mu^2)\, \ddot{\mathcal{F}}(\epsilon = \mu^2/Q^2, N)\,. \tag{4.24}$$

Evaluation of characteristic function. The characteristic function $\mathcal{F}(\epsilon)$ is obtained by considering the amplitude for an incoming and outgoing quark and for an off-shell gluon of momentum k, either virtual or real. For the combination of polarizations leading to the F_2 structure function, the amplitude-squared for off-shell gluon emission assumes the following simple form (apart from the coupling α_s and the quark colour factor $C_F/2\pi$)

$$\mathcal{M}_2 = \frac{y(1-x) - x\epsilon}{(y - x\epsilon)^2} + \frac{2x(1-y)(1-\epsilon)}{(y-x\epsilon)(1-x)} + \frac{y(1-x)-x\epsilon}{(1-x)^2} + 6x\frac{y^2(1-x)}{(y-x\epsilon)^2}\,, \tag{4.25}$$

with $y = (pk)/(pq)$. The last term in (4.25) is the one contributing to the longitudinal structure function F_L . When $\epsilon = 0$ the first and second terms are collinear singular for $y \to 0$, with coefficient equal to the quark splitting function $P(x) = (1 + x^2)/(1 - x)$.

The real part of the characteristic function contributing to F_2 is obtained by integrating y over the phase space region $\epsilon x/(1 - x) < y < 1$. Including the virtual contribution, which is now $\mathcal{V}_s(\epsilon)$ given by Eq. (4.3), one finds that

$$\mathcal{F}(x;\epsilon) = \mathcal{F}^{(r)}(x;\epsilon)\,\Theta(1 - x - \epsilon x) + \mathcal{V}_s(\epsilon)\,\delta(1-x) \tag{4.26}$$

with

$$\mathcal{F}^{(r)}(x;\epsilon) = \left[\frac{2(1-\epsilon)^2}{1-x} - (1+x) + 2(2+x+6x^2)\epsilon - 2(1+x+9x^3)\epsilon^2\right] \ln\left[\frac{(1-\epsilon x)(1-x)}{\epsilon x^2}\right]$$
$$- \frac{3+14\epsilon-15\epsilon^2}{2(1-x)} + \frac{\epsilon}{(1-x)^2} + \frac{\epsilon^2}{2(1-x)^3} + \frac{x}{1-\epsilon x}$$
$$+ 1 + 3x + 6(1-x)(1+3x)\epsilon - (8+9x+18x^2)\epsilon^2 \,. \tag{4.27}$$

The coefficient of $-\ln\epsilon$ is the quark splitting function $P(x)$, which is singular for $x \to 1$. This singularity is regularized by including the virtual contribution. The F_2 quark structure function to this order is thus (we suppress the standard parton model factor $\sum_f e_f^2$)

$$F_2(x,Q^2) \;=\; \delta(1-x) \;+\; \frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\,\alpha_{\rm eff}(\mu^2)\;\dot{\mathcal{F}}(x;\epsilon)\,. \tag{4.28}$$

The integral is convergent for $\mu^2 \to \infty$. In this limit the characteristic function $\mathcal{F}(\epsilon)$ is given by the virtual contribution alone and we have

$$\mathcal{F} = -\frac{2}{3\epsilon}\left(\ln\epsilon + \frac{11}{6}\right) \;+\; \mathcal{O}\left(\frac{\ln\epsilon}{\epsilon^2}\right)\mathcal{O}\left(\frac{\ln\epsilon}{\epsilon^3}\right)\,.. \tag{4.29}$$

In Figs. 4 and 5 we show the first few moments of $\dot{\mathcal{F}}$ and $\ddot{\mathcal{F}}$, for $N = 2,3,4$. Note that the $N = 1$ moment of $\dot{\mathcal{F}}$ vanishes owing to the following identity, which holds for any ϵ,

$$\mathcal{V}_s(\epsilon) = -\int_0^1 \mathcal{F}^{(r)}(x;\epsilon)\Theta(1-x-\epsilon x)\,dx\,. \tag{4.30}$$

This means that the Adler sum rule is satisfied identically, that is, it receives neither perturbative nor power corrections within our approach (see [32]).

Logarithmic scaling violation. The moments of the limiting function $P(x)$ in (4.22) have the meaning of the anomalous dimensions γ_N.[4] The usual perturbative scaling violation result for the moments of the DIS structure function, namely,

$$\Gamma_N(Q^2) \;=\; \gamma_N(\alpha_s(Q^2)) \;+\; \frac{d\,\alpha_s(Q^2)}{d\ln Q^2}\,\frac{\partial}{\partial\alpha_s}\,C_N(\alpha_s)\,, \tag{4.31}$$

is then reproduced as follows. Since the integrand in (3.7) is peaked at $\mu^2 = \bar{Q}^2 \sim Q^2$, one obtains the leading contribution by substituting $\alpha_{\rm eff}(\mu^2) \to \alpha_s(\bar{Q}^2)$:

$$\begin{aligned}\Gamma_N^{(1)} &= \alpha_s(\bar{Q}^2)\frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\,\ddot{\mathcal{F}}(Q^2,N;\mu^2)\\ &= \alpha_s(\bar{Q}^2)\frac{C_F}{2\pi}\left[\dot{\mathcal{F}}(Q^2,N;0) - \dot{\mathcal{F}}(Q^2,N;\infty)\right] = \alpha_s(\bar{Q}^2)\frac{C_F}{2\pi}P_N\end{aligned} \tag{4.32}$$

where P_N is the corresponding moment of $P(x)$. Keeping the next term in the expansion for the running coupling,

$$\alpha_s(\mu^2) - \alpha_s(\bar{Q}^2) \;=\; \frac{d\,\alpha_s(\bar{Q}^2)}{d\ln\bar{Q}^2}\,\ln\frac{\mu^2}{\bar{Q}^2} \;+\; \ldots,$$

[4] It should be noted that in higher orders $P(x)$ will start to depend on the integration scale through the effective coupling, $P = P(x;\alpha_{\rm eff}(\mu^2))$, while remaining Q^2-independent.

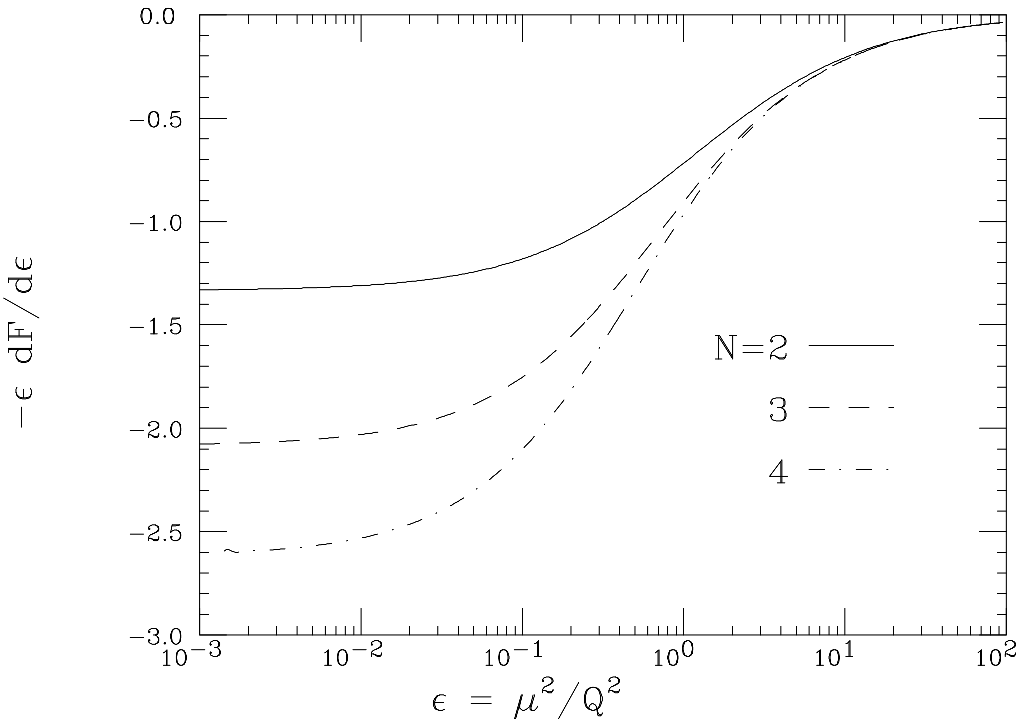

Figure 4: Derivative of characteristic function for DIS, $\dot{\mathcal{F}}_N$.

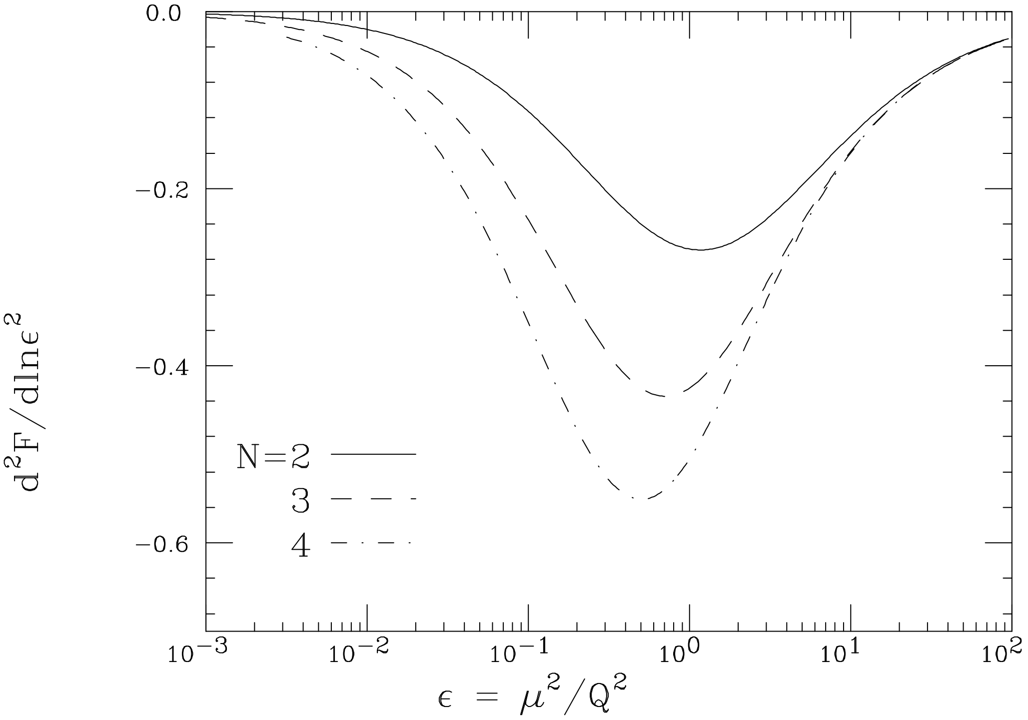

Figure 5: Second derivative of characteristic function for DIS, $\ddot{\mathcal{F}}_N$.

the first correction to (4.32) can be derived as

$$\Gamma_N - \Gamma_N^{(1)} \approx \frac{d\,\alpha_s(\bar{Q}^2)}{d\ln\bar{Q}^2}\frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\,\ln\frac{\bar{Q}^2}{\mu^2}\,\ddot{\mathcal{F}}(Q^2,N;\mu^2) = \frac{d\,\alpha_s(\bar{Q}^2)}{d\ln\bar{Q}^2}\frac{C_F}{2\pi}c_N\,, \tag{4.33}$$

where

$$c_N \;\equiv\; \lim_{\mu^2\to 0}\left\{\mathcal{F}(Q^2,N;\mu^2) - \ln\frac{\bar{Q}^2}{\mu^2}\cdot P_N\right\} = \mathcal{F}_{reg}(Q^2,N;0) \tag{4.34}$$

is the coefficient function, the value of which clearly depends on the choice of the expansion scale $\bar{Q}^2$. An obvious (first-order) identification is as follows,

$$\gamma_N(\alpha_s) \Longleftrightarrow \alpha_s C_F P_N/2\pi\,, \qquad C_N(\alpha_s) \Longleftrightarrow \alpha_s C_F c_N/2\pi\,. \tag{4.35}$$

Notice that in the original expression (4.24) there is no arbitrariness in the choice of the hard scale (scheme dependence), which only emerges when one tries to evaluate the area under the $\ddot{\mathcal{F}}$ curves in Fig. 5 (weighted with $\alpha_{\rm eff}(\mu^2)$) in terms of a series expansion around a given point $\bar{Q}^2$.

Another point that is clearly seen from Fig. 5 concerns the actual hardness scale of the process. The region from which the scaling violation rate receives its main contribution shifts to lower momentum scales with increasing N; in fact the peak is around $\bar{Q}^2 \sim Q^2/N$, or, equivalently, $Q^2(1-x) \approx W^2$. To avoid confusion, let us stress that identification of W^2 with the proper physical hardness scale is true for non-singlet structure functions only. Therefore it should not be applied to the small-x region, which is dominated by the singlet contribution, the scale of which does exceed Q^2 but by far less than the kinematically allowed limit $W^2 \approx Q^2/x \gg Q^2$.

To obtain the power corrections we have to consider the small-ϵ limit of the characteristic function.

Small-ϵ behaviour. When taking this limit, one has to exercise some care in treating the singular functions and distributions in Eq. (4.27). It follows from the identity (4.30) that for any test function $f(x)$ we have

$$\begin{aligned}\int_0^1 \mathcal{F}(x;\epsilon)\,f(x)\,dx &= \int_0^1 \mathcal{F}^{(r)}(x;\epsilon)\Theta(1-x-\epsilon x)\,[f(x)-f(1)]\;dx\\ &= \int_0^1 \mathcal{F}^{(r)}(x;\epsilon)\,[f(x)-f(1)]\;dx + f'(1)\int_{1/(1+\epsilon)}^1 \mathcal{F}^{(r)}(x;\epsilon)(1-x)\,dx + \ldots\,.\end{aligned} \tag{4.36}$$

Defining '+' and '++' prescriptions such that

$$\begin{aligned}F(x)_+ &= F(x) - \delta(1-x)\int_0^1 F(z)\,dz\,,\\ F(x)_{++} &= F(x)_+ + \delta'(1-x)\int_0^1 F(z)(1-z)\,dz\,,\end{aligned} \tag{4.37}$$

and recalling that

$$\int_0^1 \delta'(1-x)\,f(x)\,dx = f'(1)\,, \tag{4.38}$$

we should therefore define the singular terms in Eq. (4.27) at small ϵ, up to terms of order ϵ, as follows:

$$\begin{aligned}
\frac{1}{1-x} &\to \frac{1}{(1-x)_+} + \epsilon\,\delta'(1-x) \\
\frac{\ln(1-x)}{1-x} &\to \left(\frac{\ln(1-x)}{1-x}\right)_+ + (\epsilon\ln\epsilon - \epsilon)\,\delta'(1-x) \\
\frac{\epsilon}{(1-x)^2} &\to \frac{\epsilon}{(1-x)^2_{++}} + \epsilon\ln\epsilon\,\delta'(1-x) \\
\frac{\epsilon^2}{(1-x)^3} &\to -\epsilon\,\delta'(1-x)\,.
\end{aligned} \tag{4.39}$$

The small-ϵ behaviour of $\mathcal{F}$ is thus of the form

$$\mathcal{F}(x;\epsilon) = -P(x)\ln\epsilon + c(x) + g(x)\,\epsilon\ln\epsilon + h(x)\,\epsilon + \mathcal{O}\left(\epsilon^2\ln\epsilon\right)\,, \tag{4.40}$$

where

$$P(x) = \frac{2}{(1-x)_+} - (1+x) + \frac{3}{2}\,\delta(1-x) = \left(\frac{1+x^2}{1-x}\right)_+ \tag{4.41a}$$

$$\begin{aligned}
c(x) &= 2\left(\frac{\ln[(1-x)/x^2]}{1-x}\right)_+ - (1+x)\ln\left(\frac{1-x}{x^2}\right) + 1 + 4x \\
&\quad - \frac{3}{2(1-x)_+} - \frac{9}{4}\,\delta(1-x)
\end{aligned} \tag{4.41b}$$

$$g(x) = \frac{4}{(1-x)_+} - 2(2+x+6x^2) + 9\,\delta(1-x) + \delta'(1-x) \tag{4.41c}$$

$$\begin{aligned}
h(x) &= -4\left(\frac{\ln[(1-x)/x^2]}{1-x}\right)_+ + 2(2+x+6x^2)\ln\left(\frac{1-x}{x^2}\right) \\
&\quad - \frac{9}{(1-x)_+} + \frac{1}{(1-x)^2_{++}} + 8 + 13x - 16x^2 - 8\,\delta(1-x) - 4\,\delta'(1-x)\,.
\end{aligned} \tag{4.41d}$$

The corresponding formula in moment space is

$$\mathcal{F}_N(\epsilon) = -P_N\ln\epsilon + c_N + g_N\,\epsilon\ln\epsilon + h_N\,\epsilon + \mathcal{O}\left(\epsilon^2\ln\epsilon\right)\,, \tag{4.42}$$

where

$$P_N = -2S_1 + \frac{3}{2} - \frac{1}{N} - \frac{1}{N+1} \tag{4.43a}$$

$$c_N = \left(\frac{3}{2} + \frac{1}{N} + \frac{1}{N+1}\right)S_1 + S_1^2 - 3S_2 - \frac{9}{4} + \frac{2}{N} + \frac{3}{N+1} - \frac{1}{N^2} - \frac{1}{(N+1)^2} \tag{4.43b}$$

$$g_N = N + 8 - \frac{4}{N} - \frac{2}{N+1} - \frac{12}{N+2} - 4S_1 \tag{4.43c}$$

$$\begin{aligned}
h_N &= \left(N + 8 - \frac{4}{N} - \frac{2}{N+1} - \frac{12}{N+2}\right)S_1 - 2S_1^2 + 6S_2 \\
&\quad -5N - 3 + \frac{3}{N+1} + \frac{2}{N+2} + \frac{4}{N^2} + \frac{2}{(N+1)^2} + \frac{12}{(N+2)^2}
\end{aligned} \tag{4.43d}$$

with

$$S_p = \sum_{j=1}^{N-1}\frac{1}{j^p}\,; \quad S_1 = \psi(N) + \gamma_E = \ln N + \mathcal{O}(1/N)\,, \quad S_2 = \frac{\pi^2}{6} - \psi'(N)\,. \tag{4.44}$$

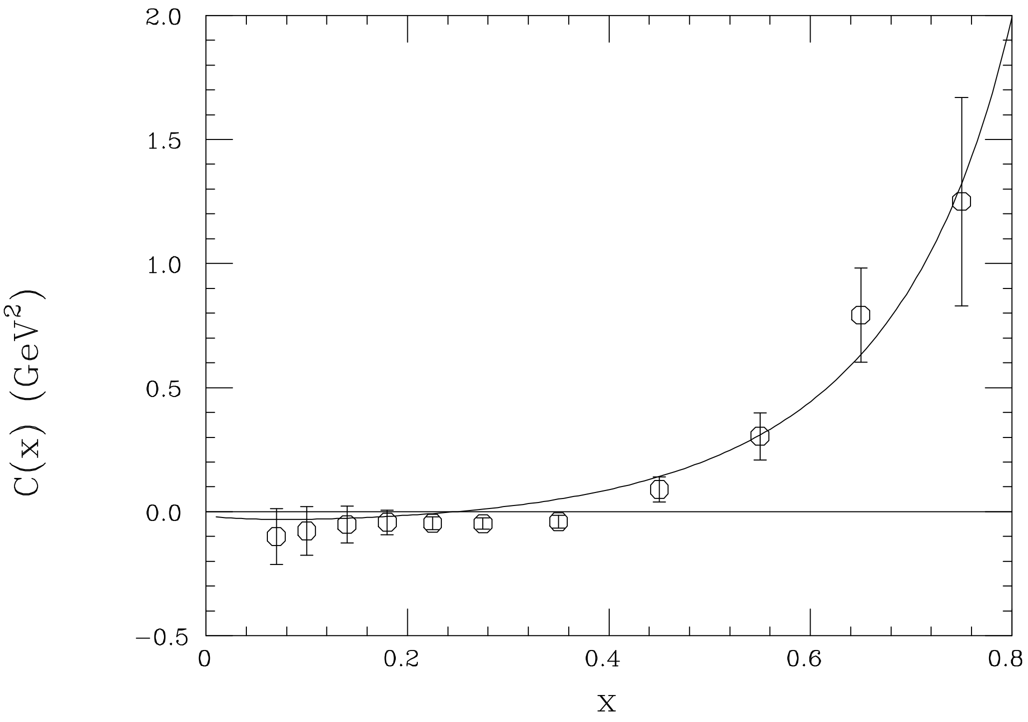

Figure 6: Coefficient of higher-twist contribution to DIS.

Power corrections. From (4.40) and (4.42) we can now evaluate the dispersively-generated power corrections to the usual (logarithmic) scaling violation using Eq. (4.24). The leading non-analytic term of $\ddot{\mathcal{F}}(\epsilon, N)$ for small ϵ is $g_N \epsilon \ln \epsilon$, leading to a power-behaved contribution of the form

$$\frac{d}{d \ln Q^2} \ln F_2^{\mathrm{NP}}(N, Q^2) \;=\; g_N \frac{A_2'}{Q^2} . \tag{4.45}$$

We shall not undertake detailed phenomenological studies in this paper, but only indicate how the above result could be applied. In the analysis of DIS data, the non-perturbative contribution is normally parametrized as a simple multiplicative correction of the form

$$F_2(x, Q^2) = F_2^{\mathrm{PT}}(x, Q^2) \left(1 + \frac{C(x)}{Q^2} \right) . \tag{4.46}$$

The 'higher-twist' coefficient $C(x)$ is then found to be a steeply increasing function of x, as illustrated by the data points in Fig. 6 [33].

In our approach, the leading non-perturbative contribution is of the form

$$F_2^{NP}(x, Q^2) = -A_2' \, g(x) \otimes F_2^{\mathrm{PT}}(x, Q^2)/Q^2 , \tag{4.47}$$

where '$\otimes$' represents the convolution corresponding to a product in moment space. Thus we predict a coefficient in Eq. (4.46) with a weak (logarithmic) Q^2 dependence,

$$C(x, Q^2) = -A_2' \, g(x) \otimes F_2^{\mathrm{PT}}(x, Q^2)/F_2^{\mathrm{PT}} . \tag{4.48}$$

The curve in Fig. 6 shows the prediction of Eq. (4.48) at $Q^2 = 10$ GeV^2, assuming $A_2' = -1$ GeV^2. For F_2^{PT} we have used the valence part of the MRSA parametrization [34]. We see that the form of the observed higher-twist correction is well reproduced. The steep rise at large x results mainly from the most singular term $\delta'(1-x)$ in $g(x)$, which generates the behaviour

$$C(x, Q^2) \sim A_2' \frac{\partial}{\partial x} \ln F_2^{\mathrm{PT}}(x, Q^2) \tag{4.49}$$

at $x \to 1$. However, the less singular terms also play a significant role in the region of x shown in Fig. 6.

4.3 DIS sum rules

Adler sum rule. As we have already mentioned above, due to the identity (4.30) the Adler sum rule (for a quark as a target)

$$\int_0^1 dx\, F_2(Q^2, x) = 1 \tag{4.50}$$

does not acquire either perturbative or non-perturbative correction to this order.

Gross-Llewellyn-Smith sum rule. This sum rule concerns the first moment of the F_3 structure function. For a target hadron h one has

$$G(Q^2) = \int_0^1 dx[F_3^{\nu h}(Q^2, x) + F_3^{\bar{\nu} h}(Q^2, x)] \;=\; G_0 \int_0^1 dx F_3(Q^2, x)\,, \tag{4.51}$$

where $G_0 = 6B - 2S$, with B and S the baryon number and strangeness of the hadron h, is the parton-model value. Here $F_3(Q^2, x)$ stands for the non-singlet structure function for lepton scattering by a single quark.

According to our procedure we first compute the matrix element squared for massive gluon emission corresponding to F_3. A simple calculation gives (apart from the coupling α_s and the colour factor $C_F/2\pi$)

$$\mathcal{M}_d = \mathcal{M}_3 - \mathcal{M}_2 = \frac{6\,yx - 4\,x + 2\,x\epsilon - 2\,y}{(y - x\epsilon)^2} y^2\,, \tag{4.52}$$

where $\mathcal{M}_2$ is given by (4.25). Since the virtual contributions to F_2 and F_3 are identical, the difference between their characteristic functions is given by

$$\begin{aligned} \mathcal{F}_3(x, \epsilon) - \mathcal{F}_2(x, \epsilon) &= \int_{x\epsilon/(1-x)}^1 dy\, \mathcal{M}_d(x, y, \epsilon) = -2\,(9\,x\epsilon - \epsilon - 4)\, x^2 \epsilon \ln\left(\frac{x^2\epsilon}{(1 - x\epsilon)(1 - x)}\right) \\ &- (1 + x) - \frac{2\,x^2\epsilon}{1 - x\epsilon} + \epsilon \left(18\,x^2\epsilon + 18\,x^2 + 7\,x\epsilon - 10\,x + 5\,\epsilon - 4 - \frac{2\,\epsilon}{(1 - x)^2} + \frac{4 - 3\,\epsilon}{1 - x}\right)\,. \end{aligned} \tag{4.53}$$

This difference, determined by real emission only, is defined for $\epsilon \le (1 - x)/x$ and vanishes at the phase space boundary as

$$\mathcal{F}_3(x, \epsilon) - \mathcal{F}_2(x, \epsilon) = -(1 + x) \left(\frac{\epsilon}{1 - x} - \frac{1}{x}\right)^2 + \ldots\,.$$

The first moment reads

$$\begin{aligned} \mathcal{F}_3(\epsilon, N\!=\!1) &= \int_0^{1/(1+\epsilon)} dx\, \mathcal{F}_3(x) \\ &= -\frac{3}{2} - \frac{\epsilon}{2} + \epsilon \left(\frac{4}{3} + \frac{5}{6}\,\epsilon\right) \ln\frac{\epsilon + 1}{\epsilon} - \left(\frac{1}{6} + \frac{2\epsilon}{3}\right) \left(\epsilon^{-2} \ln(1 + \epsilon) - \epsilon^{-1} + \frac{1}{2}\right)\,. \end{aligned} \tag{4.54}$$

It has the following small-ϵ behaviour, relevant for power corrections:

$$\mathcal{F}_3(\epsilon, N\!=\!1) = -\frac{3}{2} - \epsilon \left(\frac{4}{3} \ln\epsilon + \frac{5}{9}\right) - \epsilon^2 \left(\frac{5}{6} \ln\epsilon - \frac{83}{72}\right) + \mathcal{O}\left(\epsilon^3\right)\,. \tag{4.55}$$

Thus the one-loop correction to the first moment of the quark structure function F_3 is

$$F_3(Q^2, N=1) = \frac{C_F}{2\pi} \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\rm eff}(\mu^2) \dot{\mathcal{F}}_3(\epsilon, N=1)\,, \tag{4.56}$$

where

$$\dot{\mathcal{F}}_3(\epsilon, N=1) = \frac{1}{2} + \frac{5\,\epsilon}{3} - \epsilon\left(\frac{5}{3}\,\epsilon + \frac{4}{3}\right) \ln\frac{\epsilon+1}{\epsilon} - \frac{(2\,\epsilon+1)\ln(\epsilon+1) - \epsilon}{3\,\epsilon^2}\,. \tag{4.57}$$

As usual (for a collinear-safe quantity) this function vanishes for both large and small ϵ. The area under the curve gives the well-known one-loop perturbative contribution

$$G^{\rm PT}(Q^2) = G_0\,\left[1 - \frac{3C_F}{4\pi}\alpha_s(Q^2) + \mathcal{O}\left(\alpha_s^2(Q^2)\right)\right]. \tag{4.58}$$

We may now evaluate the dispersively-generated power corrections, which are determined by the small-ϵ behaviour

$$\dot{\mathcal{F}}_3(N=1, \epsilon) = \epsilon\left(\frac{4}{3}\ln\epsilon + \frac{17}{9}\right) + \epsilon^2\left(\frac{5}{3}\ln\epsilon - \frac{53}{36}\right) + \text{regular terms}\,. \tag{4.59}$$

The two non-analytic terms generate power-behaved contributions of the form

$$G^{\rm NP}(Q^2) \simeq G_0\left(\frac{4}{3}\frac{A_2'}{Q^2} + \frac{5}{3}\frac{A_4'}{Q^4}\right)\,. \tag{4.60}$$

The estimated coefficient of $1/Q^2$ [35] used in a recent analysis at $Q^2 \sim 3$ GeV^2 by the CCFR collaboration [36] is -0.09 ± 0.05 GeV^2. This implies a small negative value of A_2'. However, at such a low value of Q^2 the $1/Q^4$ contribution could also be significant.

4.4 e^+e^- fragmentation function

Our procedure for e^+e^- annihilation is similar to that presented for deep inelastic scattering. The e^+e^- fragmentation function $\tilde{F}(x, Q^2)$ is given in terms of the characteristic function

$$\mathcal{F}(x;\epsilon) = \mathcal{F}^{(r)}(x;\epsilon)\Theta(1-\epsilon-x) + \mathcal{V}_t(\epsilon)\delta(1-x) \tag{4.61}$$

where $\mathcal{F}^{(r)}$ is the real contribution from annihilation into quark-antiquark-gluon,

$$\begin{aligned}\mathcal{F}^{(r)}(x,\epsilon) = &\left[2\frac{(1+\epsilon)^2}{1-x} - (1+x) - 2\epsilon\right]\ln\left[\frac{(x+\epsilon)(1-x)}{\epsilon}\right] \\ &+ \frac{1}{2}(1+x) + \epsilon - \frac{3+4\epsilon+3\epsilon^2}{2(1-x)} + \frac{\epsilon(1+\epsilon)}{(1-x)^2} + \frac{\epsilon^2}{2(1-x)^3} + \frac{\epsilon}{x+\epsilon}\,,\end{aligned} \tag{4.62}$$

and $\mathcal{V}_t(\epsilon)$ is the timelike virtual correction given in Eq. (4.3b). In this case we do not have an exact sum rule of the form (4.30): instead

$$\mathcal{V}_t(\epsilon) = \frac{3}{2} - \int_0^1 \mathcal{F}^{(r)}(x;\epsilon)\Theta(1-\epsilon-x)\,dx + \mathcal{O}\left(\epsilon^2\right)\,. \tag{4.63}$$

However, this is sufficient for our purposes, because it means that for any test function $f(x)$ we can still write

$$\begin{aligned}\int_0^1 \mathcal{F}(x;\epsilon)\,f(x)\,dx = &\frac{3}{2}f(1) \\ &+ \int_0^1 \mathcal{F}^{(r)}(x;\epsilon)\,[f(x) - f(1)]\,\,dx + f'(1)\int_{1-\epsilon}^1 \mathcal{F}^{(r)}(x;\epsilon)(1-x)\,dx + \ldots\end{aligned} \tag{4.64}$$

up to terms of order ϵ. Using the dictionary of singular terms (4.39), we thus obtain in this case the small-ϵ behaviour (4.40) with the same splitting function $P(x)$ but with $c(x)$, $g(x)$ and $h(x)$ replaced by

$$\begin{aligned}
\tilde{c}(x) &= 2\left(\frac{\ln[x(1-x)]}{1-x}\right)_+ - (1+x)\ln[x(1-x)] + \frac{1}{2}(1+x) \\
&\quad -\frac{3}{2(1-x)_+} - \frac{9}{4}\,\delta(1-x) \\
\tilde{g}(x) &= -\frac{4}{(1-x)_+} + 2 + \delta'(1-x) \\
\tilde{h}(x) &= 4\left(\frac{\ln[x(1-x)]}{1-x}\right)_+ - 2\ln[x(1-x)] + \frac{1}{(1-x)^2_{++}} + \frac{2}{x} \\
&\quad -5\,\delta(1-x) - 4\,\delta'(1-x)\,.
\end{aligned} \tag{4.65}$$

The term $2/x$ in $\tilde{h}(x)$ needs some interpretation. It results from setting

$$\ln\left(1+\frac{\epsilon}{x}\right) + \frac{\epsilon}{x+\epsilon} \simeq \frac{2\epsilon}{x} \tag{4.66}$$

which is not valid unless $x \gg \epsilon$. Hence the expression for $\tilde{h}(x)$ is not to be used in the small-x region, or for moments that are sensitive to it. In particular, in moment space we have a result of the form (4.42) with $\mathcal{F}_N = \frac{3}{2} + \mathcal{O}\left(\epsilon^2\right)$ for $N=1$, while for $N>1$

$$\tilde{c}_N = \left(\frac{3}{2}+\frac{1}{N}+\frac{1}{N+1}\right)S_1 + S_1^2 + 3S_2 - \frac{9}{4} + \frac{3}{2N} - \frac{1}{2(N+1)} + \frac{2}{N^2} + \frac{2}{(N+1)^2} \tag{4.67a}$$

$$\tilde{g}_N = N - 1 + \frac{2}{N} + 4S_1 \tag{4.67b}$$

$$\tilde{h}_N = \left(N-1+\frac{2}{N}\right)S_1 + 2S_1^2 + 6S_2 - 5N + \frac{2}{N-1} + \frac{4}{N^2}\,. \tag{4.67c}$$

For phenomenological applications, taking the same approach as for DIS, we may parametrize the leading power correction to the fragmentation function as

$$\tilde{F}(x,Q^2) = \tilde{F}^{\rm PT}(x,Q^2)\left(1+\frac{\tilde{C}(x,Q^2)}{Q^2}\right) \tag{4.68}$$

where

$$\tilde{C}(x,Q^2) = -A_2'\,\tilde{g}(x)\otimes\tilde{F}^{\rm PT}(x,Q^2)/\tilde{F}^{\rm PT}(x,Q^2)\,. \tag{4.69}$$

Fig. 7 shows the resulting prediction using the same parameter value as for DIS, $A_2' = -1$ GeV^2. For the perturbative contribution $\tilde{F}^{\rm PT}$ we used the parametrization of the light quark fragmentation function at $Q = 22$ GeV given in Ref. [37]. The coefficient function $\tilde{C}$ is larger than that for DIS, but qualitatively similar in form. The values of Q^2 in e^+e^- annihilation being much larger, the predicted power correction is probably too small to be detectable.

In Ref. [37], the non-perturbative contribution to the fragmentation function was parametrized as a small shift in the value of x,

$$x \to x + h_0\left(\frac{1}{Q} - \frac{1}{Q_0}\right) \tag{4.70}$$

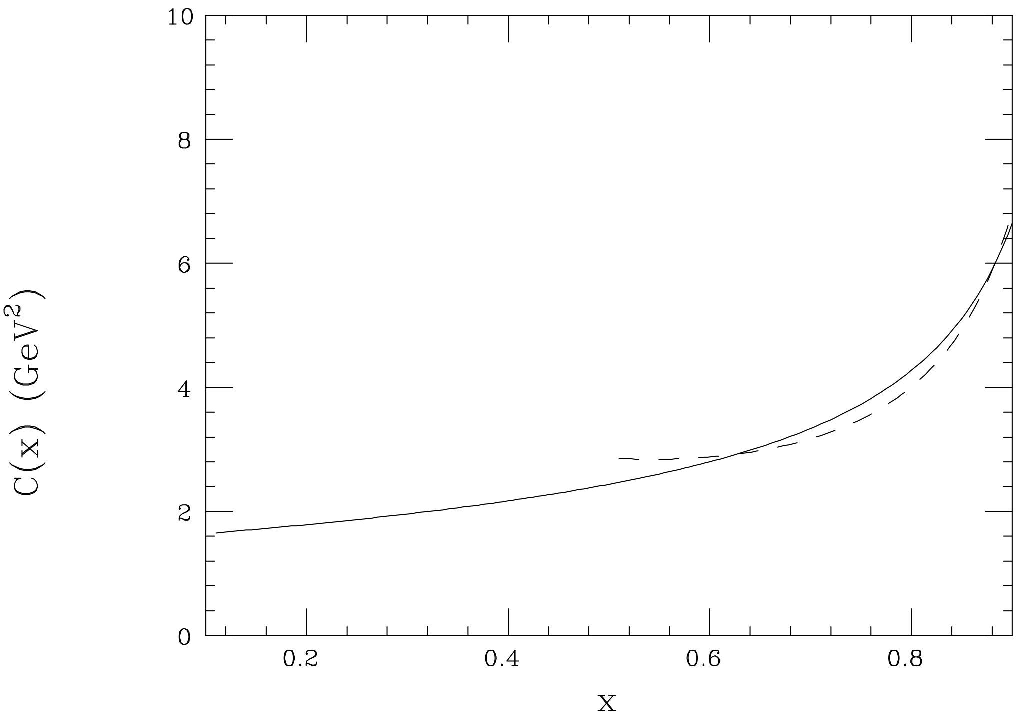

Figure 7: Coefficient of power correction to e^+e^- fragmentation function.

where $h_0 = -0.14 \pm 0.10$ GeV. However, a parametrization with $1/Q^2$ in the place of $1/Q$ was also found to be acceptable, the magnitude of the correction being practically consistent with zero. A small shift in x is clearly equivalent to a correction proportional to $\tilde{F}'$, the x-derivative of $\tilde{F}$, as expected from the δ' term in $\tilde{g}(x)$. As shown by the dashed curve in Fig. 7, a correction of this form is similar to our prediction at large x. The dashed curve corresponds to $\tilde{C}(x) = -0.35\tilde{F}'/\tilde{F}$, which, at 22 GeV, would be equivalent to $h_0 \simeq -0.016$ in Eq. (4.70).

4.5 Event shape variables

The analysis of power corrections to event shapes in e^+e^- final states proceeds in the same way as for hard cross sections, except that in this case the observables are constructed in such a way that there is no virtual contribution. Thus for the mean value of some generic shape variable y, defined so as to vanish in the two-jet limit, we have the characteristic function

$$\mathcal{F}(\epsilon) = \mathcal{F}^{(r)}(\epsilon) = \int dx\, d\bar{x}\, \mathcal{M}_{ee}(x,\bar{x},\epsilon)\, y(x,\bar{x},\epsilon)\, \Theta[(1-x)(1-\bar{x})-\epsilon]\, \Theta[x+\bar{x}-1+\epsilon]\,, \qquad (4.71)$$

where $\mathcal{M}_{ee}$ is the matrix element given in Eq. (4.6).

Thrust. In the case of the thrust variable T, for example, we define $y = 1-T$, which vanishes in the two-jet limit, and

$$y(x,\bar{x},\epsilon) = \min\{(1-x),(1-\bar{x}),(1-\sqrt{x_g^2-4\epsilon})\}\,. \qquad (4.72)$$

where $x_g = 2-x-\bar{x}$. As before, the quantity to be inserted in the fundamental equation (3.5) is $\dot{\mathcal{F}}$, the logarithmic derivative of the function (4.71).

We see from Fig. 8 that $\dot{\mathcal{F}}$ for the mean thrust decreases much more slowly at small ϵ than the corresponding function for the quantities studied earlier. In fact the behaviour at small ϵ is

$$\dot{\mathcal{F}}(\epsilon) \sim 4\sqrt{\epsilon} + \mathcal{O}\left(\epsilon \ln^2 \epsilon\right)\,, \qquad (4.73)$$

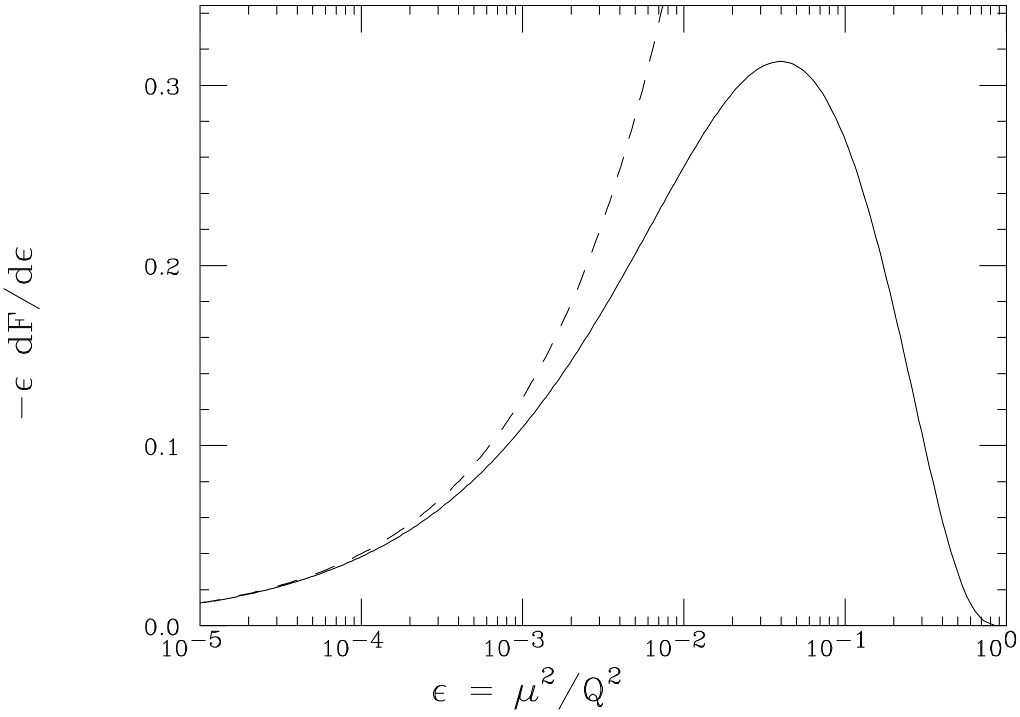

Figure 8: Derivative of characteristic function for mean value of thrust.

which is shown by the dashed curve. Thus the leading term is non-analytic and of square-root type. It follows that the leading power correction to $\langle 1-T\rangle$ will be of order $1/Q$, as observed [1].

It is easy to see how the $\sqrt{\epsilon}$ behaviour arises. It comes from the contribution to the derivative from the first theta-function in Eq. (4.71), *i.e.*, from the soft phase-space boundary, which gives

$$\begin{aligned}\dot{\mathcal{F}}(\epsilon) &\sim \epsilon \int dx\, d\bar{x}\, \mathcal{M}_{ee}(x,\bar{x},0)\, \min\{(1-x),(1-\bar{x})\}\, \delta[(1-x)(1-\bar{x})-\epsilon]\, \Theta[x+\bar{x}-1] \\ &\sim 4\int_{1-\sqrt{\epsilon}}^{1} dx \;=\; 4\sqrt{\epsilon}\,. \end{aligned} \tag{4.74}$$

Notice that we obtain in this approximation

$$\langle 1-T\rangle \sim \frac{C_F}{2\pi}\int_0^1 dx \int_{1-x}^1 d\bar{x}\; \alpha_{\mathrm{eff}}[(1-x)(1-\bar{x})Q^2]\, \mathcal{M}_{ee}(x,\bar{x},0)\, \min\{(1-x),(1-\bar{x})\}\;, \tag{4.75}$$

which corresponds to the result obtained in Ref. [2]: the argument of α_{eff} is the maximum gluon virtuality, which is equal to its transverse momentum.

In terms of the non-perturbative parameters (4.1), the power correction to $\langle 1-T\rangle$ is given by the $p=\frac{1}{2}$ moment of $\delta\alpha_{\mathrm{eff}}$ as

$$\langle 1-T\rangle^{\mathrm{NP}} \simeq 4\frac{A_1}{Q}\;. \tag{4.76}$$

The analysis performed in Ref. [2] showed that this gives a good description of the data for

$$A_1 \simeq 0.25\,\mathrm{GeV}. \tag{4.77}$$

In Ref. [7], the same $1/Q$ dependence as in (4.76) was obtained, but with a different coefficient. The difference arises from the normalization factor in the definition of the thrust: we have normalized to the sum of the final-state energies, whereas in [7] the sum of momenta is used, corresponding to inserting a factor of

$$2/\left(2-x_g+\sqrt{x_g^2-4\epsilon}\right) \tag{4.78}$$

into Eq. (4.72). Even though the thrust as usually defined is normalized to the sum of final-state momenta, the factor (4.78) should not be included, because it corresponds to finding a real massive gluon in the final state. The massive gluon in the calculation always decays into massless quarks or gluons, and so the sum of the final-state momenta should be set equal to the sum of the energies.

As mentioned in Sect. 3, the dispersive method applies directly to quantities that are fully inclusive with respect to gluon branching. Then the entire effect of branching is to make the coupling run. However, event shapes are sensitive to the structure of the final state, and branching may lead to a different value of the observable. In the case of thrust, for example, the value is unchanged only if the products of gluon branching fall into the same hemisphere, which includes quasi-collinear branching and therefore gives the dominant contribution to $\langle 1-T\rangle \sim \alpha_{\rm eff}/\pi$. Branching into opposite hemispheres, corresponding to a genuine four-parton contribution to the thrust, gives a correction $\sim (\alpha_{\rm eff}/\pi)^2$. Both give rise to $1/Q$ power terms [27]. The extent to which terms of higher order in $\alpha_{\rm eff}$ affect the magnitude of the power-behaved contribution remains to be established phenomenologically. The result (4.77) suggests a typical value of the effective coupling in the small momentum region such that $\alpha_{\rm eff}/\pi \sim 0.2$. This gives grounds for optimism that higher-order terms may be controllable.

Similar results to (4.76), with different coefficients of A_1/Q, may be obtained for the mean values of a variety of e^+e^- and DIS final-state event shapes [2,38].

Since the presence of a $1/Q$ correction introduces a large non-perturbative contribution, one would prefer for some purposes to define event shape variables for which the predicted coefficient of $1/Q$ is zero. Such variables should be more suitable for testing perturbative predictions and for measuring α_s. As suggested in [1], one can find linear combinations of variables such that the predicted $1/Q$ terms cancel, at least in the mean value. In addition, there are shape variables for which such terms vanish because the small-ϵ behaviour of the characteristic function is not of the square-root type.

Three-jet resolution. An example of a shape variable without a leading-order $1/Q$ correction is the mean value of the three-jet resolution variable y_3, defined according to the Durham or $k_\perp$ algorithm [39]. In lowest order, in the region $x > \bar{x}$, we have

$$y_3 = \min\{y_{q\bar{q}}, y_{\bar{q}g}\} \tag{4.79}$$

where

$$\begin{aligned} y_{q\bar{q}} &= \tfrac{1}{2}\bar{x}^2\,(1-\cos\theta_{q\bar{q}}) \\ y_{\bar{q}g} &= \tfrac{1}{2}\min\{\bar{x}^2, x_g^2\}(1-\cos\theta_{\bar{q}g})\,, \end{aligned} \tag{4.80}$$

with

$$\begin{aligned} \cos\theta_{q\bar{q}} &= \frac{1-x_g-\epsilon}{x\,\bar{x}} \\ \cos\theta_{\bar{q}g} &= \frac{x_g - 2(1-x-\epsilon)/\bar{x}}{\sqrt{x_g^2-4\epsilon}}\,. \end{aligned} \tag{4.81}$$

The resulting behaviour of $\dot{\mathcal{F}}$ for the mean value of y_3 is shown in Fig. 9. The small-ϵ behaviour in this case is

$$\dot{\mathcal{F}}(\epsilon) \sim \epsilon\ln^2\epsilon + \mathcal{O}(\epsilon\ln\epsilon)\ . \tag{4.82}$$

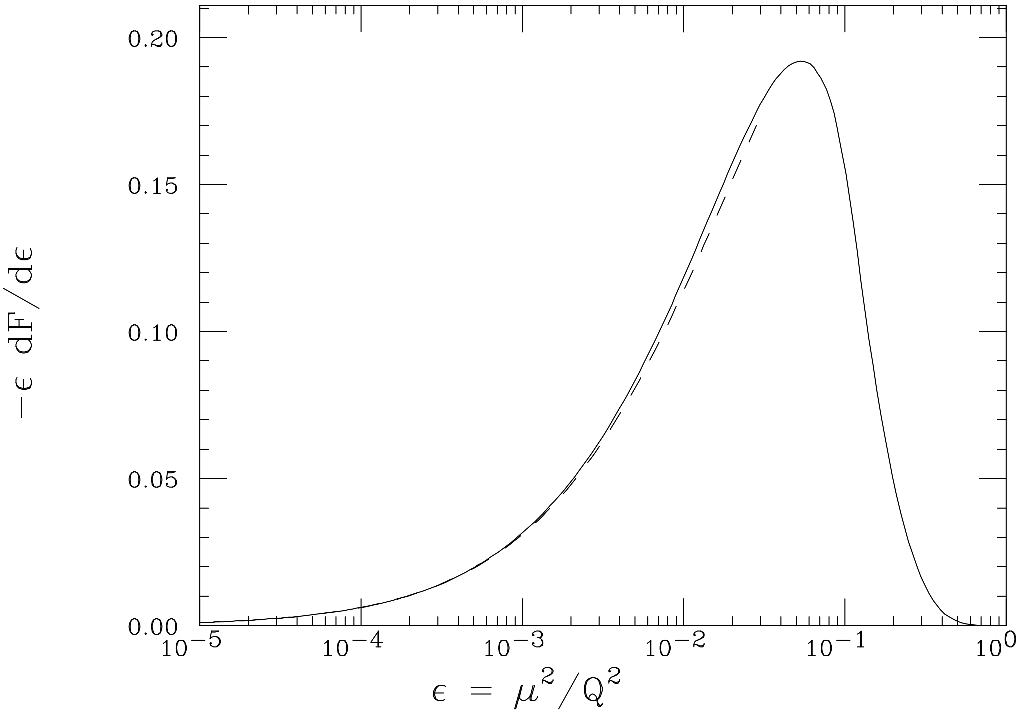

Figure 9: Derivative of characteristic function for mean value of y_3.

It follows that the leading power correction to the mean value of y_3 should be of order $\ln Q/Q^2$. We find numerically that the non-leading logarithms are such that

$$\dot{\mathcal{F}}(\epsilon) \simeq \epsilon(\ln^2\epsilon + 3\ln\epsilon + 4)\ , \tag{4.83}$$

as shown by the dashed curve. In terms of the moment integrals (4.1), we therefore expect

$$\langle y_3\rangle^{\mathrm{NP}} \simeq -(2A_2'\ln Q^2 - 3A_2' - A_2'')/Q^2\ . \tag{4.84}$$

If the three-jet resolution is defined instead according to the JADE algorithm, then Eqs. (4.80) become

$$\begin{aligned} y_{q\bar{q}} &= \tfrac{1}{2}x\,\bar{x}\,(1-\cos\theta_{q\bar{q}}) \\ y_{\bar{q}g} &= \tfrac{1}{2}\bar{x}\,x_g\,(1-\cos\theta_{\bar{q}g})\ . \end{aligned} \tag{4.85}$$

In this case the behaviour at small ϵ is of square-root type, leading to a $1/Q$ power correction. This is probably why the Durham algorithm has been found to require smaller non-perturbative corrections [40].

A final point to be noted from Figs. 8 and 9 is that the characteristic scale of hardness for event shapes, as typified by the peak of the characteristic function, lies far below Q^2, at $\bar{Q}^2 \sim 0.05Q^2$. This is a general feature of quantities which probe final-state structure in time-like processes, such as event shapes and e^+e^- fragmentation functions, and could explain why fixed-order fits to such quantities favour small scales and large values of the coupling.

4.6 Drell-Yan process and K factor

The matrix element squared for the Drell-Yan (DY) process with emission of an off-shell gluon of momentum k ($k^2 > 0$) is given by

$$M_{DY} = 2\frac{a + \tau^2 a + 2\,\tau\epsilon - 2\,\tau^2 a^2 + 2\,\epsilon^2\tau + \epsilon^2\tau^2 a}{\tau\,(\epsilon + a)^2}\ , \tag{4.86}$$

where $\epsilon = k^2/Q^2$, $\tau = Q^2/s$ and $a = q^2/Q^2$ with q^2 the momentum integration variable. The phase space integration is given by

$$d\Phi = da \frac{\tau}{\sqrt{R}}\, \Theta(R)\,,$$

with

$$R = A - 4\,\tau^2 a\,, \qquad A = 1 - 2\,\tau + \tau^2 - 2\,\tau\epsilon - 2\,\tau^2\epsilon + \epsilon^2\tau^2\,.$$

The real part of the DY characteristic function is then

$$\mathcal{F}^{(r)}_{DY}(\epsilon,\tau) = \int_0^1 d\Phi\, M_{DY} \;\; = -4\sqrt{A} + 4\,\tanh^{-1}\left(\frac{\sqrt{A}}{1-\tau-\tau\epsilon}\right)\frac{\tau^2 + 1 + 2\,\tau^2\epsilon + \tau^2\epsilon^2}{1-\tau-\tau\epsilon}\,. \tag{4.87}$$

We introduce the “rapidity” variables

$$\tau = \frac{c+\sqrt{c^2-1}}{2\,(\cosh\eta + c)}\,, \qquad c = \frac{1+y}{2\sqrt{y}}\,, \tag{4.88}$$

with the phase space

$$d\tau = \frac{c+\sqrt{c^2-1}}{2}\,\frac{\sinh\eta\, d\eta}{(\cosh\eta + c)^2}\,, \qquad \tau_{\max} = \frac{c+\sqrt{c^2-1}}{2(1+c)} = \frac{1}{(1+\sqrt{y})^2}\,.$$

One then has the simple form

$$\mathcal{F}^{(r)}_{DY} = -\frac{4\,\sinh\eta}{\cosh\eta + c} + 4\,\eta\left(1 + \frac{c}{\cosh\eta}\right)\left(1 + \frac{c^2}{(\cosh\eta + c)^2}\right)\,. \tag{4.89}$$

The complete characteristic function is obtained by including the virtual contribution. In moment space one has

$$\mathcal{F}^{DY}_N(\epsilon) = \int_0^{\tau_{\max}} d\tau\tau^{N-1}\mathcal{F}^{(r)}_{DY}(\epsilon,\tau) + \mathcal{V}_t(\epsilon)\,. \tag{4.90}$$

This function is given to order ϵ in Appendix A, Eq. (A.7). The DY distribution moments are then given by

$$\sigma^{DY}_N(Q^2) = \frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\alpha_{\rm eff}(\mu^2)\,\dot{\mathcal{F}}^{DY}_N(\epsilon)\,, \tag{4.91}$$

and the moments of the Drell-Yan K factor are

$$K_N(Q^2) = \frac{C_F}{2\pi}\int_0^\infty \frac{d\mu^2}{\mu^2}\alpha_{\rm eff}(\mu^2)\,\dot{\mathcal{K}}_N(\epsilon)\,, \tag{4.92}$$

where the characteristic function is

$$\mathcal{K}_N(\epsilon) = \mathcal{F}^{DY}_N(\epsilon) - 2\mathcal{F}^{DIS}_N(\epsilon) \tag{4.93}$$

with $\mathcal{F}^{DIS}_N(\epsilon)$ the corresponding moment of the DIS characteristic function (4.26). As is well known, the combination (4.93) is collinear safe since the collinear singularities in the Drell-Yan and DIS terms cancel. Its logarithmic derivative $\dot{\mathcal{K}}_N(\epsilon)$ vanishes both for large and small ϵ, as shown for the first four moments in Fig. 10. At small ϵ, $\mathcal{K}_N(\epsilon)$ has the form

$$\mathcal{K}_N(\epsilon) = \hat{c}_N + \epsilon(\hat{f}_N \ln^2\epsilon + \hat{g}_N \ln\epsilon + \hat{h}_N) + \mathcal{O}\left(\epsilon^2\ln^2\epsilon\right)\,, \tag{4.94}$$

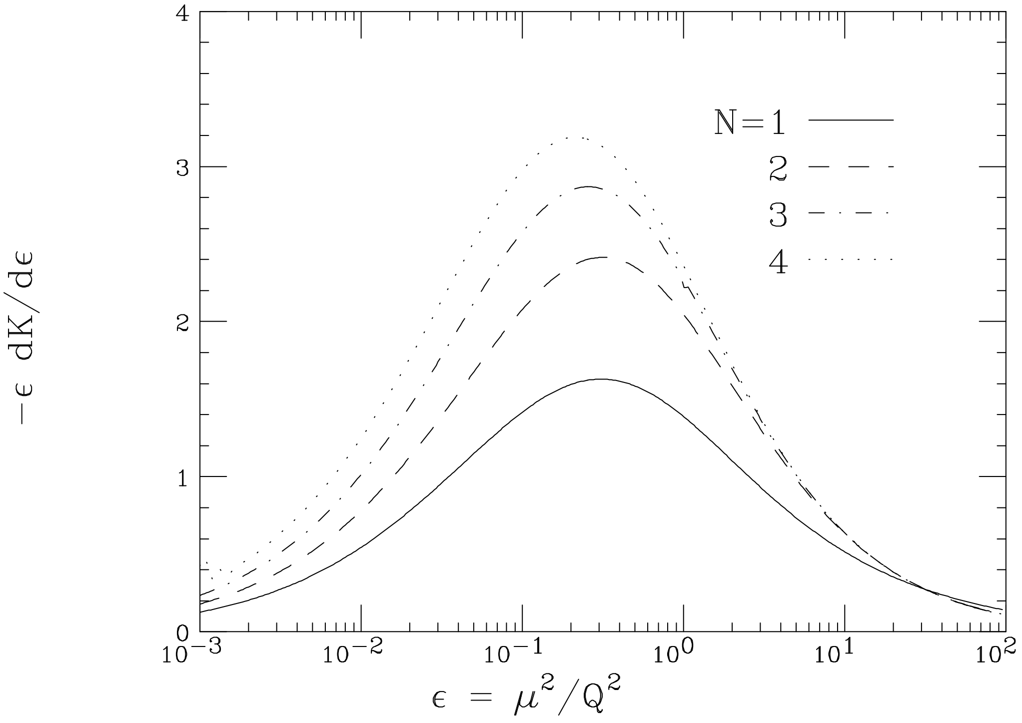

Figure 10: Derivative of characteristic function for moments of the Drell-Yan K-factor, $\dot{\mathcal{K}}_N$.

(see Appendix A, Eq. (A.8)), which shows that, apart from logarithmic enhancement, the power correction to the Drell-Yan cross section is of order $1/Q^2$, as in DIS, in agreement with the result of [7].

The expansion of $\dot{\mathcal{K}}_N(\epsilon)$ at small ϵ has the form

$$\dot{\mathcal{K}}_N(\epsilon) = \epsilon \left[\check{f}_N(\ln\epsilon + 2S_1)^2 + \check{g}_N(\ln\epsilon + 2S_1) + \check{h}_N\right] \, (1 + \mathcal{O}(\epsilon)) \,, \tag{4.95}$$

where at large N, corresponding to $\tau \to 1$, we have

$$\check{f}_N \sim N \,, \quad \check{g}_N \sim -2N^2 \,, \quad \check{h}_N \sim 4N^2 \,. \tag{4.96}$$

Thus the expansion parameter at large N becomes ϵN^2, and for $\dot{\mathcal{K}}_N$ one has approximately

$$\dot{\mathcal{K}}_N(\epsilon) = 2\epsilon N^2 \left(\ln\frac{1}{\epsilon N^2} + 2\right)\left(1 + \mathcal{O}\left(\frac{\ln(\epsilon N^2)}{N}\right)\right) \,, \tag{4.97}$$

where we have used

$$\ln\epsilon + 2S_1 \sim \ln(\epsilon N^2) \,. \tag{4.98}$$

Correspondingly, at large τ the power correction to the K factor may be expressed in terms of the non-perturbative integrals (4.1) as

$$K^{\rm NP}(\tau) = -\frac{2A_2'}{(1-\tau)^2 Q^2} \,. \tag{4.99}$$

This means that the coefficient of the power term increases much more rapidly as $\tau \to 1$ than does the DIS coefficient $C(x)$ as $x \to 1$ in Eq. (4.46).

Process	Quantity	Power	Coefficient (first order)	Eq.
e^+e^-	R	Q^{-6}	$2A_6'$	(4.14)
τ	R_τ	m_τ^{-6}	$-12A_6'L - 12A_6' + 6A_6''$	(4.21)
DIS	F_2	Q^{-2}	$-gA_2' \otimes F_2^{\rm PT}/F_2^{\rm PT}$	(4.48)
DIS	GLS	Q^{-2}	$\frac{4}{3}A_2'$	(4.60)
e^+e^-	$\tilde{F}$	Q^{-2}	$-\tilde{g}A_2' \otimes \tilde{F}^{\rm PT}/\tilde{F}^{\rm PT}$	(4.65)
e^+e^-	$\langle 1-T\rangle$	Q^{-1}	$4A_1$	(4.76)
e^+e^-	$\langle y_3\rangle$	Q^{-2}	$-2A_2'L + 3A_2' + A_2''$	(4.84)
DY	K	Q^{-2}	$2\hat{f}A_2'L - (2\hat{f}+\hat{g})A_2' - \hat{f}A_2''$	(A.8)

Table 1: Summary of power-behaved contributions.

4.7 Summary

We summarize in Table 1 the results obtained for the leading power-behaved corrections to the quantities discussed above. In each case the predictions are expressed in terms of the effective coupling moment parameters given in Eq. (4.1), and $L = \ln Q^2$ ($\ln m_\tau^2$ in the case of R_τ).

At present no systematic phenomenological tests of these predictions have been performed and the values of the moment parameters are correspondingly uncertain. The parameter A_1 specifies the lowest-order $1/Q$ contributions to a wide range of event shapes in e^+e^- (and DIS) final states, of which we show thrust only as an example. A value of $A_1 \sim 0.25$ GeV is suggested by the 1 GeV$/Q$ power-behaved contribution seen in the data on $\langle 1-T\rangle$.

As discussed in Sect. 4.2, the $1/Q^2$ contributions to the structure function F_2 in DIS appear consistent with a log-moment parameter value $A_2' \sim -1$ GeV^2. The power-behaved contribution to the Gross-Llewellyn-Smith sum rule for F_3 may suggest a smaller value. In either case the corresponding contribution to e^+e^- fragmentation functions is probably too small to be observed. The other $1/Q^2$ effects listed, for the mean three-jet resolution in e^+e^- final states and the K-factor in the Drell-Yan process, show a logarithmic enhancement and therefore involve also the additional log-moment parameter A_2''.

5 Conclusions

In this paper we have started to explore the possibility that the concept of the QCD running coupling can be extended, in a process-independent way, down to small momentum scales, at least in an effective sense. At low scales confinement physics dominates, undermining the very possibility of applying the language of quark and gluons, the only language we are able to use at the present level of our understanding of QCD. The hope is that, in spite of all the richness and complexity of the spectrum of hadrons, the bulk of the non-perturbative physics will reveal itself in a smooth way in sufficiently inclusive observables, and that it can be taken into account by extending the notion of the perturbative QCD coupling. The coupling $\alpha_{\rm eff}$ thus defined is intended to measure the effective interaction strength even at large distances, allowing the extension of the quark-gluon language beyond its original domain of applicability.

In an inclusive observable F, confinement reveals itself via departure of the measured value from the perturbative prediction $F^{\rm PT}$ at decreasing values of the relevant hard scale of the process, Q^2. At sufficiently large Q^2, $F^{\rm PT}$ varies slowly (logarithmically) with Q^2, since it is given by a finite-order expansion in $\alpha_s(Q^2)$. The departure is typically power-behaved (see Table 1)

$$F^{\rm NP} \;=\; F(Q^2) - F^{\rm PT}(Q^2) \simeq \left[C_1 A_{2p} + C_2 A'_{2p} + C_3 A''_{2p}\right]\cdot Q^{-2p}\,, \tag{5.1}$$

where the dimensionless coefficients C_k are at most linear in $\ln Q^2$. In this paper we have developed a method of calculating the process-dependent exponents p and coefficients C_k for a wide variety of hard observables which do not involve gluons in the Born approximation (quark-dominated processes). The dimensionful parameters A_{2p}, A'_{2p} and A''_{2p} are universal and are given by (log-)moment integrals of the effective coupling modification $\delta\alpha_{\rm eff}$ in the small momentum region. These are new phenomenological parameters which should be studied experimentally. We have computed the corresponding power-behaved contributions to first order in $\delta\alpha_{\rm eff}$.

Our method is based on using a dispersion relation for the running coupling α_s. The dispersive machinery then leads to the basic representation (3.5) for the observable F in terms of an integral over μ^2 of the product of a characteristic function $\dot{\mathcal{F}}$ and the effective coupling $\alpha_{\rm eff}(\mu^2)$. The latter is related to the standard α_s by the operator equation

$$\alpha_{\rm eff}(\mu^2) = \frac{\sin(\pi\mathcal{P})}{\pi\mathcal{P}}\,\alpha_s(\mu^2)\,,\quad \mathcal{P}=\mu^2\frac{d}{d\mu^2}\,.$$

This means that $\alpha_{\rm eff}$ and α_s are practically equivalent in the perturbative region. At low scales, $\alpha_{\rm eff}$ and α_s may differ substantially from each other and from expectations based on the perturbative β-function. However, we have argued that the non-perturbative modifications to $\alpha_{\rm eff}(\mu^2)$ at low μ^2 should be such that no power corrections to α_s are generated at higher scales. It follows that the μ^{2p}-moments of these deviations have to vanish for the first few integer values of p. Thus in order to generate power corrections, $\dot{\mathcal{F}}$ must contain terms which are non-analytic in μ^2.

The characteristic function $\dot{\mathcal{F}}$ is specific to a given observable and can be analysed by Feynman diagram techniques. We have computed the characteristic functions for a number of observables in various processes to first order in $\alpha_{\rm eff}$. In this order, $\dot{\mathcal{F}}$ is a function of the ratio $\epsilon=\mu^2/Q^2$ and is given by the one-loop diagrams with a gluon of mass equal to the dispersive variable μ, $0\le\mu\le\infty$. The leading non-analytic term in the $\epsilon\to 0$ behaviour of $\dot{\mathcal{F}}$ determines the power p and coefficients C_k in Eq. (5.1).

Let us stress again that we work with a massless gluonic quantum field. However, in the characteristic function the dispersive variable μ enters as a gluon mass in Feynman denominators and the phase space. The role of a small gluon mass as a trigger for long-distance contributions to hard processes has been recognized and exploited in the recent literature, see [1,7,12,17].

An attractive feature of the dispersive method is that for a given process it suffices to compute a single function to obtain power corrections to all associated observables. For example, the matrix element squared for e^+e^- annihilation into massless $q\bar{q}$ and a massive gluon gives the dispersively-generated power corrections to $R_{e^+e^-}$, to the non-singlet fragmentation function, and to all event shapes (thrust, y_3, etc). The results obtained for these and other hard process observables are summarized in Table 1.

As emphasised in Table 1, our calculation of the coefficients of power-behaved terms is first-order in the effective coupling $\alpha_{\rm eff}$. The accuracy of the first-order estimate will be reasonable if $\alpha_{\rm eff}/\pi$ happens to be numerically sufficiently small in the important kinematic region. This is a question that can only be answered by comparison with experiment. The same qualification applies even more strongly to event shapes which, by construction, are sensitive to final state structure and not fully inclusive. Multiparton contributions to such quantities are higher-order in $\alpha_{\rm eff}$, and may or may not be suppressed, depending on the typical value of $\alpha_{\rm eff}/\pi$.

On the topic of higher-order contributions, we would like to end with the following remark. A characteristic feature of the leading-order power-behaved terms that we have calculated is that they are enhanced near the boundary of phase space, where the additional gluon is soft. One thus observes the following general pattern of enhancement of power terms and the corresponding phase space boundaries:

$$\begin{aligned}
\text{DIS coefficient functions} &= \frac{N}{Q^2}\Lambda^2 &&\Longrightarrow (1-x)Q^2 \sim \Lambda^2 \\
\text{DY } K\text{-factor} &= \frac{N^2}{Q^2}\Lambda^2 &&\Longrightarrow (1-\tau)^2 Q^2 \sim \Lambda^2 \\
\text{differential thrust distribution} &= \frac{1}{(1-T)Q^2}\Lambda^2 &&\Longrightarrow (1-T)Q^2 \sim \Lambda^2 \\
\text{etc.} &
\end{aligned}$$

The power contribution becomes of order 1 at the edge of phase space, where the invariant mass of the final-state hadronic system is squeezed down to a finite value $\sim \Lambda^2$, so that the process becomes quasi-elastic and can no longer be treated as hard. As has already been remarked [4], it is possible that such effects, being closely related to the Sudakov form factor, can be shown to exponentiate and factorize universally.

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Note added: The results of Ref. [41], based on the large-N_f limit, also indicate that the GLS sum rule should receive only Q^{-2} and Q^{-4} power corrections, as in Eq. (4.60), and suggest that the power corrections to the polarized Bjorken sum rule should be the same. BW thanks D.J. Broadhurst for a discussion of this point.

Appendix A : Drell-Yan cross section

The real contributions of the moments of the Drell-Yan characteristic function are given by

$$\mathcal{F}_N^{(r)}(\epsilon) = 4\left(\frac{c+\sqrt{c^2-1}}{2\,c}\right)^N \int_0^\infty d\eta \\ \left\{\frac{\eta\sinh\eta}{\cosh\eta}\cdot\frac{c^N}{(\cosh\eta+c)^N} + \frac{\eta\sinh\eta}{\cosh\eta}\cdot\frac{c^{N+2}}{(\cosh\eta+c)^{N+2}} - \sinh^2\eta\cdot\frac{c^N}{(\cosh\eta+c)^{N+2}}\right\}\,. \tag{A.1}$$

Introducing

$$I_1(c) = 4\int_0^\infty \frac{d\eta\;\eta\,\sinh\eta}{\cosh\eta(\cosh\eta+c)} = \frac{2}{c}\left[\ln^2\left(c+\sqrt{c^2-1}\right)+\frac{\pi^2}{4}\right]\;; \tag{A.2a}$$

$$I_2(c) = -4\int_0^\infty \frac{d\eta\;\sinh^2\eta}{(\cosh\eta+c)^3} = \frac{2\ln\left(c+\sqrt{c^2-1}\right)}{(c^2-1)^{3/2}} - \frac{2c}{c^2-1}\,, \tag{A.2b}$$

we obtain

$$\mathcal{F}_N^{(r)}(\epsilon) = \left(\frac{c+\sqrt{c^2-1}}{2\,c}\right)^N \left\{[\,D_{N-1}+D_{N+1}\,]\,I_1(c) \;+\; \frac{2}{N(N+1)}\,D_{N-1}\,I_2(c)\right\}. \tag{A.3}$$

with

$$D_n \equiv \frac{c^{n+1}}{n!}\left(-\frac{d}{dc}\right)^n\,.$$

For the purpose of extracting the first power correction, the expressions (A.2) may be expanded as

$$I_1(c) = \frac{2\ln^2(2c)}{c}+\frac{\pi^2}{2c}-\frac{\ln(2c)}{c^3}+\mathcal{O}\left(c^{-5}\right)\;; \tag{A.4a}$$

$$I_2(c) = -\frac{2}{c}+\frac{2\ln(2c)}{c^3}-\frac{2}{c^3}+\mathcal{O}\left(c^{-5}\right)\,. \tag{A.4b}$$

The operator D_n acts as follows:

$$D_n\left\{\,c^{-k}\,\right\} = \frac{\Gamma(k+n)}{\Gamma(k)n!}\,c^{-k+1}\,, \tag{A.5a}$$

$$D_n\left\{\,\ln(2c)c^{-k}\,\right\} = \frac{\Gamma(k+n)}{\Gamma(k)n!}\,c^{-k+1}\left(\ln(2c)+\psi(k)-\psi(k+n)\right), \tag{A.5b}$$

$$D_n\left\{\,\ln^2(2c)c^{-k}\,\right\} = \frac{\Gamma(k+n)}{\Gamma(k)n!}\,c^{-k+1}\left(\psi'(k+n)-\psi'(k)+[\,\ln(2c)+\psi(k)-\psi(k+n)\,]^2\right). \tag{A.5c}$$

Applying the above results, one arrives at, to first order in ϵ,

$$\begin{aligned} \mathcal{F}_N^{(r)} &= M_N^{(0)}+\epsilon\cdot M_N^{(1)}+\ldots\;; \\ M_N^{(0)} &= (\ln\epsilon+2S_1)^2+\pi^2+2(\ln\epsilon+2S_1)\left[\frac{1}{N}+\frac{1}{N+1}\right]-4S_2\,, \\ M_N^{(1)} &= -N[(\ln\epsilon+2S_1)^2+\pi^2]+2(\ln\epsilon+2S_1)\left[N^2+3N-3+\frac{1}{N+1}\right] \\ &\quad -6N(N+1)+4NS_2+16-\frac{8}{N+1}\,. \end{aligned} \tag{A.6}$$

Taking into account the virtual correction, one obtains

$$\begin{aligned}
\mathcal{F}_N^{DY}(\epsilon) = &= \left(4S_1 - 3 + \frac{2}{N} + \frac{2}{N+1}\right)\ln\epsilon \\
&+ \frac{4\,\pi^2}{3} + 4(S_1^2 - S_2) + \left(\frac{4}{N+1} + \frac{4}{N}\right) S_1 - \frac{7}{2} \\
&+ \epsilon \Big\{ -(N+2)(\ln\epsilon + 2S_1)^2 + 2(\ln\epsilon + 2S_1)\left[N^2 + 3N - 3 + \frac{1}{N+1} + 4S_1\right] \\
&-6N(N+1) + 4\left(NS_2 - 2S_1^2\right) - \left(N - \frac{2}{3}\right)\pi^2 + 12 - \frac{8}{N+1}\Big\}.
\end{aligned} \tag{A.7}$$

Constructing

$$\mathcal{K}_N(\epsilon) = \mathcal{F}_N^{DY}(\epsilon) - 2\mathcal{F}_N^{DIS}(\epsilon)\,,$$

one derives the following expression for the characteristic function of the K-factor in moment space:

$$\mathcal{K}_N(\epsilon) = \hat{c}_N + \epsilon(\hat{f}_N \ln^2\epsilon + \hat{g}_N \ln\epsilon + \hat{h}_N)$$

where

$$\begin{aligned}
\hat{c}_N &= 2(S_1^2 + S_2) + \left(-3 + \frac{2}{N} + \frac{2}{N+1}\right) S_1 + \frac{4\pi^2}{3} + 1 - \frac{4}{N} - \frac{6}{N+1} + \frac{2}{N^2} + \frac{2}{(N+1)^2} \\
\hat{f}_N &= -(N+2) \\
\hat{g}_N &= 2N(N+2) - 4(N-2)S_1 - 22 + 2\,\frac{19N^2 + 30N + 8}{N(N+1)(N+2)} \\
\hat{h}_N &= -4(N-1)S_1^2 + \left(4N^2 + 10N - 28 + 8\frac{5N^2 + 8N + 2}{N(N+1)(N+2)}\right) S_1 + 4(N-3)S_2 \\
&\quad - 6N^2 + 4N + 18 - \left(N - \frac{2}{3}\right)\pi^2 - 2\,\frac{9N^5 + 61N^4 + 122N^3 + 104N^2 + 48N + 16}{N^2(N+1)^2(N+2)^2}\,.
\end{aligned} \tag{A.8}$$

Its logarithmic derivative may be written in the following form, which is suitable for studying the large N regime:

$$\dot{\mathcal{K}}_N(\epsilon) = \epsilon\left[\check{f}_N(\ln\epsilon + 2S_1)^2 + \check{g}_N(\ln\epsilon + 2S_1) + \check{h}_N\right]$$

where

$$\begin{aligned}
\check{f}_N &= N + 2 \\
\check{g}_N &= -2\left(N(N+1) + 8S_1 - 13 + \frac{19N^2 + 30N + 8}{N(N+1)(N+2)}\right) \\
\check{h}_N &= -2\left(N + 16 - \frac{4}{N} - \frac{2}{N+1} - \frac{12}{N+2}\right) S_1 + 4(N-1)^2 - 4(N-3)S_2 + 20S_1^2 \\
&\quad + \left(N - \frac{2}{3}\right)\pi^2 - 4\frac{5N^5 + 13N^4 + 7N^3 - 10N^2 - 16N - 8}{N^2(N+1)^2(N+2)^2}\,.
\end{aligned} \tag{A.9}$$

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