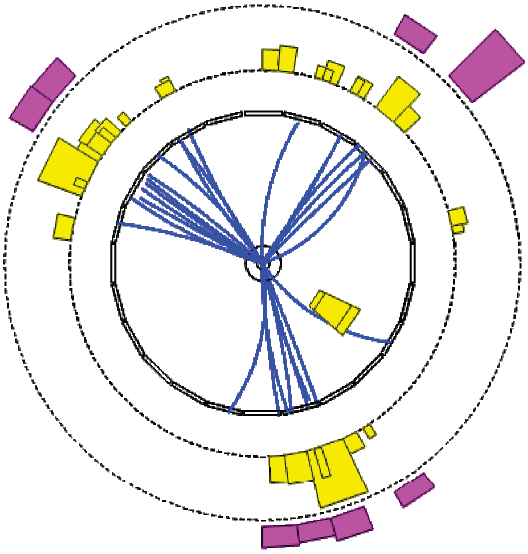
Particle Physics

Michaelmas Term 2011
Prof Mark Thomson



Handout 8 : Quantum Chromodynamics

The Local Gauge Principle

(see the Appendices A, B and C for more details)

★ All the interactions between fermions and spin-1 bosons in the SM are specified by the principle of LOCAL GAUGE INVARIANCE

★ To arrive at QED, require physics to be invariant under the local phase transformation of particle wave-functions

$$\psi \rightarrow \psi' = \psi e^{iq\chi(x)}$$

★ Note that the change of phase depends on the space-time coordinate: $\chi(t,\vec{x})$

- Under this transformation the Dirac Equation transforms as

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0} \Rightarrow \boxed{i\gamma^\mu(\partial_\mu + iq\partial_\mu\chi)\psi - m\psi = 0}$$

- To make "physics", i.e. the Dirac equation, invariant under this local phase transformation FORCED to introduce a massless gauge boson, A_μ .

+ The Dirac equation has to be modified to include this new field:

$$\boxed{i\gamma^\mu(\partial_\mu - qA_\mu)\psi - m\psi = 0}$$

- The modified Dirac equation is invariant under local phase transformations if:

$$\boxed{A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu\chi}$$ **Gauge Invariance**

★ For physics to remain unchanged – must have **GAUGE INVARIANCE** of the new field, i.e. physical predictions unchanged for $A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$

★ Hence the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (i.e. photon):

$$i\gamma^\mu(\partial_\mu \psi - qA_\mu)\psi - m\psi = 0$$

⇨ interaction vertex: $i\gamma^\mu q A_\mu$ (see p.111)

⇨ **QED !**

★ The local phase transformation of QED is a unitary **U(1)** transformation

$\psi \to \psi' = \hat{U}\psi$ i.e. $\psi \to \psi' = \psi e^{iq\chi(x)}$ with $U^\dagger U = 1$

Now extend this idea…

From QED to QCD

★ Suppose there is another fundamental symmetry of the universe, say **"invariance under SU(3) local phase transformations"**

- i.e. require invariance under $\psi \to \psi' = \psi e^{ig\vec{\lambda}.\vec{\theta}(x)}$ where

$\vec{\lambda}$ are the eight 3x3 Gell-Mann matrices introduced in handout 7

$\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time

⇨ 8 spin-1 gauge bosons

$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$ wave function is now a vector in **COLOUR SPACE**

⇨ **QCD !**

★ QCD is fully specified by require invariance under **SU(3)** **local** phase transformations

Corresponds to rotating states in colour space about an axis whose direction is different at every space-time point

⇨ interaction vertex: $-\frac{1}{2} i g_s \lambda^a \gamma^\mu$

★ Predicts 8 massless gauge bosons – the gluons (one for each λ)

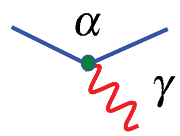
★ Also predicts exact form for interactions between gluons, i.e. the 3 and 4 gluon vertices – the details are beyond the level of this course

Colour in QCD

★ The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "colour" charges

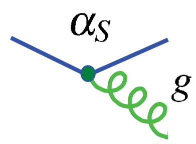
In QED:

- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless "gauge boson" – the photon



In QCD:

- quarks carry colour charge: r, g, b
- anti-quarks carry anti-charge: $\overline{r}, \overline{g}, \overline{b}$
- The force is mediated by massless gluons



★ In QCD, the strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colours

$\Rightarrow$ $\boxed{\text{SU(3) colour symmetry}}$

- This is an **exact** symmetry, unlike the approximate uds flavour symmetry discussed previously.

★ Represent r, g, b SU(3) colour states by:

$$r = \begin{pmatrix}1\\0\\0\end{pmatrix};\quad g = \begin{pmatrix}0\\1\\0\end{pmatrix};\quad b = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

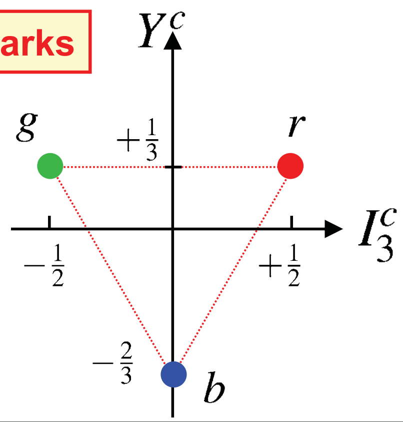
★ Colour states can be labelled by two quantum numbers:

- I_3^c colour isospin
- Y^c colour hypercharge

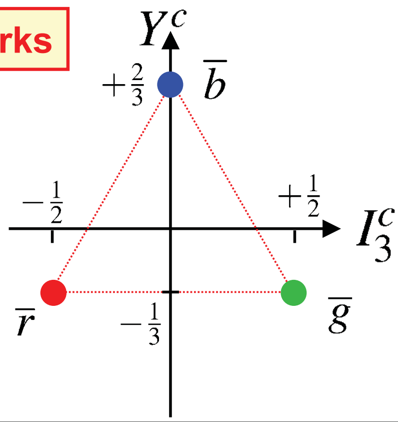
Exactly analogous to labelling u,d,s flavour states by I_3 and Y

★ Each quark (anti-quark) can have the following colour quantum numbers:

quarks



anti-quarks



Colour Confinement

★ It is believed (although not yet proven) that all observed free particles are "colourless"
- i.e. never observe a free quark (which would carry colour charge)
- consequently quarks are always found in bound states colourless hadrons

★ <u>**Colour Confinement Hypothesis:**</u>

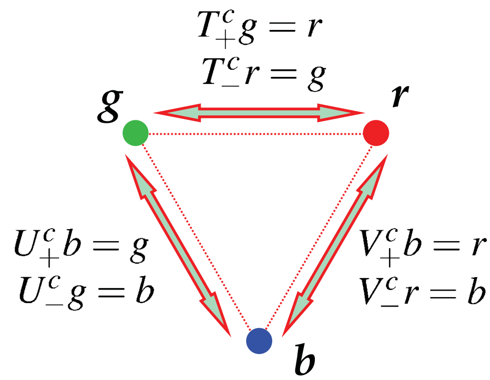
only <u>colour singlet</u> states can exist as free particles

★ All hadrons must be "colourless" i.e. colour singlets

★ To construct colour wave-functions for hadrons can apply results for SU(3) flavour symmetry to SU(3) colour with replacement

$$\boxed{\begin{array}{l} u \longrightarrow r \\ d \longrightarrow g \\ s \longrightarrow b \end{array}}$$

★ just as for uds flavour symmetry can define colour ladder operators



Colour Singlets

★ It is important to understand what is meant by a singlet state

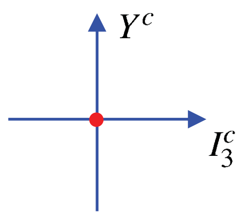
★ Consider spin states obtained from two spin 1/2 particles.
- Four spin combinations: $\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$
- Gives four eigenstates of $\hat{S}^2,\ \hat{S}_z$ $\qquad (2 \otimes 2 = 3 \oplus 1)$

$$\left.\begin{array}{l} |1,+1\rangle = \uparrow\uparrow \\ |1,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1,-1\rangle = \downarrow\downarrow \end{array}\right. \quad \boxed{\text{spin-1 triplet}} \quad \oplus \quad |0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \quad \boxed{\text{spin-0 singlet}}$$

★ The singlet state is "spinless": it has zero angular momentum, is invariant under SU(2) spin transformations and spin ladder operators yield zero

$$S_\pm |0,0\rangle = 0$$

★ In the same way COLOUR SINGLETS are "colourless" combinations:
- they have zero colour quantum numbers $I_3^c = 0,\ Y^c = 0$
- invariant under SU(3) colour transformations
- ladder operators $T_\pm,\ U_\pm,\ V_\pm$ all yield zero

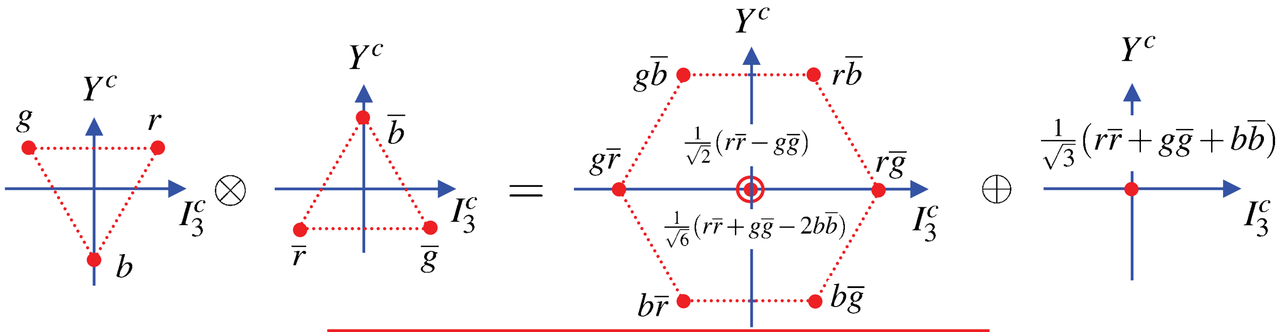


★ NOT sufficient to have $I_3^c = 0,\ Y^c = 0$: does not mean that state is a singlet

Meson Colour Wave-function

★ Consider colour wave-functions for $q\bar{q}$

★ The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with uds flavour symmetry



➡ **Coloured octet and a colourless singlet**

• Colour confinement implies that hadrons only exist in colour singlet states so the colour wave-function for mesons is:

$$\boxed{\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})}$$

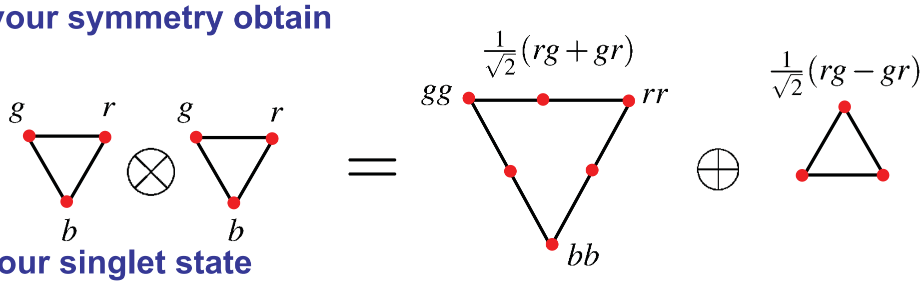
★ Can we have a $qq\bar{q}$ state ? i.e. by adding a quark to the above octet can we form a state with $Y^c = 0;\ I_3^c = 0$. The answer is clear no.

➡ $qq\bar{q}$ bound states do not exist in nature.

Baryon Colour Wave-function

★ Do **qq** bound states exist ? This is equivalent to asking whether it possible to form a colour singlet from two colour triplets ?

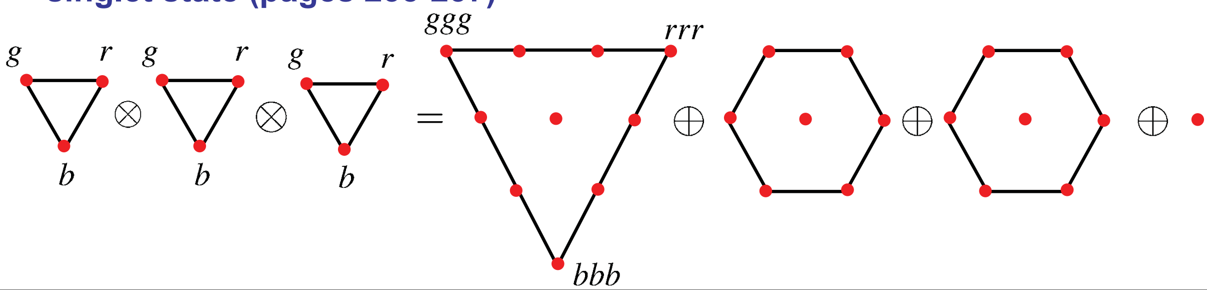
• Following the discussion of construction of baryon wave-functions in SU(3) flavour symmetry obtain



• No **qq** colour singlet state

• Colour confinement ➡ bound states of **qq** do not exist

★ BUT combination of three quarks (three colour triplets) gives a colour singlet state (pages 235-237)



★ The singlet colour wave-function is:

$$\boxed{\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)}$$

Check this is a colour singlet…

- It has $I_3^c = 0,\ Y^c = 0$: a necessary but not sufficient condition
- Apply ladder operators, e.g. T_+ (recall $T_+ g = r$)

$$T_+\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0$$

- Similarly $T_-\psi_c^{qqq} = 0;\quad V_\pm\psi_c^{qqq} = 0;\quad U_\pm\psi_c^{qqq} = 0;$

★ Colourless singlet - therefore **qqq** bound states exist !

⇨ **Anti-symmetric colour wave-function**

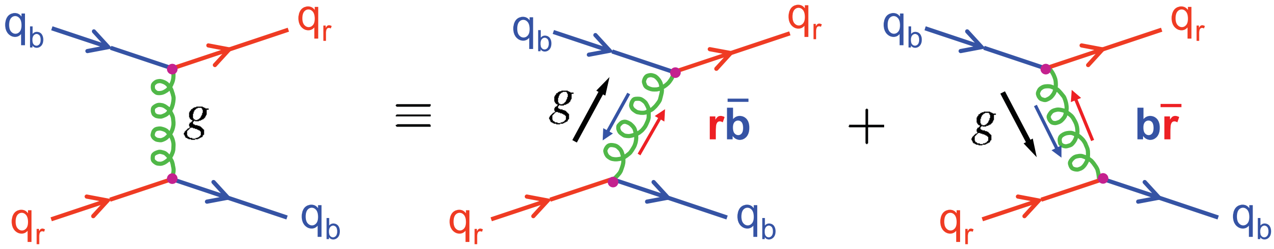
Allowed Hadrons i.e. the possible colour singlet states

- $q\bar{q},\ qqq$ — **Mesons and Baryons**
- $q\bar{q}q\bar{q},\ qqqq\bar{q}$ — **Exotic states, e.g. pentaquarks**

To date all confirmed hadrons are either **mesons** or **baryons**. However, some recent (but not entirely convincing) "evidence" for pentaquark states

Gluons

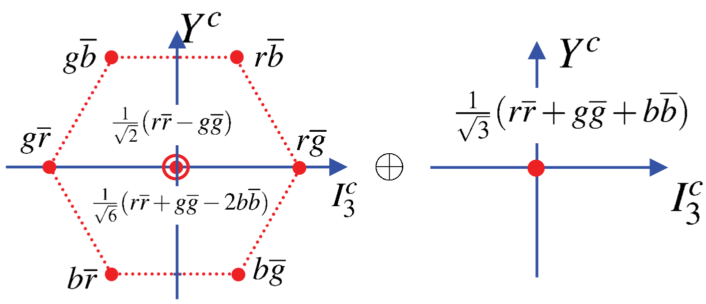
★ In QCD quarks interact by exchanging virtual massless gluons, e.g.



★ Gluons carry colour and anti-colour, e.g.

★ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour)

⇨ OCTET + "COLOURLESS" SINGLET



★ So we might expect 9 physical gluons:

OCTET: $r\bar{g},\ r\bar{b},\ g\bar{r},\ g\bar{b},\ b\bar{r},\ b\bar{g},\ \frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g}),\ \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

SINGLET: $\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$

★ **BUT**, colour confinement hypothesis:

only colour singlet states can exist as free particles ⇒ **Colour singlet gluon** would be unconfined. It would behave like a strongly interacting photon ➡ infinite range Strong force.

★ **Empirically**, the strong force is short range and therefore know that the physical gluons are confined. The colour singlet state does not exist in nature !

NOTE: this is not entirely ad hoc. In the context of gauge field theory (see minor option) the strong interaction arises from a fundamental SU(3) symmetry. The gluons arise from the generators of the symmetry group (the Gell-Mann λ matrices). There are 8 such matrices ➡ 8 gluons.
Had nature "chosen" a U(3) symmetry, would have 9 gluons, the additional gluon would be the colour singlet state and QCD would be an unconfined long-range force.

NOTE: the "gauge symmetry" determines the exact nature of the interaction
➡ FEYNMAN RULES

Gluon-Gluon Interactions

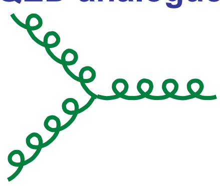
★ In QED the photon does not carry the charge of the EM interaction (photons are electrically neutral)

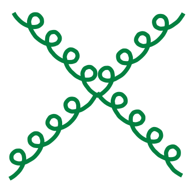
★ In contrast, in QCD the gluons do carry colour charge

➡ Gluon Self-Interactions

★ Two new vertices (no QED analogues)

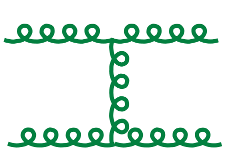
triple-gluon vertex

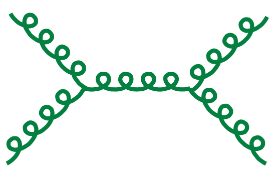




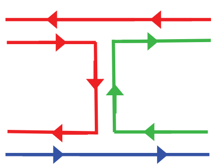
quartic-gluon vertex

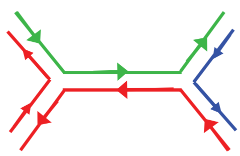
★ In addition to quark-quark scattering, therefore can have gluon-gluon scattering





e.g. possible way of arranging the colour flow



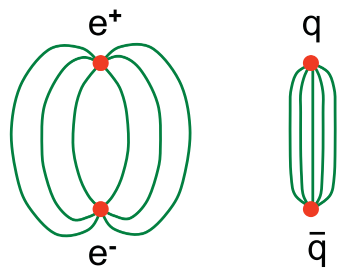


Gluon self-Interactions and Confinement

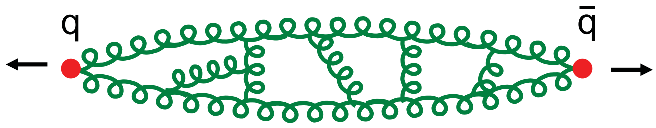
★ Gluon self-interactions are believed to give rise to colour confinement

★ Qualitative picture:

- Compare QED with QCD
- In QCD **"gluon self-interactions squeeze lines of force into a flux tube"**



★ What happens when try to separate two coloured objects e.g. $q\bar{q}$



- Form a flux tube of interacting gluons of approximately constant energy density $\sim 1\,\mathrm{GeV/fm}$

$$\Rightarrow \quad V(r) \sim \lambda r$$

- Require infinite energy to separate coloured objects to infinity
- **Coloured** quarks and gluons are always confined within colourless states
- In this way QCD provides a plausible explanation of confinement – but not yet proven (although there has been recent progress with Lattice QCD)

Hadronisation and Jets

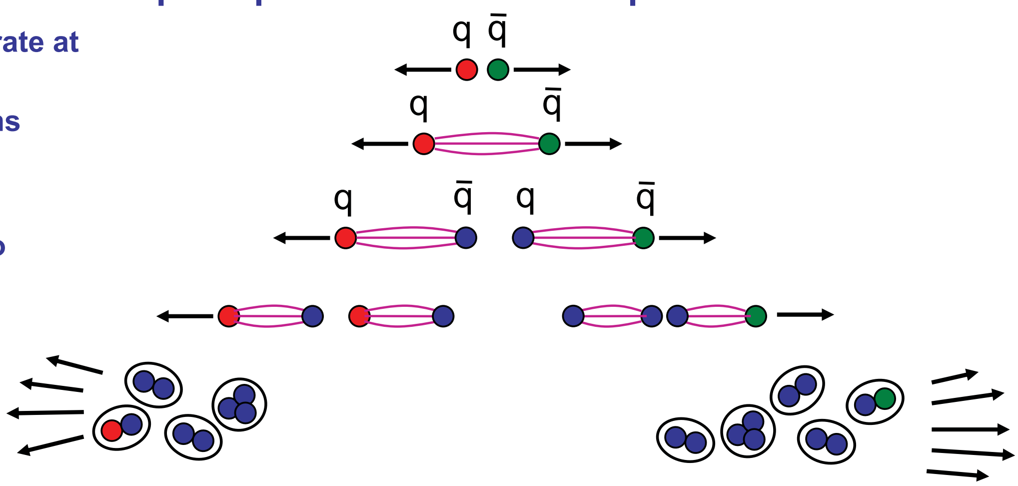
★ Consider a quark and anti-quark produced in electron positron annihilation

i) Initially Quarks separate at high velocity

ii) Colour flux tube forms between quarks

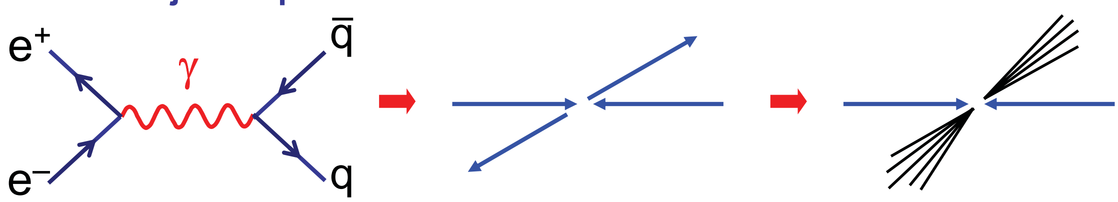
iii) Energy stored in the flux tube sufficient to produce **$q\bar{q}$** pairs

iv) Process continues until quarks pair up into jets of colourless hadrons



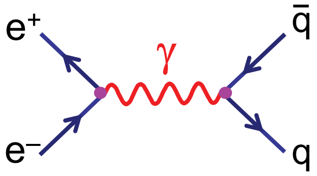
★ This process is called hadronisation. It is not (yet) calculable.

★ The main consequence is that at collider experiments **quarks** and **gluons** observed as jets of particles



QCD and Colour in e⁺e⁻ Collisions

★ e^+e^- colliders are an excellent place to study QCD

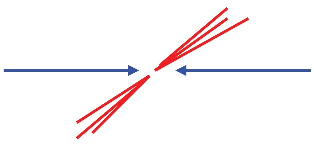


★ Well defined production of quarks
- QED process well-understood
- no need to know parton structure functions
- + experimentally very clean – no proton remnants

★ In handout 5 obtained expressions for the $e^+e^- \to \mu^+\mu^-$ cross-section

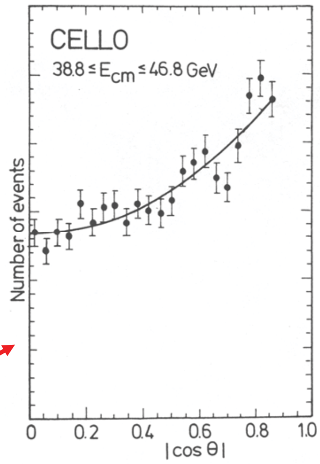
$$\sigma = \frac{4\pi\alpha^2}{3s} \qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

- In e^+e^- collisions produce all quark flavours for which $\sqrt{s} > 2m_q$
- In general, i.e. unless producing a $q\bar{q}$ bound state, produce jets of hadrons
- Usually can't tell which jet came from the quark and came from anti-quark



★ Angular distribution of jets $\propto (1+\cos^2\theta)$

➡ **Quarks are spin ½**



H.J.Behrend et al., Phys Lett 183B (1987) 400

★ Colour is conserved and quarks are produced as $r\bar{r},\ g\bar{g},\ b\bar{b}$

★ For a single quark flavour and single colour

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{4\pi\alpha^2}{3s}Q_q^2$$

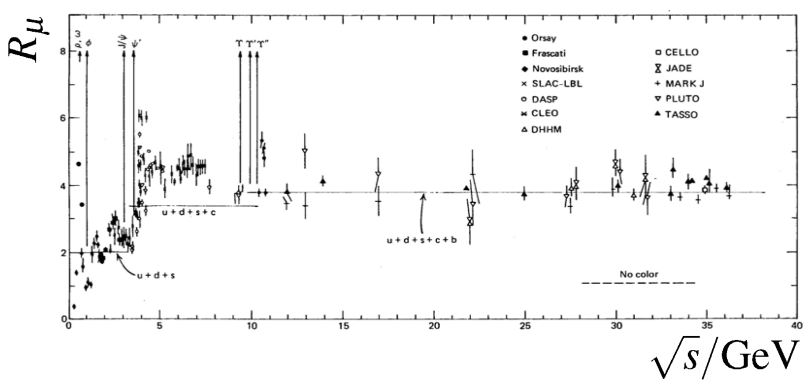
- Experimentally observe jets of hadrons:

$$\sigma(e^+e^- \to \text{hadrons}) = 3\sum_{u,d,s,..}\frac{4\pi\alpha^2}{3s}Q_q^2$$

Factor 3 comes from colours

- Usual to express as ratio compared to $\sigma(e^+e^- \to \mu^+\mu^-)$

$$R_\mu = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{u,d,s,..}Q_q^2$$



u,d,s: $R_\mu = 3\times(\frac{1}{9}+\frac{4}{9}+\frac{1}{9}) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

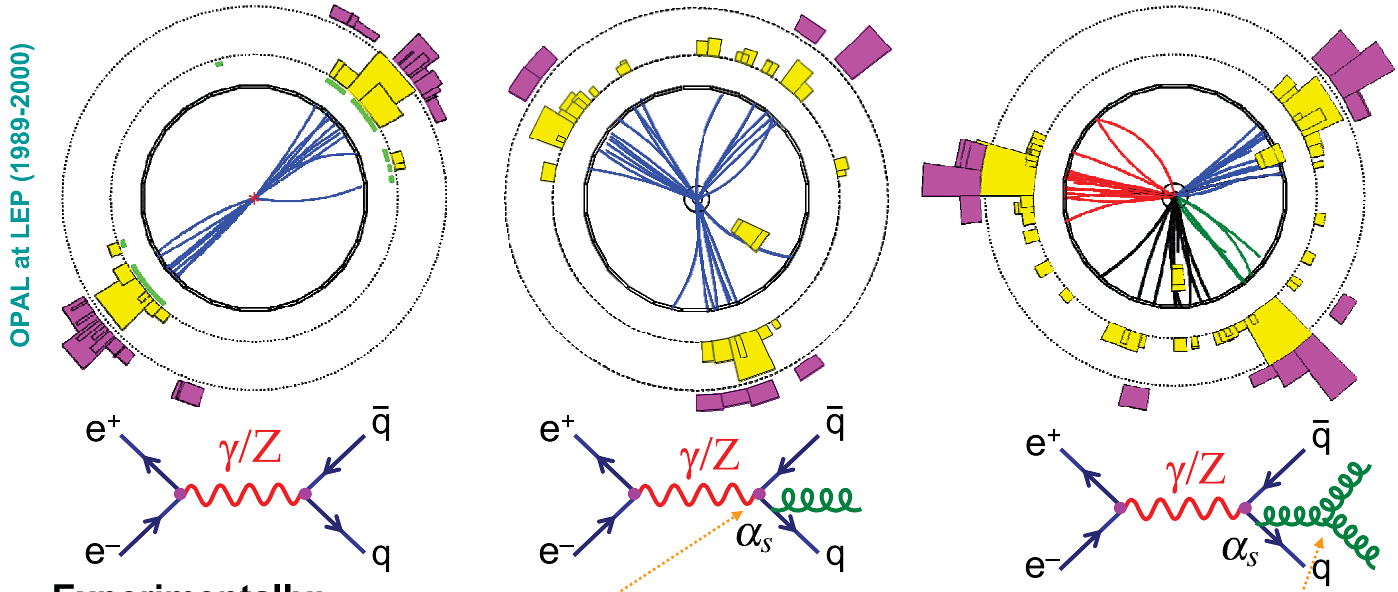
u,d,s,c,b: $R_\mu = \frac{11}{3}$

★ Data consistent with expectation with factor 3 from colour

Jet production in e+e- Collisions

★ e^+e^- colliders are also a good place to study gluons

$e^+e^- \to q\bar{q} \to 2\text{jets}$ $\quad e^+e^- \to q\bar{q}g \to 3\text{jets}$ $\quad e^+e^- \to q\bar{q}gg \to 4\text{jets}$



Experimentally:

- Three jet rate ➡ measurement of α_s
- Angular distributions ➡ gluons are spin-1
- Four-jet rate and distributions ➡ QCD has an underlying SU(3) symmetry

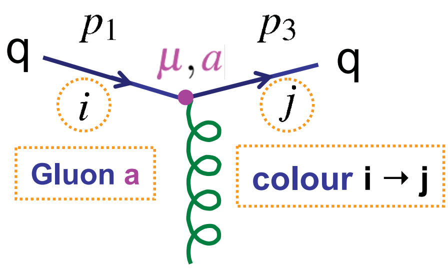
The Quark – Gluon Interaction

- Representing the colour part of the fermion wave-functions by:

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$
- The QCD qqg vertex is written:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices
- Isolating the colour part:

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix}\lambda^a_{1i}\\ \lambda^a_{2i}\\ \lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

- Hence the fundamental quark - gluon QCD interaction can be written

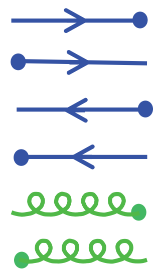
$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Feynman Rules for QCD

- **External Lines**

spin 1/2 { incoming quark $u(p)$; outgoing quark $\overline{u}(p)$; incoming anti-quark $\overline{v}(p)$; outgoing anti-quark $v(p)$ }

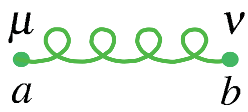
spin 1 { incoming gluon $\varepsilon^{\mu}(p)$; outgoing gluon $\varepsilon^{\mu}(p)^*$ }



- **Internal Lines (propagators)**

spin 1 gluon

$$\boxed{\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$$

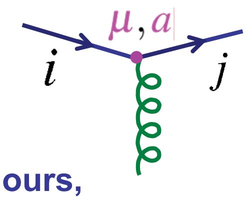


a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$\boxed{-ig_s\tfrac{1}{2}\lambda^a_{ji}\gamma^{\mu}}$$



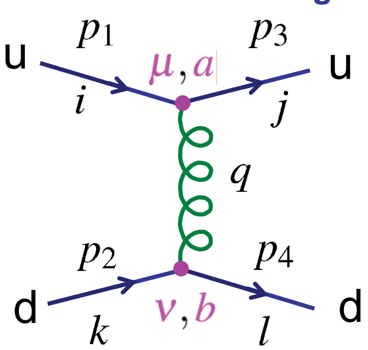
i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element -iM = product of all factors**

Matrix Element for quark-quark scattering

★ Consider QCD scattering of an up and a down quark



- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1,2,3\}$ (or $\{r,g,b\}$)
- In terms of colour this scattering is
$$ik \to jl$$
- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, \ldots, 8$
- NOTE: the δ-**function** in the propagator ensures a = b, i.e. the gluon "emitted" at a is the same as that "absorbed" at b

★ Applying the Feynman rules:

$$-iM = \left[\overline{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^{\mu}\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\overline{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^{\nu}\}u_d(p_2)\right]$$

where summation over a and b (and μ and ν) is implied.

★ Summing over a and b using the δ-function gives:

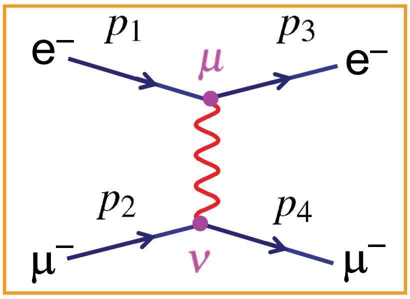
$$\boxed{M = -\frac{g_s^2}{4}\underbrace{\lambda^a_{ji}\lambda^a_{lk}}_{\text{Sum over all 8 gluons (repeated indices)}}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^{\mu}u_u(p_1)][\overline{u}_d(p_4)\gamma^{\nu}u_d(p_2)]}$$

QCD vs QED

QED

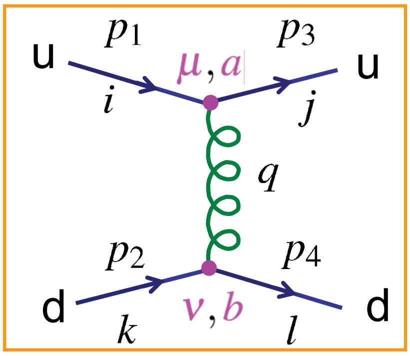
$$-iM = [\overline{u}(p_3) ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\overline{u}(p_4) ie\gamma^\nu u(p_2)]$$

$$\boxed{M = -e^2 \frac{1}{q^2} g_{\mu\nu} [\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]}$$



QCD

$$\boxed{M = -\frac{g_s^2}{4} \lambda^a_{ji}\lambda^a_{lk} \frac{1}{q^2} g_{\mu\nu} [\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$



★ **QCD Matrix Element = QED Matrix Element with:**

- $\boxed{e^2 \rightarrow g_s^2}$ **or equivalently** $\boxed{\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}}$

+ QCD Matrix Element includes an additional "colour factor"

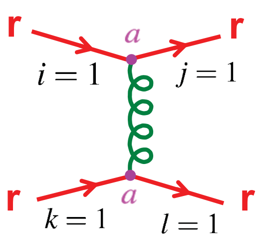
$$\boxed{C(ik \rightarrow jl) \equiv \frac{1}{4} \sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}}$$

Evaluation of QCD Colour Factors

• **QCD colour factors reflect the gluon states that are involved**

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\overline{g}, g\overline{r}$	$r\overline{b}, b\overline{r}$	$g\overline{b}, b\overline{g}$	$\frac{1}{\sqrt{2}}(r\overline{r} - g\overline{g})$ $\frac{1}{\sqrt{6}}(r\overline{r} + g\overline{g} - 2b\overline{b})$

❶ **Configurations involving a single colour**

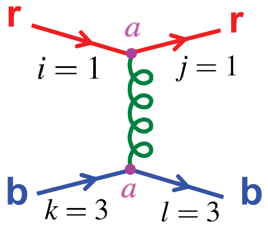


• **Only matrices with non-zero entries in 11 position are involved**

$$\begin{aligned} C(rr \rightarrow rr) &= \frac{1}{4} \sum_{a=1}^{8} \lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11}) \\ &= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3} \end{aligned}$$

Similarly find $\boxed{C(rr \rightarrow rr) = C(gg \rightarrow gg) = C(bb \rightarrow bb) = \frac{1}{3}}$

❷ Other configurations where quarks don't change colour e.g. $rb \to rb$

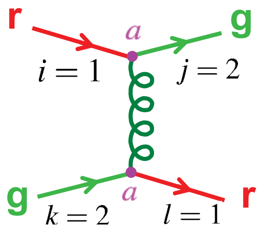


- Only matrices with non-zero entries in **11** and **33** position are involved

$$C(rb \to rb) = \frac{1}{4}\sum_{a=1}^{8}\lambda_{11}^{a}\lambda_{33}^{a} = \frac{1}{4}(\lambda_{11}^{8}\lambda_{33}^{8})$$
$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}.\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb\to rb) = C(rg\to rg) = C(gr\to gr) = C(gb\to gb) = C(br\to br) = C(bg\to bg) = -\frac{1}{6}$

❸ Configurations where quarks swap colours e.g. $rg \to gr$



- Only matrices with non-zero entries in **12** and **21** position are involved

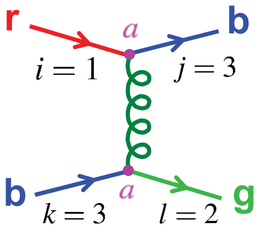
Gluons $r\bar{g},\, g\bar{r}$

$$C(rg \to gr) = \frac{1}{4}\sum_{a=1}^{8}\lambda_{21}^{a}\lambda_{12}^{a} = \frac{1}{4}(\lambda_{21}^{1}\lambda_{12}^{1} + \lambda_{21}^{2}\lambda_{12}^{2})$$
$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}_{+}^{(ij)}\hat{T}_{-}^{(kl)}$

$C(rb\to br) = C(rg\to gr) = C(gr\to rg) = C(gb\to bg) = C(br\to rb) = C(bg\to gb) = \frac{1}{2}$

❹ Configurations involving 3 colours e.g. $rb \to bg$



- Only matrices with non-zero entries in the **13** and **32** position
- But none of the λ matrices have non-zero entries in the **13** and **32** positions. Hence the colour factor is zero

★ colour is conserved

Colour Factors : Quarks vs Anti-Quarks

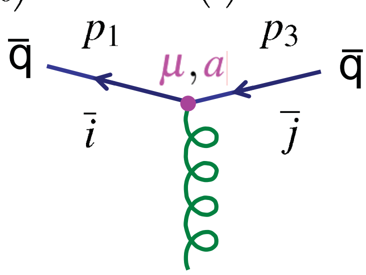
- Recall the colour part of wave-function: $r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$
- The QCD qqg vertex was written:

$$\bar{u}(p_3)c_j^{\dagger}\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$

★Now consider the anti-quark vertex

- The QCD $\bar{q}\bar{q}$g vertex is:

$$\bar{v}(p_1)c_i^{\dagger}\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3)$$



Note that the incoming anti-particle now enters on the LHS of the expression

- For which the colour part is

$$c_i^{\dagger}\lambda^a c_j = c_i^{\dagger}\begin{pmatrix}\lambda_{1j}^a\\ \lambda_{2j}^a\\ \lambda_{3j}^a\end{pmatrix} = \lambda_{ij}^a$$

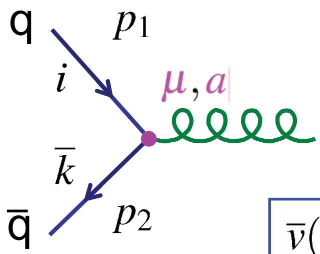
i.e indices *ij* are swapped with respect to the quark case

- **Hence** $\bar{v}(p_1)c_i^{\dagger}\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \bar{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda_{ij}^a\gamma^\mu\}v(p_3)$

- c.f. the quark - gluon QCD interaction

$$\bar{u}(p_3)c_j^{\dagger}\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda_{ji}^a\gamma^\mu\}u(p_1)$$

★ **Finally we can consider the quark – anti-quark annihilation**

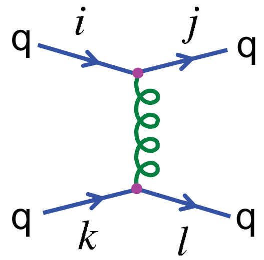


QCD vertex: $\overline{v}(p_2)c_k^{\dagger}\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

with $c_k^{\dagger}\lambda^a c_i = \lambda^a_{ki}$

$$\boxed{\overline{v}(p_2)c_k^{\dagger}\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)}$$

• **Consequently the colour factors for the different diagrams are:**



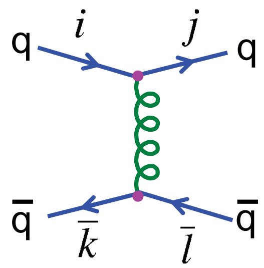
$$\boxed{C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}}$$

e.g.

$C(rr \to rr) = \tfrac{1}{3}$

$C(rg \to rg) = -\tfrac{1}{6}$

$C(rg \to gr) = \tfrac{1}{2}$

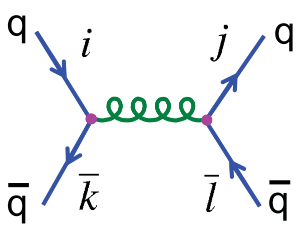


$$\boxed{C(i\overline{k} \to j\overline{l}) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{kl}}$$

$C(r\overline{r} \to r\overline{r}) = \tfrac{1}{3}$

$C(r\overline{g} \to r\overline{g}) = -\tfrac{1}{6}$

$C(r\overline{r} \to g\overline{g}) = \tfrac{1}{2}$



$$\boxed{C(i\overline{k} \to j\overline{l}) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ki}\lambda^a_{jl}}$$

$C(r\overline{r} \to r\overline{r}) = \tfrac{1}{3}$

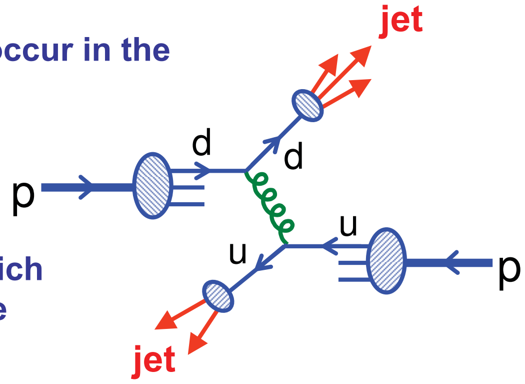
$C(r\overline{g} \to r\overline{g}) = \tfrac{1}{2}$

$C(r\overline{r} \to g\overline{g}) = -\tfrac{1}{6}$

Colour index of adjoint spinor comes first

Quark-Quark Scattering

- Consider the process $u + d \to u + d$ which can occur in the high energy proton-proton scattering
- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states



$$\langle |M_{fi}|^2 \rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^{3} |M_{fi}(ij \to kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\langle |C|^2 \rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |C(ij \to kl)|^2$$

- For $qq \to qq$ $\boxed{rr\to rr,..}$ $\boxed{rb\to rb,..}$ $\boxed{rb\to br,..}$

$$\langle |C|^2 \rangle = \frac{1}{9}\left[3\times\left(\frac{1}{3}\right)^2 + 6\times\left(-\frac{1}{6}\right)^2 + 6\times\left(\frac{1}{2}\right)^2\right] = \frac{2}{9}$$

- Previously derived the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit (handout 6).

$\boxed{\text{QED}}$

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s}\right)^2\right]$$

- For ud → ud in QCD replace $\alpha \to \alpha_s$ and multiply by $\langle |C|^2 \rangle$

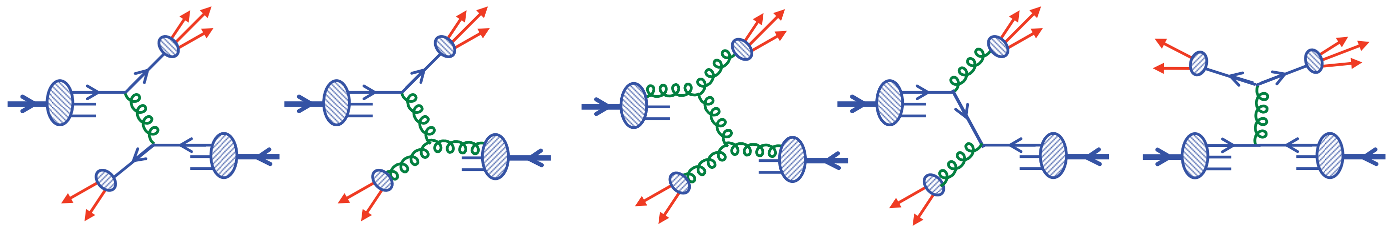
$\boxed{\text{QCD}}$

$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_S^2}{q^4}\left[1 + \left(1 + \frac{q^2}{\hat{s}}\right)^2\right]$$

Never see colour, but enters through colour factors. Can tell QCD is SU(3)

- Here $\hat{s}$ is the centre-of-mass energy of the quark-quark collision
- The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions

e.g. two jet production in proton-antiproton collisions

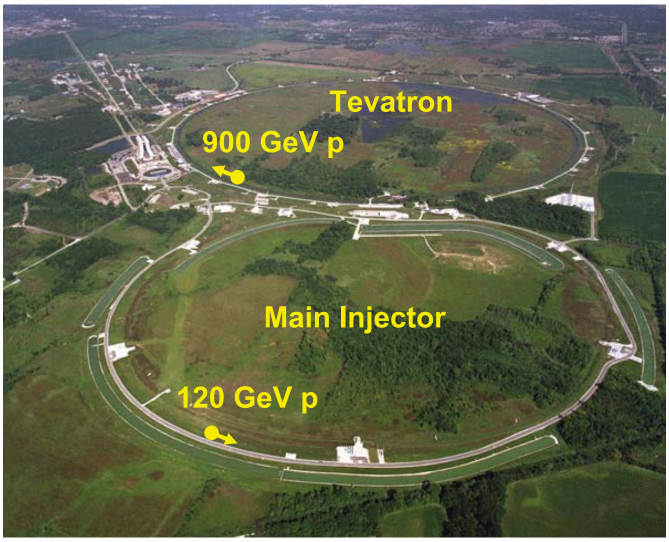


$q\bar{q} \to q\bar{q}$ $\quad qg \to qg$ $\quad gg \to gg$ $\quad q\bar{q} \to gg$ $\quad q\bar{q} \to q\bar{q}$

e.g. p$\bar{\text{p}}$ collisions at the Tevatron

★ Tevatron collider at Fermi National Laboratory (FNAL)
- located ~40 miles from Chigaco, US
- started operation in 1987 (will run until 2009/2010)

★ p$\bar{\text{p}}$ collisions at $\sqrt{s}$ = 1.8 TeV   c.f. 14 TeV at the LHC



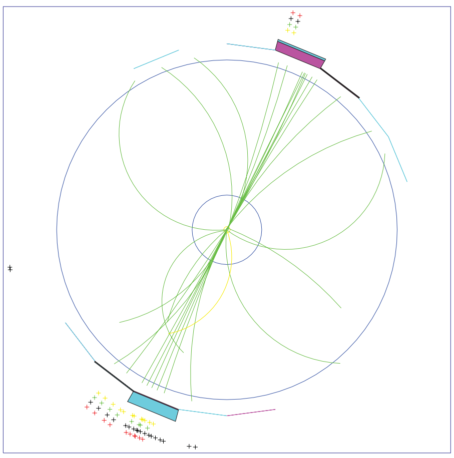
Two main accelerators:

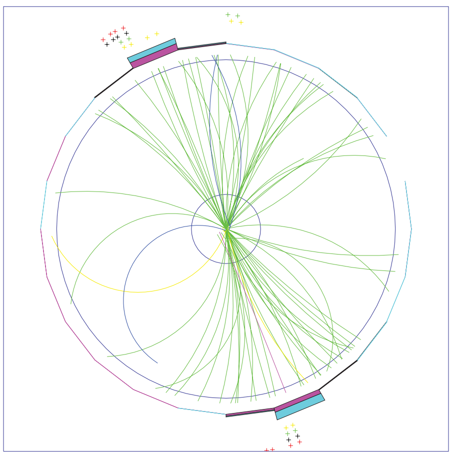
★Main Injector
- Accelerates 8 GeV p to 120 GeV
- also $\overline{p}$ to 120 GeV
- Protons sent to Tevatron & MINOS
- $\overline{p}$ all go to Tevatron

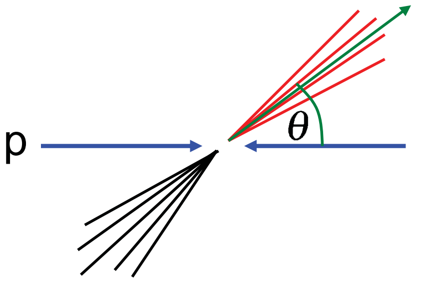
★Tevatron
- 4 mile circumference
- accelerates $p/\overline{p}$ from 120 GeV to 900 GeV

★ Test QCD predictions by looking at production of pairs of high energy jets

p$\bar{\text{p}}$ → jet jet + X



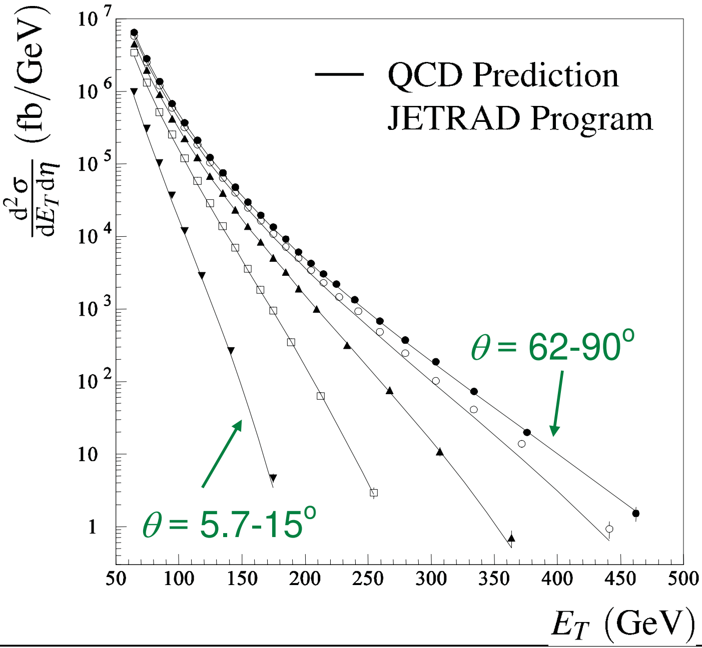




★ Measure cross-section in terms of

- "transverse energy" $E_T = E_{\text{jet}} \sin\theta$
- "pseudorapidity" $\eta = \ln\left[\cot\left(\frac{\theta}{2}\right)\right]$

...don't worry too much about the details here, what matters is that...



D0 Collaboration, Phys. Rev. Lett. 86 (2001)

★ QCD predictions provide an excellent description of the data

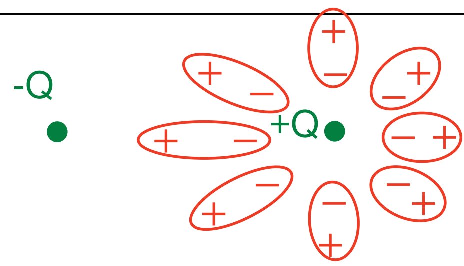
★ **NOTE:**

- at low E_T cross-section is dominated by low x partons
  i.e. **gluon-gluon** scattering
- at high E_T cross-section is dominated by high x partons
  i.e. **quark-antiquark** scattering

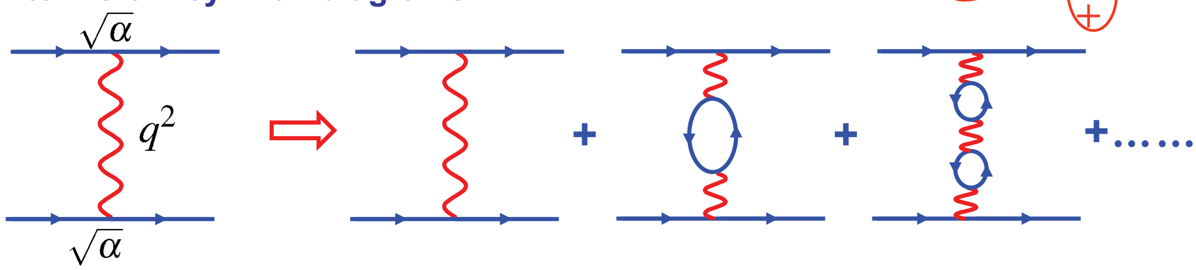
Running Coupling Constants

QED

- "bare" charge of electron screened by virtual e^+e^- pairs
- behaves like a polarizable dielectric



★ In terms of Feynman diagrams:



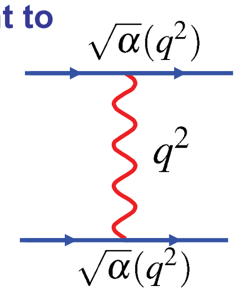
★ Same final state so add matrix element **amplitudes**: $M = M_1 + M_2 + M_3 + \ldots$

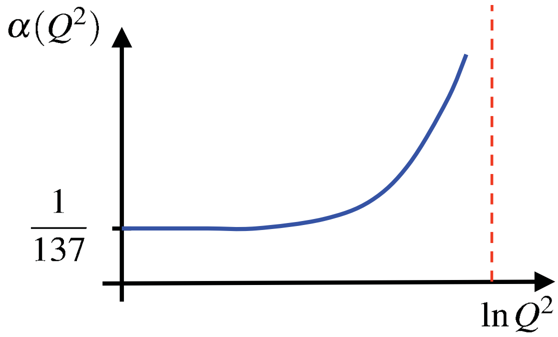
★ Giving an infinite series which can be summed and is equivalent to a single diagram with "running" coupling constant

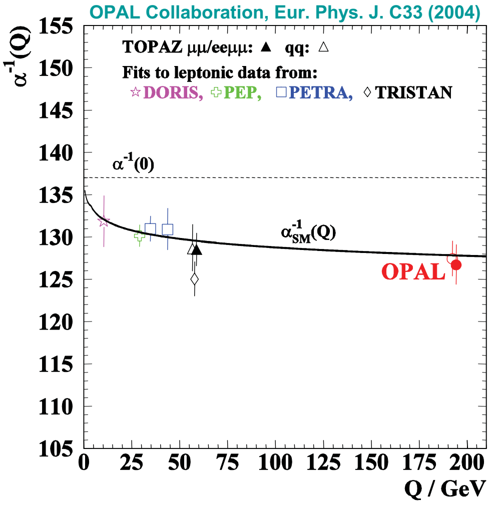
$$\alpha(Q^2) = \alpha(Q_0^2) \Big/ \left[1 - \frac{\alpha(Q_0^2)}{3\pi}\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

$$Q^2 \gg Q_0^2$$

Note sign







★ Might worry that coupling becomes infinite at

$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at $Q \sim 10^{26}\,\mathrm{GeV}$

- But quantum gravity effects would come in way below this energy and it is highly unlikely that QED "as is" would be valid in this regime

★ In QED, running coupling increases very slowly

- Atomic physics: $Q^2 \sim 0$

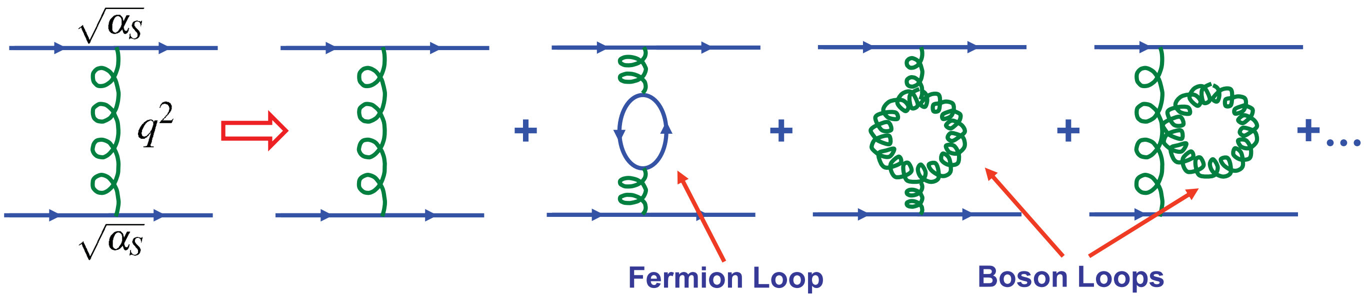
$$1/\alpha = 137.03599976(50)$$

- High energy physics:

$$1/\alpha(193\,\mathrm{GeV}) = 127.4 \pm 2.1$$

Running of α_s

QCD Similar to QED but also have gluon loops



★ Remembering **adding amplitudes**, so can get negative interference and the sum can be smaller than the original diagram alone

★ Bosonic loops "interfere negatively"

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Bigg/ \left[1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

with $B = \dfrac{11N_c - 2N_f}{12\pi}$ $\begin{cases} N_c & = \text{no. of colours} \\ N_f & = \text{no. of quark flavours} \end{cases}$

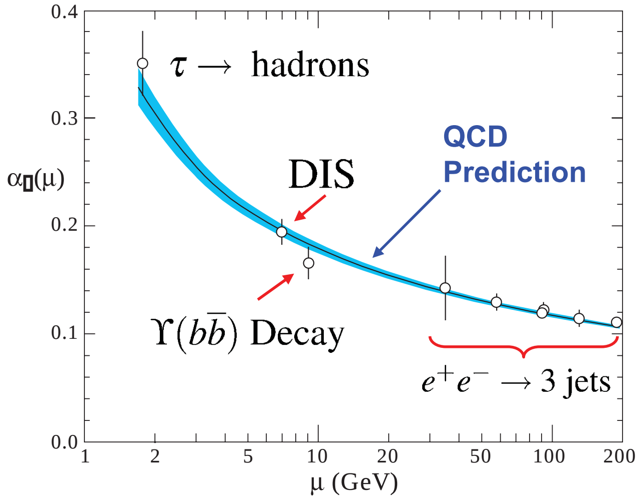
$N_c = 3;\ N_f = 6$ ➡ $B > 0$

➡ $\boxed{\alpha_S \text{ decreases with } Q^2}$

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

★ Measure α_s in many ways:
- jet rates
- DIS
- tau decays
- bottomonium decays
- +...

★ As predicted by QCD, α_s decreases with Q^2



$\mu \approx Q$

★ At low Q^2: α_s is large, e.g. at $Q^2 = 1\,\mathrm{GeV}^2$ find $\alpha_s \sim 1$

- Can't use perturbation theory ! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,...

★ At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_Z^2$ find $\alpha_s \sim 0.12$

➡ **Asymptotic Freedom**

- Can use perturbation theory and this is the reason that in DIS at high Q^2 quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons)

Summary

★ Superficially QCD very similar to QED
★ But gluon self-interactions are believed to result in colour confinement
★ All hadrons are colour singlets which explains why only observe

Mesons **Baryons**

★ A low energies $\alpha_S \sim 1$

➡ Can't use perturbation theory !

Non-Perturbative regime

★ Coupling constant runs, smaller coupling at higher energy scales

$$\alpha_S(100\,\mathrm{GeV}) \sim 0.1$$

➡ Can use perturbation theory

Asymptotic Freedom

★ Where calculations can be performed, QCD provides a good description of relevant experimental data

Appendix A1 : Electromagnetism

(Non-examinable)

★ In Heaviside-Lorentz units $\varepsilon_0 = \mu_0 = c = 1$ Maxwell's equations in the vacuum become

$$\vec{\nabla}.\vec{E} = \rho; \quad \vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}; \quad \vec{\nabla}\cdot\vec{B} = 0; \quad \vec{\nabla}\wedge\vec{B} = \vec{J} + \frac{\partial\vec{E}}{\partial t}$$

★ The electric and magnetic fields can be expressed in terms of scalar and vector potentials

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi; \quad \vec{B} = \vec{\nabla}\wedge\vec{A} \qquad \text{(A1)}$$

★ In terms of the 4-vector potential $A^\mu = (\phi, \vec{A})$ and the 4-vector current $j^\mu = (\rho, \vec{J})$ Maxwell's equations can be expressed in the covariant form:

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \text{(A2)}$$

where $F^{\mu\nu}$ is the anti-symmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad \text{(A3)}$$

- Combining (A2) and (A3)

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu \qquad \text{(A4)}$$

which can be written $\Box^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$ **(A5)**

where the D'Alembertian operator

$$\Box^2 = \partial_\nu\partial^\nu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

- Acting on equation (A5) with ∂_ν gives

$$\partial_\nu j^\nu = \partial_\nu\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial_\nu\partial^\nu A^\mu = 0$$

$$\Rightarrow \quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

Conservation of Electric Charge

- Conservation laws are associated with symmetries. Here the symmetry is the GAUGE INVARIANCE of electro-magnetism

Appendix A2 : Gauge Invariance

(Non-examinable)

★ The electric and magnetic fields are unchanged for the gauge transformation:

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi; \quad \phi \rightarrow \phi' = \phi - \frac{\partial\chi}{\partial t}$$

where $\chi = \chi(t, \vec{x})$ is any finite differentiable function of position and time

★ In 4-vector notation the gauge transformation can be expressed as:

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\chi$$

★ Using the fact that the physical fields are gauge invariant, choose χ to be a solution of

$$\Box^2\chi = -\partial_\mu A^\mu$$

★ In this case we have

$$\partial^\mu A'_\mu = \partial^\mu(A_\mu + \partial_\mu\chi) = \partial^\mu A_\mu + \Box^2\chi = 0$$

★ Dropping the prime we have a chosen a gauge in which

$$\boxed{\partial_\mu A^\mu = 0} \qquad \text{The Lorentz Condition} \qquad \text{(A6)}$$

★ With the Lorentz condition, equation (A5) becomes:

$$\boxed{\Box^2 A^\mu = j^\mu} \qquad \text{(A7)}$$

★ Having imposed the Lorentz condition we still have freedom to make a further gauge transformation, i.e.

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\Lambda$$

where $\Lambda(t,\vec{x})$ is any function that satisfies

$$\Box^2\Lambda = 0 \qquad \text{(A8)}$$

★ Clearly (A7) remains unchanged, in addition the Lorentz condition still holds:

$$\partial^\mu A'_\mu = \partial^\mu(A_\mu + \partial_\mu\Lambda) = \partial^\mu A_\mu + \Box^2\Lambda = \partial^\mu A_\mu = 0$$

Appendix B : Local Gauge Invariance

(Non-examinable)

★The Dirac equation for a charged particle in an electro-magnetic field can be obtained from the free particle wave-equation by making the minimal substitution

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi \qquad (q = \text{charge})$$ (see p.113)

In QM: $i\partial_\mu \to i\partial_\mu - qA_\mu$ and the Dirac equation becomes

$$\boxed{\gamma^\mu(i\partial_\mu - qA_\mu)\psi - m\psi = 0} \qquad \text{(B1)}$$

★ In Appendix A2 : saw that the physical EM fields where invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi \qquad \Box^2\chi = 0$$

★ Under this transformation the Dirac equation becomes

$$\gamma^\mu(i\partial_\mu - qA_\mu + q\partial_\mu\chi)\psi - m\psi = 0$$

which is not the same as the original equation. If we require that the Dirac equation is invariant under the Gauge transformation then under the gauge transformation we need to modify the wave-functions

$$\psi \to \psi' = \psi e^{iq\chi}$$

A Local Phase Transformation

★ To prove this, applying the gauge transformation :

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi \qquad \psi \to \psi' = \psi e^{iq\chi}$$

to the original Dirac equation gives

$$\gamma^\mu (i\partial_\mu - qA_\mu + q\partial_\mu \chi)\psi e^{iq\chi} - m\psi e^{iq\chi} = 0 \qquad \text{(B2)}$$

★ But $\quad i\partial_\mu(\psi e^{iq\chi}) = ie^{iq\chi}\partial_\mu \psi - q(\partial_\mu \chi)e^{iq\chi}\psi$

★ Equation (B2) becomes

$$\gamma^\mu e^{iq\chi}(i\partial_\mu - qA_\mu + q\partial_\mu \chi - q\partial_\mu \chi)\psi - m\psi e^{iq\chi} = 0$$

$$\Rightarrow \quad \gamma^\mu e^{iq\chi}(i\partial_\mu - qA_\mu)\psi - m\psi e^{iq\chi} = 0$$

$$\Rightarrow \quad \gamma^\mu (i\partial_\mu - qA_\mu)\psi - m\psi = 0$$

which is the original form of the Dirac equation

Appendix C : Local Gauge Invariance 2

(Non-examinable)

★ Reverse the argument of Appendix B. Suppose there is a fundamental symmetry of the universe under local phase transformations

$$\psi(x) \to \psi'(x) = \psi(x)e^{iq\chi(x)}$$

★ Note that the local nature of these transformations: the phase transformation depends on the space-time coordinate $x = (t, \vec{x})$

★ Under this transformation the free particle Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

becomes

$$i\gamma^\mu \partial_\mu(\psi e^{iq\chi}) - m\psi e^{iq\chi} = 0$$

$$ie^{iq\chi}\gamma^\mu(\partial_\mu \psi + iq\psi\partial_\mu \chi) - m\psi e^{iq\chi} = 0$$

$$i\gamma^\mu(\partial_\mu + iq\partial_\mu \chi)\psi - m\psi = 0$$

Local phase invariance is not possible for a free theory, i.e. one without interactions

★ To restore invariance under local phase transformations have to introduce a massless "gauge boson" A^μ which transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

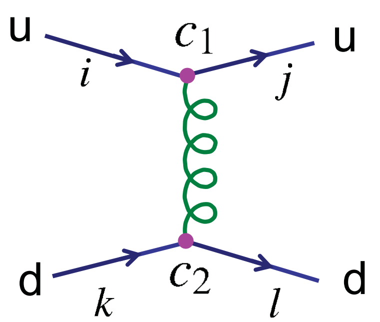
and make the substitution

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

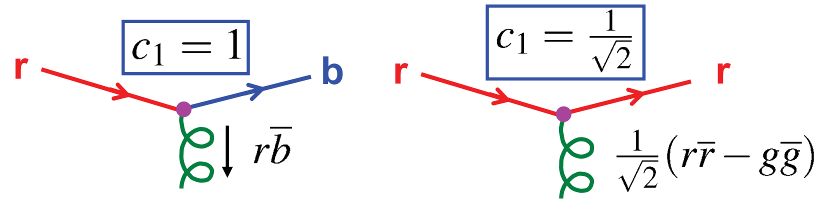
Appendix D: Alternative evaluation of colour factors

"Non-examinable" but can be used to derive colour factors.

★ **The colour factors can be obtained (more intuitively) as follows :**



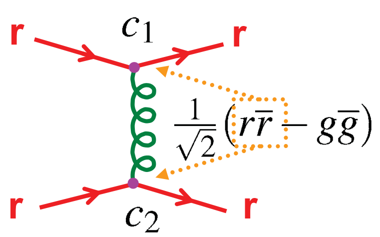
- **Write** $C(ik \to jl) = \frac{1}{2} c_1 c_2$
- **Where the colour coefficients at the two vertices depend on the quark and gluon colours**

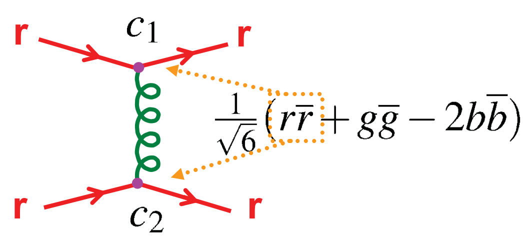


- **Sum over all possible exchanged gluons conserving colour at both vertices**

① Configurations involving a single colour

e.g. $rr \to rr$ **: two possible exchanged gluons**

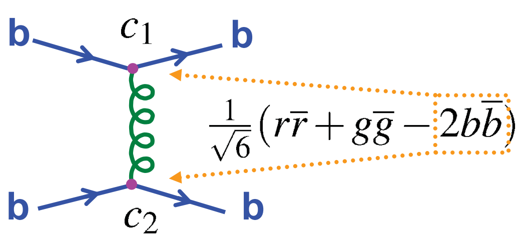




$$c_1 = c_2 = \frac{1}{\sqrt{2}} \qquad c_1 = c_2 = \frac{1}{\sqrt{6}}$$

$$C(rr \to rr) = \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{6}\right) = \tfrac{1}{3}$$

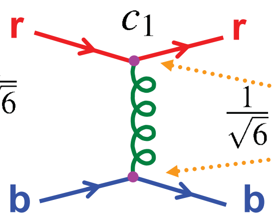
e.g. $bb \to bb$ **: only one possible exchanged gluon**



$$c_1 = c_2 = -\frac{2}{\sqrt{6}}$$

$$\Rightarrow \quad C(bb \to bb) = \tfrac{1}{2}\left(\tfrac{2}{\sqrt{6}}\tfrac{2}{\sqrt{6}}\right) = \tfrac{1}{3}$$

② Other configurations where quarks don't change colour



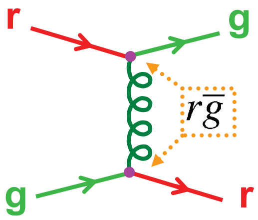
$c_1 = \frac{1}{\sqrt{6}}$

$$\frac{1}{\sqrt{6}}(\boxed{r\bar{r}} + g\bar{g} - \boxed{2b\bar{b}})$$

$c_2 = -\frac{2}{\sqrt{6}}$

$$C(rb \to rb) = \tfrac{1}{2}\left(-\tfrac{1}{\sqrt{6}}\tfrac{2}{\sqrt{6}}\right) = -\tfrac{1}{6}$$

③ Configurations where quarks swap colours



$$c_1 = c_2 = 1$$

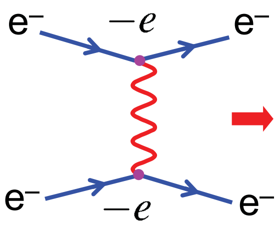
$$C(rg \to gr) = \tfrac{1}{2}$$

Appendix E: Colour Potentials

Non-examinable

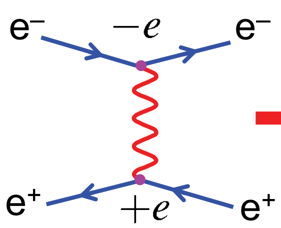
- Previously argued that gluon self-interactions lead to a $+\lambda r$ long-range potential and that this is likely to explain colour confinement
- Have yet to consider the short range potential – i.e. for quarks in mesons and baryons does QCD lead to an attractive potential?
- Analogy with QED: (NOTE this is very far from a formal proof)

QED



$$\boxed{V(r) = +\frac{\alpha}{r}}$$

Repulsive Potential



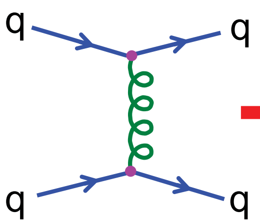
$$\boxed{V(r) = -\frac{\alpha}{r}}$$

Attractive Potential

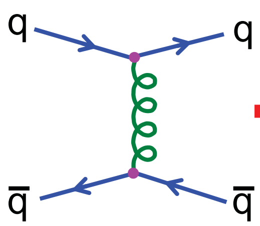
Static

QCD

★ by analogy with QED expect potentials of form



$$\boxed{V(r) = +C\frac{\alpha_S}{r}}$$



$$\boxed{V(r) = -C\frac{\alpha_S}{r}}$$

★ Whether it is a attractive or repulsive potential depends on sign of colour factor

★ Consider the colour factor for a $q\bar{q}$ system in the colour singlet state:

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

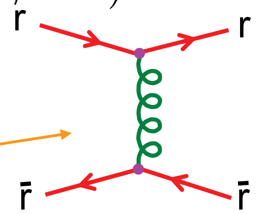
with colour potential $\langle V_{q\bar{q}}\rangle = \langle\psi|V_{\rm QCD}|\psi\rangle$

$$\Rightarrow \quad \langle V_{q\bar{q}}\rangle = \frac{1}{3}\left(\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle + \ldots + \langle r\bar{r}|V_{\rm QCD}|b\bar{b}\rangle + \ldots\right)$$

• Following the QED analogy:

$$\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle = -C(r\bar{r}\to r\bar{r})\frac{\alpha_S}{r}$$

which is the term arising from $r\bar{r}\to r\bar{r}$



• Have 3 terms like $r\bar{r}\to r\bar{r},\, b\bar{b}\to b\bar{b},\ldots$ and 6 like $r\bar{r}\to g\bar{g},\, r\bar{r}\to b\bar{b},\ldots$

$$\langle V_{q\bar{q}}\rangle = -\frac{1}{3}\frac{\alpha_S}{r}\left[3\times C(r\bar{r}\to r\bar{r}) + 6\times C(r\bar{r}\to g\bar{g})\right] = -\frac{1}{3}\frac{\alpha_S}{r}\left[3\times\frac{1}{3} + 6\times\frac{1}{2}\right]$$

$$\Rightarrow \quad \boxed{\langle V_{q\bar{q}}\rangle = -\frac{4}{3}\frac{\alpha_S}{r}}$$

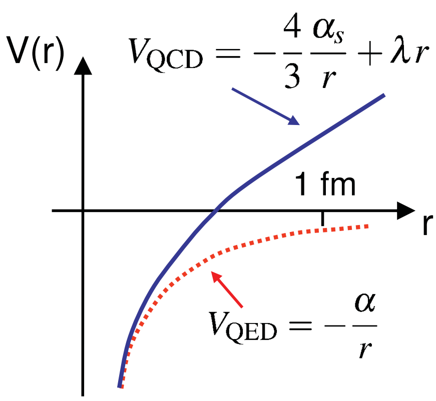
NEGATIVE ➡ ATTRACTIVE

• The same calculation for a $q\bar{q}$ colour octet state, e.g. $r\bar{g}$ gives a positive repulsive potential: $C(r\bar{g}\to r\bar{g}) = -\frac{1}{6}$

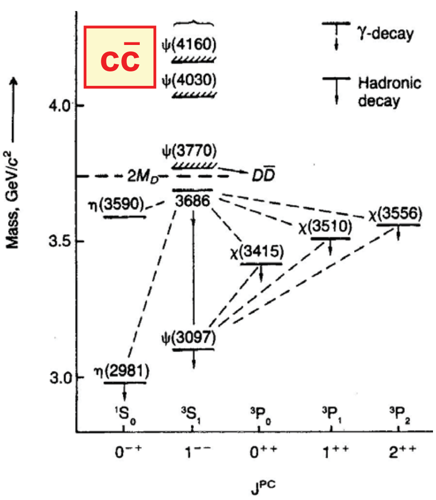
★ Whilst not a formal proof, it is comforting to see that in the colour singlet $q\bar{q}$ state the QCD potential is indeed attractive. (question 15)

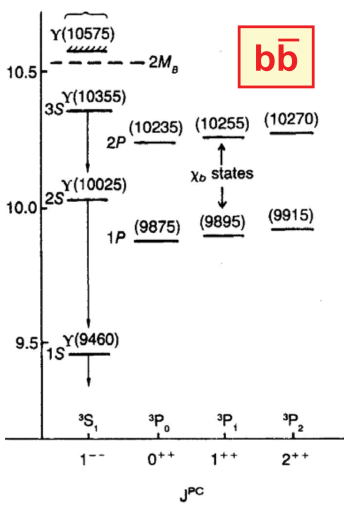
★ Combining the short-range QCD potential with the linear long-range term discussed previously:

$$\boxed{V_{\rm QCD} = -\frac{4}{3}\frac{\alpha_s}{r} + \lambda r}$$



★ This potential is found to give a good description of the observed charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) bound states.





NOTE:
- c, b are heavy quarks
- approx. non-relativistic
- orbit close together
- probe **1/r** part of $V_{\rm QCD}$

Agreement of data with prediction provides strong evidence that $V_{\rm QCD}$ has the Expected form