Modern Particle Physics
Solutions and Hints
version 1.02

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Contents

Preface

This short guide gives numerical answers and hopefully helpful hints to all questions in the first edition of Modern Particle Physics. Comments are always welcome.

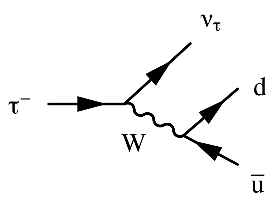
Course instructors can obtain fully-worked solutions in the Instructor's Manual to Modern Particle Physics (available from Cambridge University Press).

Mark Thomson, Cambridge, January 4th 2014

1 Introduction

1.1 **Answer:** Of the sixteen vertices, the only valid Standard Model vertices are: a), d), f), j), n) and o). It should be remembered that only the weak charged current (W) interaction changes the flavour of the fermion.

1.2 **Answer:** Since the decay involves a change of flavour it can only be a weak charged-current interaction ($W^{\pm}$):



1.3 **Hint:** Try drawing Feynman diagrams for each process using the SM vertices and think about charge, flavour and particle/antiparticle.

1.4 **Answer:** All other things being equal, strong decays will dominate over EM decays, and EM decays will dominate over weak decays and with the appropriate Feynman diagrams the order is a), b), c).

1.5 **Hint:** In the decay of the π^0, which has a quark flavour wavefunction:

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}\left(u\overline{u} + d\overline{d}\right) ,$$

the Feynman diagram can be considered as either annihilation of $u\overline{u}$ or $d\overline{d}$. With the exception of the branching ratio to e^+e^-, the predictions based on counting vertices are in reasonable agreement with the observed ratios. It should be noted that considering only the vertex factors, addresses only one of the contributions to the matrix element squared, other factors may be just as important.

1.6 **Hint:** With the exception of b) there are two possible lowest-order diagrams. In a) these are the s- and u-channel diagrams. In c), d) and e) there are s- and t-channel diagrams.

1.7 **Answer:**

a) Ionisation and bremsstrahlung/pair production processes become equally likely (in standard rock) when $a = bE$, i.e. $E_\mu = 714\,\mathrm{GeV}$.

b) Integrating the energy loss equation from $E = 100\,\mathrm{GeV} \rightarrow 0\,\mathrm{GeV}$ should give a range $L = 141$ m.

1.8 Answer: The number of particles in a shower doubles every radiation length of material traversed until the critical energy is reached, for a 500 GeV EM shower the critical energy is (on average) reached after 16 radiation lengths, i.e. 5.6 cm of Tungsten.

1.9 Answer: The momentum of the particle can be obtained from $\mathrm{p}\cos\lambda = 0.3\,\mathrm{B}R$ and if the particle were a kaon, it would have a velocity $\beta_\mathrm{K} = 0.73$, which would not give a Čerenkov signal and hence the particle is a pion.

1.10 Answer: To achieve a centre-of-mass energy of 14 TeV in a fixed-target collision $E_\mathrm{p} = 1.05 \times 10^5$ TeV.

1.11 Answer: Using the values in the original text $\mathcal{L} = 6 \times 10^{30}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$.

Note that there is an updated and clearer version of the original problem.
At the LEP e^+e^- collider, which had a circumference of 27 km, the electron and positron beams consisted of four equally spaced bunches in the accelerator. Each bunch corresponded to a beam current of 1.0 mA. The beams collided head-on at the interaction point, where the beam spot had an rms profile of $\sigma_x \approx 250\,\mu$m and $\sigma_y \approx 4\,\mu$m, giving an effective area of $1.0 \times 10^3\,\mu\mathrm{m}^2$. Calculate the instantaneous luminosity and estimate the event rate for the process $e^+e^- \rightarrow Z$, which has a cross section of about 40 nb.

2 Underlying Concepts

2.1 **Answer:** To restore the correct dimensions a factor of $\hbar$ needs to be inserted, giving $\tau = 3.3 \times 10^{-25}$ s .

2.2 **Answer:** $\sigma = 2.6 \times 10^{-9}$ GeV^{-2} .

2.3 **Hint:** This problem can be solved using a number of approaches, the most easy is to considering the reaction in the rest frame of the e^+e^- pair, namely the frame in which the total momentum is zero.

2.4 **Hint:** Here the key equations are (in natural units):

$$E = \gamma m\,, \quad \mathrm{p} = \gamma m\beta \quad \text{and} \quad E^2 = p^2 + m^2\,.$$

2.5 **Hint:** Remember that $\gamma^2 = 1/(1-\beta^2)$ or equivalently $\gamma^2(1-\beta^2) = 1$, and use the explicit energy-momentum Lorentz transformations

$$E' = \gamma(E - \beta p_z)\,, \quad p'_x = p_x\,, \quad p'_y = p_y \quad \text{and} \quad p'_z = \gamma(p_z - \beta E)\,.$$

2.6 **Hint:** Start from$m_a^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2$.

2.7 **Answer:**

a) $m_\Lambda = 1.115$ GeV.

b) Accounting for relativistic time dilation the mean distance travelled will be

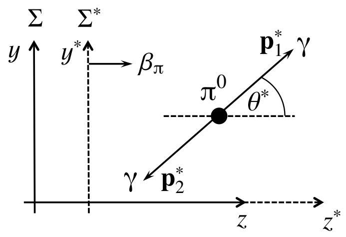
$$d = \gamma\beta c\tau\,,$$

from which

$$\tau = 0.35/4.47c = 2.6 \times 10^{-10}\,\mathrm{s}\,.$$

2.8 **Hint:** The lowest energy configuration is where all four final-state particles are at rest *in the centre-of-mass frame*, then use the fact that the Lorentz invariant quantity s is identical in all frames.

2.9 **Answer:** The momenta of the photons in the π^0 rest frame can be boosted into the laboratory frame giving the extreme values are $\cos\theta = -1$ and $\cos\theta =$

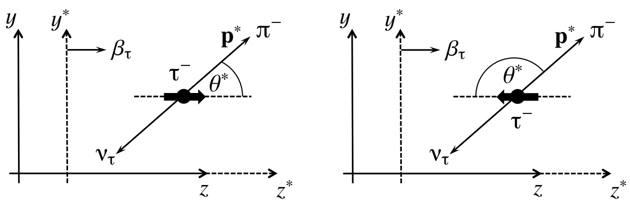


$2\beta^2 - 1$, where θ is the opening angle in the laboratory frame. The minimum opening angle is $\theta_{\text{min}} = 0.027\,\text{rad} \equiv 1.5°$.

2.10 **Answer:** $m_\Delta = \sqrt{s} = 1.23\,\text{GeV}$.

2.11 **Hint:** The angular dependence arises from the chiral nature of the weak interaction, which implies that the ν_τ is left-handed. The two cases are indicated below. In the rest frame of the four-momentum of the π^- are respectively given by:

$$p^* = (E^*_\pi, 0, \text{p}^*_\pi \sin\theta^*, \text{p}^*_\pi \cos\theta^*) \quad \text{and} \quad p^* = (E^*_\pi, 0, \text{p}^*_\pi \sin\theta^*, -\text{p}^*_\pi \cos\theta^*)\,.$$



The Lorentz transformation can then be used to determine the dependence of E_π on θ^* and the distribution can be found using

$$\frac{dN}{dE_\pm} = \frac{dN}{d(\cos\theta^*)} \times \left|\frac{dE_\pm}{d(\cos\theta^*)}\right|^{-1} .$$

Answer: For the two different spin orientations

$$\frac{dN}{dE_\pi} = \frac{E_\pi - E_{\text{min}}}{E_{\text{max}} - E_{\text{min}}} \quad \text{and} \quad \frac{dN}{dE_\pi} = \frac{E_{\text{max}} - E_\pi}{E_{\text{max}} - E_{\text{min}}}\,.$$

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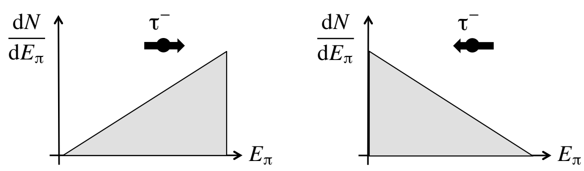


2.12 **Hint:** $s + u + t = (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2$.

2.13 **Answer:** $\sqrt{s} = 300\,\mathrm{GeV}$.

2.14 **Hint:** Using four-vectors, this is a fairly straightforward problem. Write $p' = k - k' + p$ and squaring (the four-vectors).

2.15 **Hint:** Use

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y\,, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \quad \text{and} \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x\,,$$

and $[\hat{z}, \hat{p}_z] = i$ etc.

2.16 **Hint:** Show that

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hat{S}_z\,,$$

and

$$\hat{\mathbf{S}}^2 = \tfrac{1}{4}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) = \tfrac{3}{4}I\,.$$

2.17 **Answer:**

$$T_{fi} = \langle f|\hat{H}'|i\rangle + \sum \frac{\langle k|\hat{H}'|i\rangle\langle f|\hat{H}'|k\rangle}{(E_k - E_i)} + \sum\sum \frac{\langle f|\hat{H}'|k\rangle\langle k|\hat{H}'|j\rangle\langle j|\hat{H}'|i\rangle}{(E_k - E_j)(E_j - E_i)}\,.$$

3 Decay Rates and Cross Sections

3.1 **Answer:** $E_\mu^2 = m_\mu^2 + \mathrm{p}^2 = 110\,\mathrm{MeV}$.

3.2 **Hint:** Write $m_a - E_2 = E_1$ and square to eliminate E_1, then rearrange to give an expression for E_2 and square again.

3.3 **Answer:** $BR(\mathrm{K}^+ \to \pi^+\pi^0) = 21\,\%$.

3.4 **Answer:** The total number of events is given by

$$N = \int \sigma \mathcal{L}\,\mathrm{d}t\,,$$

therefore in five years of operation with 50 % lifetime, a total of 394000 $\mathrm{e}^+\mathrm{e}^- \to$ HZ events would be accumulated.

3.5 **Answer:** $\sigma = 8.9 \times 10^{-8}\,\mathrm{GeV}^{-2} \times 0.197^2 \times 0.01\,\mathrm{b} = 34\,\mathrm{pb}$.

3.6 **Answer:** The average number of interactions for the single neutrino traversing the block is approximately 7×10^{-10} and therefore the interaction probability is less than 10^{-9}.

3.7 **Hint:** First consider the low energy limit of the four-vector product $p_a \cdot p_b = E_a E_b - \mathbf{p}_a \cdot \mathbf{p}_b$ where, (as expected) the non-relativistic limit of the particle energy and momentum are (in natural units)

$$E = \gamma m = m\left(1 - \beta^2\right)^{-1/2} \approx m(1 + \tfrac{1}{2}\beta^2) = m + \tfrac{1}{2}m\beta^2\,.$$

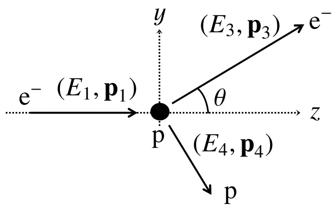
3.8 **Hint:** Here $p_a = (E_a, 0, 0, \mathrm{p}_a)$ and $p_b = (m_b, 0, 0, 0)$.

3.9 **Hint:** First write $\sqrt{s} - E_1^* = E_2^*$ and square to eliminate E_2^* and then eliminate E_1^* by again squaring.

3.10 **Hint: a**) Differentiating $E_3^2 = \mathrm{p}_3^2 + m_3^2$ with respect to $\cos\theta$ gives

$$2E_3 \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = 2\mathrm{p}_3 \frac{\mathrm{d}\mathrm{p}_3}{\mathrm{d}(\cos\theta)}\,. \tag{3.1}$$

Then equate the expressions for the Mandelstam t variable written in terms of the electron and proton four-momenta $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$.



b) From (3.37) of the main text,

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s\, p_i^{*2}} |\mathcal{M}_{fi}|^2 .$$

This can be related to the differential cross section in terms of solid angle using

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt} .$$

4 The Dirac Equation

4.1 **Hint:** The commutator of $\hat{\mathbf{p}}^2$ with the x-component of $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y .$$

can be written

$$\begin{aligned}
\left[\hat{\mathbf{p}}^2, \hat{L}_x\right] &= \left[\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right] \\
&= \left[\hat{p}_y^2, \hat{y}\hat{p}_z\right] - \left[\hat{p}_z^2, \hat{z}\hat{p}_y\right] \\
&= \left[\hat{p}_y^2, \hat{y}\right]\hat{p}_z - \left[\hat{p}_z^2, \hat{z}\right]\hat{p}_y .
\end{aligned}$$

4.2 **Hint:** Use

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}.$$

4.3 **Hint:** In matrix form $\gamma^\mu p_\mu - m$ is given by

$$\begin{aligned}
\gamma^\mu p_\mu - m &= E\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - p_x\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} - p_y\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} - p_z\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - mI \\
&= \begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -(E+m) & 0 \\ p_x+ip_y & -p_z & 0 & -(E+m) \end{pmatrix}.
\end{aligned}$$

4.4 **Hint:** Use the free particle spinor $u_1(p)$ and recall that for *arbitrary* spinors

ψ and ϕ, with spinor components ψ_i and ϕ_i, matrix multiplication gives,

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2\,.
\end{aligned}$$

For the final part of the question, note that a particle spinor $u(p)$ can always be expressed as a linear combination of the basis spinors $u_1(p)$ and $u_2(p)$:

$$u = \alpha_1 u_1 + \alpha_2 u_2\,, \quad \text{with} \quad |\alpha_1|^2 + |\alpha_2|^2 = 1\,.$$

4.5 **Hint:** Here we are looking for the general order of magnitude of the relative size of the upper and lower components. So for simplicity, consider a particle travelling in the z-direction, which from the definition of u_1 has

$$u_A = N\begin{pmatrix}1\\0\end{pmatrix}, \quad \text{and} \quad u_B = N\begin{pmatrix}\frac{\mathrm{p}}{E+m}\\0\end{pmatrix}.$$

4.6 **Hint:** Just consider the cases $\mu = \nu = 0$, $\mu = \nu = k = 1, 2, 3$ and $\mu \neq \nu$ and use the commutation relations.

4.7 **Hint:** Remember that ψ satisfies the Dirac equation $i\gamma^\nu\partial_\nu\psi = m\psi$.

4.8 **Hint:** Use $\gamma^{0\dagger} = \gamma^0$, $\gamma^{k\dagger} = \gamma^k$, $\gamma^0\gamma^0 = I$ and $\gamma^0\gamma^k = -\gamma^k\gamma^0$.

4.9 **Hint:** In part b) consider the $\overline{u}\gamma^\nu\times$ the Dirac equation and Dirac equation for the adjoint spinor $\times\gamma^\nu u$

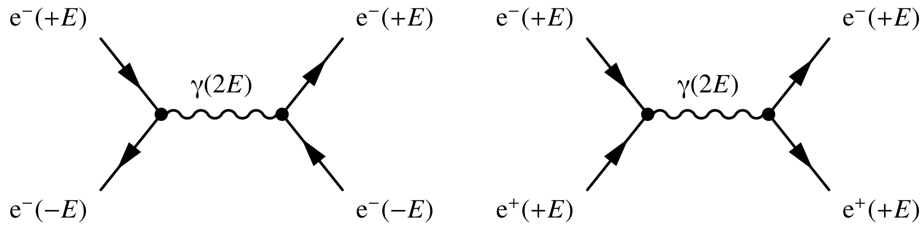
$$\overline{u}\gamma^\nu(\gamma^\mu p_\mu - m)u = 0 \quad \text{and} \quad \overline{u}(\gamma^\mu p_\mu - m)\gamma^\nu u = 0$$

and take the sum.

4.10 **Hint:** This can be demonstrated either by writing out the explicit form of $\boldsymbol{\sigma}\cdot\mathbf{p}$ using the Pauli spin matrices or (more elegantly) by using the properties of the matrices, namely $\sigma_k^2 = 1$ and $\sigma_x\sigma_y = -\sigma_y\sigma_x$.

4.11 **Answer:**
a) In the first interpretation (left diagram), the intial-state positive e^- of energy $+E$ emits a photon of energy 2E. To conserve energy it is now a negative energy e^- and therefore propagates backwards in time. At the other vertex, the photon interacts with a negative energy e^-, which is propagating backwards in time and scattering results in a positive energy e^-.

b) In the Feynman-Stückelberg interpretation (right diagram), the intial-state positive e^- of energy $+E$ annihilates with a positive energy e^+ to produce a photon of energy 2E. At the second vertex the photon produces an e^+e^- pair. All particles propagate forwards in time.



4.12 **Hint:** In the Pauli-Dirac representation

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix},$$

and since β contains the identity matrix it is clear that $\left[\hat{h}, m\beta\right] = 0$ and therefore it is only necessary to consider $\left[\hat{h}, \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\right]$.

4.13 **Answer:** The action of the parity operator has the effect that $\mathbf{p} \to -\mathbf{p}$ (reversing the direction of the particle), but leaves the orientation of the spin unchanged in space, this transforming a RH particle into a LH particle travelling in the opposite direction.

4.14 **Hint:** This is mostly an algebraic exercise to show that

$$\hat{C}\hat{P}u_{\uparrow}(\theta, \phi) = -e^{i\phi}v_{\downarrow}(\pi - \theta, \pi + \phi)\,.$$

Note this overall (unobservable phase) could have been included in the original definition of the $v_{\downarrow}$.

4.15 **Hint:** Start with the Dirac equation for the spinor $u(p)$ and the corresponding equation for the adjoint spinor $\overline{u}(p')$:

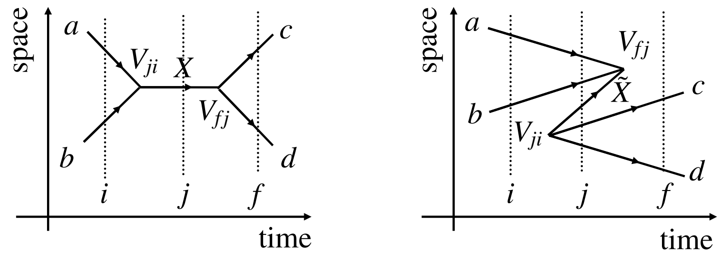
$$(\gamma^{\mu}p_{\mu} - m)u(p) = 0 \quad \text{and} \quad \overline{u}(p')(\gamma^{\mu}p'_{\mu} - m) = 0\,,$$

giving

$$\gamma^{\mu}p_{\mu}u(p) = mu(p) \quad \text{and} \quad \overline{u}(p')\gamma^{\mu}p'_{\mu} = m\overline{u}(p')\,.$$

5 Interaction by Particle Exchange

5.1 Hint: The two possible time-orderings are shown below. In the first $a + b$ annihilate into X and then X produces $c + d$. In the second time-ordering, the three particles $c + d + \tilde{X}$ "pop out" of the vacuum and subsequently $a + b + \tilde{X}$ annihilate into the vacuum.



Following the same arguments as in the many text, you should find

$$\mathcal{M} = \frac{g^2}{(E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2 - m_X^2}$$
$$= \frac{g^2}{q^2 - m_X^2},$$

where (here) $q^2 = (p_a + p_b)^2$.

5.2 Hint: The lowest-order diagrams have just two QED eeγ interaction vertices. Here there is a t-channel and an s-channel diagram.

5.3 Answer:

$$-i\mathcal{M}_t = \left[\varepsilon_\mu^*(p_3) ie\gamma^\mu u(p_1)\right] \cdot \left[-\frac{i(\gamma^\rho q_\rho + m_e)}{q^2 - m_e}\right] \cdot \left[\overline{v}(p_2) ie\gamma^\nu \varepsilon_\nu^*(p_4)\right]$$
$$-i\mathcal{M}_u = \left[\varepsilon_\mu^*(p_4) ie\gamma^\mu u(p_1)\right] \cdot \left[-\frac{i(\gamma^\rho q_\rho + m_e)}{q^2 - m_e}\right] \cdot \left[\overline{v}(p_2) ie\gamma^\nu \varepsilon_\nu^*(p_3)\right].$$

6 Electron-Positron Annihilation

6.1 **Hint:** Remember that $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$.

6.2 **Hint:** Remember that $(\gamma^5)^2 = 1$.

6.3 **Hint:** In the first part of the question, you may need to realise that $p_\mu p_\nu \gamma^\mu\gamma^\nu$ is a symmetric tensor and that it can be written as $\frac{1}{2}p_\mu p_\nu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$.

6.4 **Hint:** In the Dirac-Pauli representation, the relevant matrices are

$$\hat{S}_k = \frac{1}{2}\hat{\Sigma}_k = \frac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where $k = 1, 2, 3$ and $\hat{S}_k$ are the components of the spin operator *for a Dirac spinor*.

6.5 **Answer:** Because s-channel QED cross sections decrease as $1/s$, as the centre-of-mass energy increases, higher instantaneous luminosities are required to obtain a reasonable event rate, $Rate = \sigma\mathcal{L}$.

6.6 **Hint:** This question is a fairly straightforward but requires care with the algebra. Firstly, it should be noted that $\alpha^2 + \beta^2 + \gamma^2 = 1$. Secondly, by definition $\hat{S}_n|1,+1\rangle_\theta = +|1,+1\rangle_\theta$, where $\hat{S}_n = \mathbf{n}\cdot\hat{\mathbf{S}}$ and, without loss of generality $\mathbf{n}$ taken to lie in the xz plane.

$$\hat{S}_n = \mathbf{n}\cdot\hat{\mathbf{S}} = \sin\theta\hat{S}_x + \cos\theta\hat{S}_z\,.$$

The operator $\hat{S}_n$ can be written in terms of operators in terms of the $|s,m\rangle$ states using the angular momentum ladder operators,

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y \quad \text{and} \quad \hat{S}_+ = \hat{S}_x - i\hat{S}_y\,.$$

6.7 **Hint:** This is a fairly involved question, but with the exception of part c) it is just algebra. In part c) it should be realised that the parity operator reverses the momentum of a particle (a vector quantity) but leaves the spin (an axial-vector quantity) unchanged, and therefore has the effect $\hat{P}u_\uparrow(E,\mathbf{p}) = u_\downarrow(E,-\mathbf{p})$. This can be utilised here to obtain the possible muon currents from the electron currents (once the different masses have been accounted for), since in the centre-of-mass frame, the initial- and final-state electron and muon momenta are equal and opposite.

6.8* **Hint:** In part a), note that $g_{\mu\nu}$ is a symmetric tensor and thus

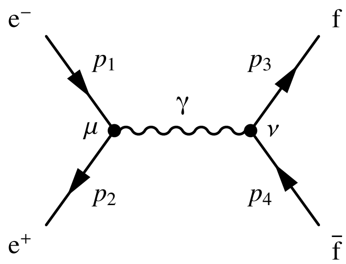
$$\begin{aligned}\gamma^\mu\gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\nu \\ &= \tfrac{1}{2}g_{\mu\nu}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) .\end{aligned}$$

6.9* **Hint:** Start from

$$\left[\overline{\psi}\gamma^\mu\gamma^5\phi\right]^\dagger = [\psi^\dagger\gamma^0\gamma^\mu\gamma^5\phi]^\dagger .$$

6.10* **Hint:** The QED matrix element for the Feynman diagram shown below is

$$\mathcal{M}_{fi} = \frac{Q_\mathrm{f}e^2}{q^2}\left[\overline{v}(p_2)\gamma^\mu u(p_1)\right] g_{\mu\nu}\left[\overline{u}(p_3)\gamma^\nu v(p_4)\right] .$$



Noting the order in which the spinors appear in the matrix element (working backwards along the arrows on the fermion lines), the spin-summed matrix element squared is given by

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{Q_\mathrm{f}^2 e^4}{q^4}\,\mathrm{Tr}\left([\not{p}_2 - m_\mathrm{e}]\gamma^\mu[\not{p}_1 + m_\mathrm{e}]\gamma^\nu\right)\,\mathrm{Tr}\left([\not{p}_3 + m_\mathrm{f}]\gamma_\mu[\not{p}_4 - m_\mathrm{f}]\gamma_\nu\right) .$$

Then, remember that the trace of an odd number of gamma-matrices is zero.

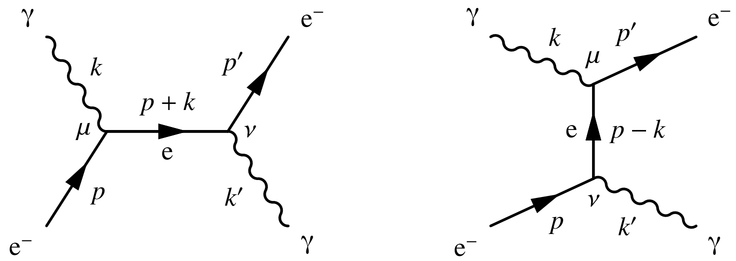
6.11* **Hint:** The spin averaged matrix element squared for the s-channel process $\mathrm{e^+e^-} \to \mathrm{f\overline{f}}$ is given in (6.63) of the main text:

$$\langle|\mathcal{M}_{fi}|^2\rangle_s = 2\frac{Q_\mathrm{f}^2 e^4}{(p_1\cdot p_2)^2}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_\mathrm{f}^2(p_1\cdot p_2)\right] .$$

6.12* **Answer:** The two lowest-order Feynman diagrams for the Compton scattering process $\mathrm{e}^-(p) + \gamma(k) \to \mathrm{e}^- p' + \gamma(k')$ are shown below. In both diagrams the vertex with the incoming photon is labelled μ.

From the QED Feynman rules, the matrix element for the s-channel diagram is given by

$$\mathcal{M}_s = -e^2\varepsilon^*_\mu(k)\varepsilon_\nu(k')\,\overline{u}(p')\left[\gamma^\nu\,\frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\gamma^\mu\right]u(p) ,$$



where $q = k + p$ and the slashed notation has been used. Similarly, the matrix element for the second diagram is

$$\mathcal{M}_t = -e^2 \varepsilon^*_\mu(k)\varepsilon_\nu(k')\,\overline{u}(p')\left[\gamma^\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m_\mathrm{e}^2}\gamma^\nu\right] u(p)\,.$$

For the spin sums, you will need to use the completeness relation for photons (see Appendix D) and after some manipulation:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\sum_{r=1,2}\sum_{r'=1,2}\{\overline{u}_{r'}(p')\,[\gamma^\nu\,\Gamma_+\gamma^\mu + \gamma^\mu\,\Gamma_-\gamma^\nu]\,u_r(p)\} \times \left\{\overline{u}_{r'}(p')\left[\gamma_\nu\Gamma_+\gamma_\mu + \gamma_\mu\,\Gamma_-\gamma_\nu\right]u_r(p)\right\}^\dagger\,,$$

where

$$\Gamma_\pm = \frac{\not{p} + \not{k} + m}{(p+k)^2 - m_\mathrm{e}^2}\,.$$

7 Electron-Proton Elastic Scattering

7.1 Hint: Using the expression for κ

$$(\gamma+1)(1-\kappa^2) = (\gamma+1) - \frac{\beta^2\gamma^2}{(\gamma+1)} .$$

7.2 Note: *This question should be ignored - unless a particular limit is taken finding a general solution is non-trivial and involves a lot of uninteresting algebra.*

7.3 Answer: a) Elastically scattered electrons would have an energy of 373.3 GeV, consistent with the observed value. b) $Q = 541$ MeV.

7.4 Hint: You will need to use the expansion $\sin \mathrm{qr} \simeq \mathrm{qr} - \frac{1}{3!}(\mathrm{qr})^3 + ...$ and use

$$\int 4\pi \mathrm{r}^2 \rho(\mathrm{r})\,\mathrm{dr} = 1 \quad \text{and} \quad \int 4\pi \mathrm{r}^2 \mathrm{r}^2 \rho(\mathrm{r})\,\mathrm{dr} = \langle R^2 \rangle .$$

7.5 Answer: Using the gradient at $Q^2 = 0$ of Figure 7.8a gives the rms charge radius of the proton of approximately 0.8 fm.

7.6 Answer:

$$G_M(Q^2 = 0.292\,\mathrm{GeV}^2) \simeq 1.26 \quad \text{and} \quad G_E(Q^2 = 0.292\,\mathrm{GeV}^2) \simeq 0.52 .$$

7.7 Answer: This is quite involved and the exact answers obtained will depend on how the interpolation between different data points is performed. The cross section values corresponding to $Q^2 = 500\,\mathrm{MeV}^2$ can be found from Equation (7.32) which can be rearranged to give a quadratic equation in E_1

$$2m_\mathrm{p}(1-\cos\theta)E_1^2 - Q^1(1-\cos\theta)E_1 - m_\mathrm{p}Q^2 = 0 .$$

Hence for each of the values of θ, shown in the plot, the corresponding value of E_1 for $Q^2 = 500\,\mathrm{MeV}^2$ can be obtained, enabling the cross sections to be read off from the lines, these can then be compared to the expected Mott cross section for a point-like charge. A plot of the ratio of the measured (interpolated) cross section to

$d\sigma/d\Omega_0$ plotted against $\tan^2(\theta/2)$ should be approximately linear with an intercept of c and gradient m, where The form factors can be obtained from

$$m = 2\tau\left[G_M(Q^2)\right]^2 \quad \text{and} \quad c = \frac{\left[G_E(Q^2)\right]^2 + \tau\left[G_M(Q^2)\right]^2}{(1+\tau)}\,.$$

where $\tau = Q^2/4m_p^2 = 0.142$. The analysis of the data should give

$$G_M(Q^2 = 0.5\,\mathrm{GeV}^2) \simeq 0.99 \quad \text{and} \quad G_E(Q^2 = 0.5\,\mathrm{GeV}^2) \simeq 0.41\,,$$

roughly in the expected ratio of 2.79.

7.8 Answer:

$$1/a = \lambda = \sqrt{0.71\,\mathrm{GeV}^2} = 0.84\,\mathrm{GeV}\,.$$

8 Deep Inelastic Scattering

8.1 Answer:

$$\tau = 1/\Gamma \approx 1\,\text{GeV}^{-1} \equiv 6.6 \times 10^{-25}\,\text{s}\,.$$

8.2 Hint: In part b) there is only one independent variable in elastic scattering, the differential cross sections in terms of $\mathrm{d}Q^2$ and $\mathrm{d}\Omega$ are related by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \left|\frac{\mathrm{d}\Omega}{\mathrm{d}Q^2}\right| \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi \left|\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}Q^2}\right| \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,.$$

8.3 Hint: In part a) first change variables from $d\Omega = 2\pi d(\cos\theta)$ using

$$Q^2 = -q^2 = 2E_1E_3(1-\cos\theta)\,,$$

and then relate

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}\Omega} \quad \text{to} \quad \frac{\mathrm{d}^2\sigma}{\mathrm{d}E_3\,\mathrm{d}Q^2}\,,$$

remembering that E_1 is the fixed initial-state electron energy. Finally change variables from ν to x using

$$x = \frac{Q^2}{2m_\mathrm{p}\nu}\,.$$

Parts b) and c) should be relatively straightforward but part d) requires some thought. Given that we wish to measure the structure functions at $x = 0.2$ and $Q^2 = 2\,\text{GeV}^2$, the electron energies E_1 and E_3 are constrained via

$$E_1 - E_3 = \frac{Q^2}{2Mx} \quad \text{and} \quad E_1E_3 = \frac{Q^2}{4\sin^2\theta/2}\,.$$

Here it helps to think in terms of graphical solutions of on a plot of E_3 versus E_1. The experimental limitations, $E_1 < 20\,\text{GeV}$ and $E_3 > 2\,\text{GeV}$, then lead to constraints on the scattering angle θ.

Answer:

$$\theta_\text{max} = 21.3^\circ\,.$$

The experimental strategy is to choose several values of θ between approximately 5° and 20°, and for each angle, measure the *reduced* cross section,

$$\frac{d^2\sigma}{dE_3\,d\Omega} \times \frac{4E_1^2 \sin^4\theta/2}{\alpha^2\cos^2\theta/2} = \left[\frac{F_2}{\nu} + \frac{2F_1}{m_p}\tan^2\frac{\theta}{2}\right],$$

and plot this versus $\tan^2\theta/2$. This should give a straight line (since ν is fixed here) with slope $2F_1/m_p$ and intercept F_2/ν. Each θ value requires a different beam energy given by solving

$$E_1(E_1 - 5.33) = \frac{Q^2}{4\sin^2\theta/2}$$

8.4 **Answer:** If quarks were spin-0 particles, there would be no magnetic contribution to this QED scattering process. Consequently $F_1^{ep}(x)$, which is associated with the $\sin^2\theta/2$ angular dependence, would be zero.

8.5 **Answer:** 2 .

8.6 **Answer:** You should find

$$f_d/f_u \simeq 0.52\,,$$

which is consistent with the result quoted in Chapter 8.

8.7 **Answer:** The measured value can be interpreted as

$$\int_0^1 (\overline{u}(x) - \overline{d}(x))\,dx = \frac{3}{2}\,[0.24 - 0.33 \pm 0.03] = -0.14 \pm 0.05\,,$$

demonstrating that there is a deficit of $\overline{u}$ quarks relative to $\overline{d}$ quarks in the proton, as can be seen in the global fit to a wide range of data shown in Figure 8.17.

8.8 **Answer:** For the event shown in the text $\theta \approx 150°$, and

$$Q^2 \simeq 3 \times 10^4\,\text{GeV}^2 \quad \text{and} \quad x \simeq 0.7\,.$$

9 Symmetries and the Quark Model

9.1 Hint: For compactness, writing $x = i\boldsymbol{\alpha} \cdot \hat{\mathbf{G}}$, the required expression can be written

$$\left(1 + \frac{x}{n}\right)^n = 1 + n\frac{x}{n} + \frac{1}{2!}n(n-1)\left(\frac{x}{n}\right)^2 + \frac{1}{3!}n(n-1)(n-2)\left(\frac{x}{n}\right)^3 + ...$$

In the limit $n \to \infty$ terms such as $n(n-1)(n-2)/n^3 \to 1$.

9.2 Hint: For a infinitesimal rotation of the x and y axes

$$x \to x' = x\cos\epsilon + y\sin\epsilon \simeq x - \epsilon y$$
$$y \to y' = y\cos\epsilon - x\sin\epsilon \simeq y + \epsilon x\,.$$

Under this coordinate transformation, wavefunctions transform as

$$\begin{aligned}\psi(x,y,z) \to \psi'(x,y,z) &= \psi(x+\epsilon y, y - \epsilon x, z)\\ &= \psi(x,y,z) + y\epsilon\frac{\partial\psi}{\partial x} - x\epsilon\frac{\partial\psi}{\partial y}\,.\end{aligned}$$

9.3 Hint: We have asserted that SU(2) flavour symmetry is an exact symmetry of the strong interaction. One consequence is that isospin and the third component of isospin is conserved in strong interactions. Furthermore, from the point of view of the strong interaction the Δ^-, Δ^0, Δ^- and Δ^{++} are indistinguishable. The amplitudes for the above decays can be written as

$$\mathcal{M}(\Delta \to \pi \mathrm{N}) \sim \langle \pi\mathrm{N}|\hat{H}_{\mathrm{strong}}|\Delta\rangle\,,$$

which in the case of an exact SU(2) light quark flavour symmetry can be written as

$$\mathcal{M}(\Delta \to \pi \mathrm{N}) \sim A\langle \phi(\pi\mathrm{N})|\phi(\Delta)\rangle\,,$$

where A is a constant and ϕ represents the isospin wavefunctions. Here $\langle \phi(\pi\mathrm{N})|\phi(\Delta)\rangle$ expresses conservation of isospin in the interaction. The question therefore boils down to determining the isospin values for the states involved. For example, the decay $\Delta^- \to \pi^-\mathrm{n}$ corresponds to

$$\phi\left(\tfrac{3}{2}, -\tfrac{3}{2}\right) \to \phi\,(1,-1)\phi\left(\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

The decay rate will depend on the isospin of the combined $\pi^-\mathrm{n}$ system. Since I_3

is an additive quantum number the third component of the combined π^-n system is $-3/2$ and this implies that the total isospin must be *at least* 3/2. But since the total isospin lies between $|1 - 1/2| < I < |1 + 1/2|$, the isospin of the π^-n system is uniquely identified as

$$\phi(\pi^-\mathrm{n}) = \phi\,(1,-1)\phi\left(\tfrac{1}{2},-\tfrac{1}{2}\right) = \phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right).$$

Consequently the amplitude for the decay is given by

$$\mathcal{M}(\Delta^- \to \pi^-\mathrm{n}) \sim A\langle\phi(\pi^-\mathrm{n})\,|\,\phi(\Delta^-)\rangle = A\left\langle\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right)|\phi\left(\tfrac{3}{2},-\tfrac{3}{2}\right)\right\rangle = A\,.$$

The isospin assignments for the other decays can be obtained using the isospin ladder operator $\hat{T}_+$.

9.4 **Answer:** Since the colour quantum numbers of the quarks has nothing to do with spin, the colour singlet states are still

$$\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}) \qquad \text{and} \qquad \frac{1}{\sqrt{6}}(rgb - grb + gbr - bgr + brg - rbg)\,.$$

Hence, due to colour confinement, we still expect to see *mesons* containing a quark and an antiquark and *baryons* containing three quarks. For the mesons we would expect to see nonets (with the total angular momentum equal to L) with

$$J^P = 0^+, 1^-, 2^+, 3^-, \ldots \quad \text{nonets}\,.$$

For baryons made from spin-0 quarks, the wavefunction would become

$$\psi = \phi_{\text{flavour}}\,\xi_{\text{colour}}\,\eta_{\text{space}}\,.$$

and the overall wavefunction ψ would be totally *symmetric* under quark interchange since quarks are now bosons. In this model, the baryon multiplets would be

$$J^P = 0^+, 1^-, 2^+, 3^-, \ldots \quad \text{singlets}\,.$$

9.5 **Hint:** The underlying process is the QED annihilation process $\mathrm{q\bar{q}} \to \mathrm{e^+e^-}$, where the matrix element can be expressed as

$$\mathcal{M}(\mathrm{q\bar{q}} \to \mathrm{e^+e^-}) \sim \langle \mathrm{e^+e^-}|\hat{Q}_\mathrm{q}|\mathrm{q\bar{q}}\rangle = AQ_\mathrm{q}\,,$$

where A is assumed to be a constant and Q_q is the charge of the annihilating quark-pair. For the ϕ which is a pure $\mathrm{s\bar{s}}$ state, the matrix element

$$\mathcal{M}(\phi \to \mathrm{e^+e^-}) \sim \langle \mathrm{e^+e^-}|\hat{Q}_\mathrm{q}|\mathrm{s\bar{s}}\rangle = AQ_\mathrm{s} = -\tfrac{1}{3}A\,.$$

For the ρ^0 with wavefunction $\left|\rho^0,=\right\rangle \frac{1}{\sqrt{2}}(\mathrm{u\bar{u}} - \mathrm{d\bar{d}})$, the phases of the two components are important and the total amplitude depends on the coherent sum of the contributions from the decays of the $\mathrm{u\bar{u}}$ and $\mathrm{d\bar{d}}$.

9.6 **Answer:** The meson mass formulae works well for all of the mesons in the question, with the exception of the η', where the prediction of approximately 350 MeV is very different from the measured mass of 958 MeV. However it should be noted that the η' is a flavour singlet state and in principle it could mix with flavourless purely gluonic bound states and given the special nature of the η', it is not surprising that the simple mass formula does not work.

9.7 Answer: Using

$$m_{\mathrm{d}} = m_{\mathrm{u}} = 0.365\,\mathrm{GeV}\,, \quad m_{\mathrm{s}} = 0.540\,\mathrm{GeV} \quad \text{and} \quad A' = 0.026\,\mathrm{GeV}^3\,,$$

the predicted masses are: $m_\Delta = 1.241\,\mathrm{GeV}$, $m_{\Sigma^*} = 1.385\,\mathrm{GeV}$, $m_{\Xi^*} = 1.533\,\mathrm{GeV}$ and $m_\Omega = 1.687\,\mathrm{GeV}$, which are in good agreement with the measured values.

9.8 Answer: This is a difficult question that requires some insight.

a) If the SU(3) flavour symmetry were exact, the Λ(uds) and Σ^0(uds) baryons would have the same mass – they don't. The situation is similar to the that of the neutral mesons, where the quark flavour wavefunctions for the π^0 and η can be obtained from the operation of the ladder operators on the six states around the "edges" of the octet. The physical states are linear combinations of these states. How treat this ambiguity is not *a priori* obvious. Following the discussion of the light meson states, one expects that the u and d quarks in the uds baryon wavefunction obey an exact SU(2) flavour symmetry. Making this assumption the Σ^0(dds) wavefunction can be obtained directly from that of the Σ^-(dds), which has the same form as that of the proton, giving

$$|\Sigma^0\uparrow\rangle \propto 2\mathrm{d}\!\uparrow\mathrm{u}\!\uparrow\mathrm{s}\!\downarrow - \mathrm{d}\!\uparrow\mathrm{u}\!\downarrow\mathrm{s}\!\uparrow - \mathrm{u}\!\downarrow\mathrm{d}\!\uparrow\mathrm{s}\!\uparrow + \text{ cyclic combinatorics}\,.$$

Note that in this wavefunction the d quarks appear in symmetric spin states - this fact can be used to construct the orthogonal wavefunction for the Λ:

$$|\Lambda\uparrow\rangle \propto \mathrm{d}\!\uparrow\mathrm{u}\!\downarrow\mathrm{s}\!\uparrow - \mathrm{u}\!\downarrow\mathrm{d}\!\uparrow\mathrm{s}\!\uparrow + \text{ cyclic combinatorics}$$

b) Following the above arguments the total spin of the ud system assume that the ud quarks in the Λ and Σ^0 are either in a spin-0 or spin-1 state, $s_{\mathrm{ud}} = 0$ or $s_{\mathrm{ud}} = 1$. This allows the scalar products to be determined from the total spin of the three quark system:

$$\mathbf{S} = \mathbf{S}_{\mathrm{u}} + \mathbf{S}_{\mathrm{d}} + \mathbf{S}_{\mathrm{s}}\,.$$

The resulting masses are predicted to be

$$\begin{aligned} s_{\mathrm{ud}} = 0 \quad &: \quad m(\Lambda) = 1.124\,\mathrm{GeV} \\ s_{\mathrm{ud}} = 1 \quad &: \quad m(\Sigma^0) = 1.187\,\mathrm{GeV}\,, \end{aligned}$$

In reasonable agreement with the observed values of $m(\Lambda) = 1.116\,\mathrm{GeV}$ and $m(\Sigma^0) = 1.193\,\mathrm{GeV}$.

9.9 **Answer:** Using the given magnetic moments:

$$\mu_{\rm u} = (+1.68 \pm 0.01)\mu_N = +\frac{2m_{\rm p}}{3m_{\rm u}}\mu_N \quad \Rightarrow \quad m_{\rm u} = 0.39 m_{\rm p} \simeq 370\,\text{MeV}\,,$$
$$\mu_{\rm d} = (-1.04 \pm 0.02)\mu_N = -\frac{m_{\rm p}}{3m_{\rm d}}\mu_N \quad \Rightarrow \quad m_{\rm d} = 0.32 m_{\rm p} \simeq 300\,\text{MeV}\,,$$
$$\mu_{\rm s} = (-0.673 \pm 0.02)\mu_N = -\frac{m_{\rm p}}{3m_{\rm s}}\mu_N \quad \Rightarrow \quad m_{\rm s} = 0.50 m_{\rm p} \simeq 465\,\text{MeV}\,.$$

9.10 **Hint:** If the colour did not exist, baryon wavefunctions would be constructed from

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\;\eta_{\text{space}}\,.$$

For the $L = 0$ baryons, the spatial wavefunction is symmetric and the requirement that the overall wavefunction is anti-symmetric implies that the combination of $\phi_{\text{flavour}} \times \chi_{\text{spin}}$ must be anti-symmetric under the interchange of any two quarks. The linear combination

$$\psi = \alpha\phi_S\chi_A + \beta\phi_A\chi_S$$

is clearly anti-symmetric under the interchange of quarks $1 \leftrightarrow 2$ and for the right choice of α and β is anti-symmetric under the interchange of any two quarks. By finding α and β the "nucleon" wavefunctions can be obtained.

Answer: Taking $m_{\rm u} \sim m_{\rm d}$,

$$\frac{\mu_{\rm n}}{\mu_{\rm p}} = \frac{\mu_{\rm u}}{\mu_{\rm d}} = -2\,.$$

This colourless model, therefore, does not predict the observed ratio of magnetic moments of the proton and neutron.

10 Quantum Chromodynamics (QCD)

10.1 Answer: In the absence of colour, the overall wavefunction has the following degrees of freedom:

$$\psi = \phi_{\text{flavour}} \chi_{\text{spin}} \eta_{\text{space}} .$$

The overall wavefunction must be anti-symmetric under the interchange of any two quarks (since they are fermions). For the a state with zero orbital angular momentum ($\ell = 0$), the spatial wavefunction is symmetric. The flavour wavefunction sss is clearly symmetric under the interchange of any two quarks. Therefore, the required overall anti-symmetric wavefunction would imply a totally anti-symmetric spin wavefunction, however, there is no totally anti-symmetic spin wavefunction for the combination of three spin-half particles ($2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$). Hence, without an additional degree of freedom, in this case colour, the Ω^- would not exist.

10.2 Answer: $q_\infty \approx 200\,\text{MeV}$.

10.3 Answer:

$$\begin{aligned}\langle |C|^2 \rangle &= \frac{1}{4} \sum_{i,j,k,l=1}^{2} |C(ij \to kl)|^2 \\ &= \frac{3}{16} .\end{aligned}$$

10.4 Answer: The NRQCD potential between two quarks can be expressed as

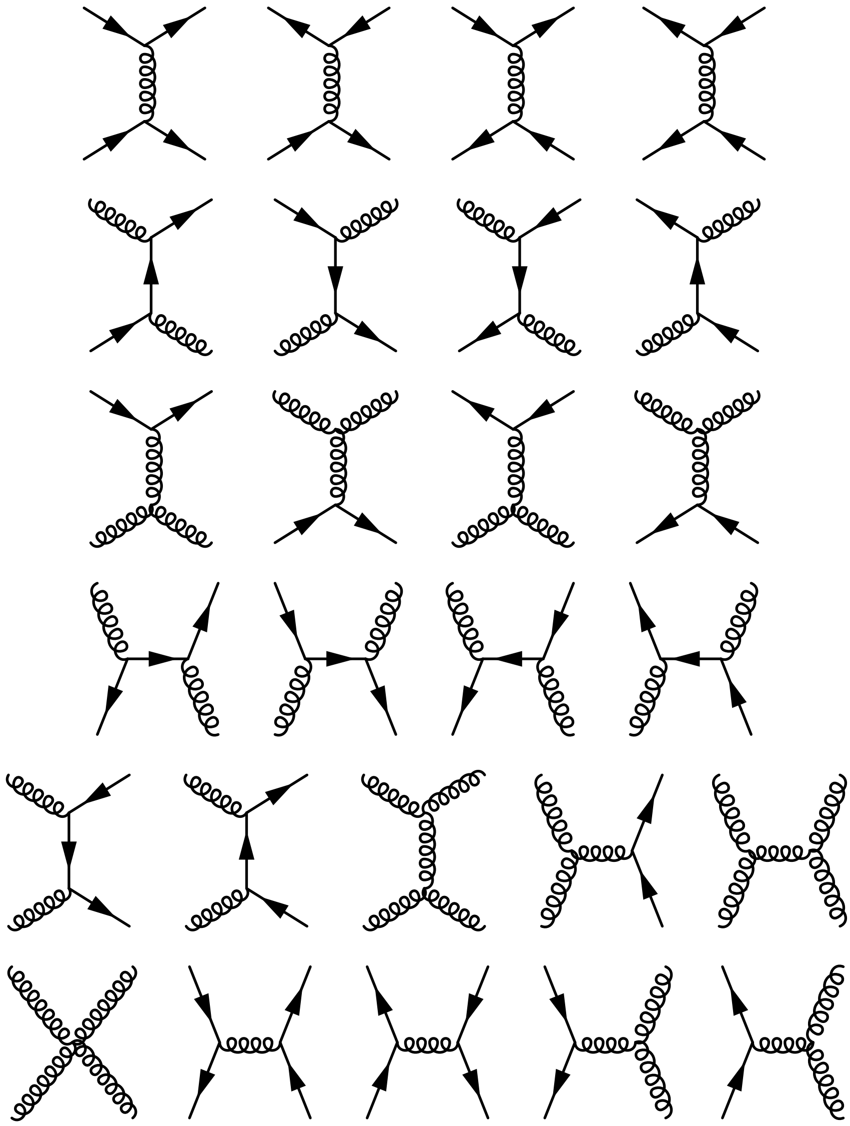
$$V_{\text{qq}}(\mathbf{r}) = +C \frac{\alpha_S}{\text{r}} ,$$

where C is the appropriate colour factor, consideration of the colour exchange processes involved then gives

$$\langle V_{\text{qq}}^{12} \rangle = -\frac{2\alpha_S}{3\text{r}} .$$

Hence, in the non-relativistic limit, the QCD potential between any two quarks in a baryon is attractive.

10.5 Answer: There are diagrams involving: i) the scattering of quarks and anti-quarks, ii) the scattering of a quark/antiquark and a gluon and iii) the scattering of gluons, where the anti-quarks/quarks can either be from the valance or sea content of the proton and antiproton.



10.6 Hint: Remember to assume that the jets are effectively massless, $E^2 = p_{\mathrm{T}}^2 + p_z^2$ and neglect the masses of the quarks $p_1^2 = 0$ etc. The rest is just algebra.

10.7 **Hint:** First obtain an expression for the Jacobian (it terms of p_T rather than p_T^2):

$$J = \frac{\partial(x_1, x_2, Q^2)}{\partial(y_3, y_4, p_T)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_3} & \frac{\partial x_1}{\partial y_4} & \frac{\partial x_1}{\partial p_T} \\ \frac{\partial x_2}{\partial y_4} & \frac{\partial x_2}{\partial y_4} & \frac{\partial x_2}{\partial p_T} \\ \frac{\partial Q^2}{\partial y_3} & \frac{\partial Q^2}{\partial y_4} & \frac{\partial Q^2}{\partial p_T} \end{vmatrix}$$

Multiplying out the terms in the determinant leads to

$$J = -2p_T x_1 x_2 .$$

The next step is to transform from $Q^2 \to q^2 = -Q^2$ and from $p_T \to p_T^2$.

10.8 **Answer:** Assuming that $u_V(x) = 2d_V(x)$ then

$$d^2\sigma_{DY}^{p\overline{p}} = \frac{4\pi\alpha^2}{81 s x_1 x_2}\{17 d_V(x_1) d_V(x_2) + \\ 9 d_V(x_1) S(x_2) + 9S(x_1) d_V(x_2) + 10 S(x_1) S(x_2)\} dx_1 dx_2 .$$

Answer: b) For pp collisions, the Drell-Yan cross section is

$$d^2\sigma_{DY}^{pp} = \frac{4\pi\alpha^2}{81 s x_1 x_2}\{9 d_V(x_1) S(x_2) + 9S(x_1) d_V(x_2) + 10 S(x_1) S(x_2)\} dx_1 dx_2$$

Hint: c) Remember that $\hat{s} = x_1 x_2 s$ and lines of constant $\hat{s}$ define hyperbolae in the $\{x_1, x_2\}$ plane.

10.9 **Hint:** The PDFs for the $\pi^+(u\overline{d})$ can be written in terms of valance and sea quark distributions:

$$\begin{aligned}
u^{\pi^+}(x) &= u_V^{\pi^+}(x) + S^{\pi^+}(x) \equiv u_V^{\pi}(x) + S^{\pi}(x) \\
\overline{d}^{\pi^+}(x) &= \overline{d}_V^{\pi^+}(x) + S^{\pi^+}(x) \equiv \overline{d}_V^{\pi}(x) + S^{\pi}(x) \\
d^{\pi^+}(x) &= S^{\pi^+}(x) \equiv S^{\pi}(x) \\
\overline{u}^{\pi^+}(x) &= S^{\pi^+}(x) \equiv S^{\pi}(x) ,
\end{aligned}$$

where the symbols with a superscript π implicitly refer to the PDFs for the π^+. Then assuming isospin symmetry, e.g. the down-quark PDF in the $\pi^-(d\overline{u})$ is identical to the up-quark PDF in the π^+.

11 The Weak Interaction

11.1 Hint: Consider conservation of angular momentum, parity and the symmetry of the $\pi^0\pi^0$ wavefunction (indetical bosons).

11.2 Hint: Conservation of angular momentum implies that

$$\begin{aligned} \mathbf{J}_{\mathrm{D}} + \mathbf{J}_{\pi} + \boldsymbol{\ell} &= \mathbf{L} + \mathbf{S}_{\mathrm{nn}} \\ \mathbf{1} &= \mathbf{L} + \mathbf{S}_{\mathrm{nn}} \end{aligned} \tag{11.1}$$

where $\mathbf{S}_{\mathrm{nn}} = \mathbf{0}$ or $\mathbf{1}$, is the total spin of the neutron-neutron system. Since the final state consists of identical fermions the overall wavefunction of the neutron-neutron system must be anti-symmetric

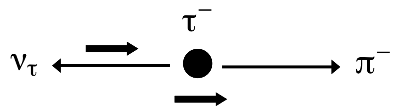
$$\psi_{\mathrm{space}} \times \psi_{\mathrm{spin}} : \text{anti-symmetric}\,.$$

11.3 Answer:

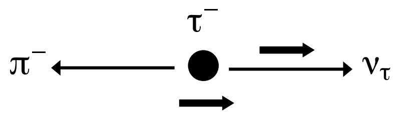
a) $P = \mathbf{F} \cdot \mathbf{v}$: **scalar** - scalar product of two vectors ;
b) $\mathbf{F}$: **vector**;
c) $\mathbf{G} = \mathbf{r} \times \mathbf{F}$: **axial-vector** - cross product of two vectors ;
d) $\mathbf{\Omega} = \mathbf{\nabla} \times \mathbf{v}$: **axial-vector** - cross product of two vectors (even if one is a vector operator) ;
e) magnetic flux, $\phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S}$: **pseudo-scalar** - scalar product an axial vector ($\mathbf{B}$) with a vector ;
f) divergence of the electric field strength, $\mathbf{\nabla} \cdot \mathbf{E}$: **scalar** - scalar product of two vectors .

11.4 Answer: For for either a pure scalar interaction or pure pseudoscalar interaction, the chiral combinations that contribute to the annihilation process are $LL \to LL$, $LL \to RR$, $RR \to LL$ and $RR \to RR$. For an $S - P$ interaction, the only non-zero contribution to the amplitude comes from $RR \to LL$.

11.5 Answer: i) Here the $V - A$ for of the interaction projects out LH particle states and RH antiparticle states. Hence in the decay $\tau^- \to \pi^-\nu_\tau$, the neutrino is produced in a LH chiral state. Since the neutrino is almost massless, it is highly



relativistic and the chiral and helicity states are the same. Hence the neutrino must be produced in a LH helicity state and the allowed spin combination is:
ii) Here the $V + A$ for of the interaction projects out RH particle states and LH antiparticle states. Hence in the decay $\tau^- \rightarrow \pi^-\nu_\tau$, the neutrino would now be produced in a RH chiral state:



11.6 **Answer:** The expression for the decay rate in this case is

$$\Gamma = \frac{4\pi}{32\pi^2 m_\pi^2} \mathrm{p} \langle |\mathcal{M}_{fi}|^2 \rangle = \frac{G_X^2 f_\pi^2}{8\pi} \mathrm{p}^2 .$$

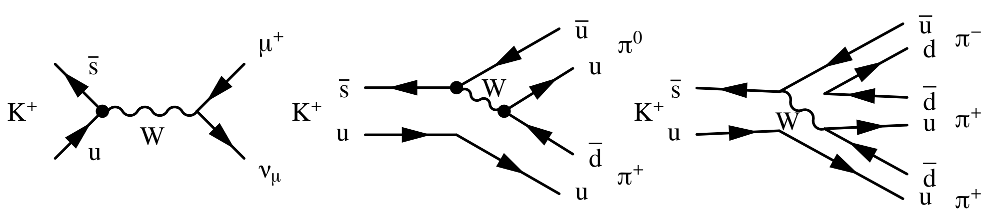
Therefore, to lowest order, the predicted ratio of the $\pi^- \rightarrow e^-\overline{\nu}_e$ to $\pi^- \rightarrow \mu^-\overline{\nu}_\mu$ decay rates is

$$\frac{\Gamma(\pi^- \rightarrow e^-\overline{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^-\overline{\nu}_\mu)} = \frac{p_e^2}{p_\mu^2} = \left[\frac{(m_\pi^2 - m_e^2)}{(m_\pi^2 - m_\mu^2)}\right]^2 = 5.49 .$$

11.7 **Answer:** From the result derived in the text for pion decay, the predicted ratio of the two leptonic decays of the charged kaon is

$$\frac{\Gamma(K^- \rightarrow e^-\overline{\nu}_e)}{\Gamma(K^- \rightarrow \mu^-\overline{\nu}_\mu)} = \left[\frac{m_e(m_K^2 - m_e^2)}{m_\mu(m_K^2 - m_\mu^2)}\right]^2 = 2.55 \times 10^{-5} .$$

11.8 **Answer:** a) The lowest-order quark-level Feynman diagrams are:



b) The numerical answer you obtain will depend on the assumptions made. Making the assumption that $f_K \approx f_\pi$ (both are pseudo scalar mesons) and putting in the numbers

$$\begin{aligned}\tau_{K^+} &\approx 0.05\tau_{\pi^+}\\ &= 1.3\times 10^{-9}\,\mathrm{s}\end{aligned}$$

This is a factor 10 shorter than the measured value of $\tau_{K^+} = 1.2\times10^{-8}$ s because the K^+ decay rate is suppressed by a factor of $\tan^2\theta_C = 0.053$ (see Chapter 14) relative to the π^+ decay rate; in charged kaon decay the weak decay vertex is $\bar{s} \rightarrow \bar{u}$, whereas for pion decay it is $\bar{d} \rightarrow \bar{u}$.

11.9
Answer: $f_\pi \approx 0.135\,\mathrm{GeV}$, and thus $f_\pi \sim m_\pi$.

11.10 **Answer:** e) Taking $f_\pi = m_\pi$,

$$\Gamma(\tau^- \rightarrow \pi^-\nu_\tau) = 2.93\times 10^{-13}\,\mathrm{GeV}\,.$$

f) $BR(\tau^- \rightarrow \pi^-\nu_\tau) = \frac{\Gamma(\tau^-\rightarrow\pi^-\nu_\tau)}{\Gamma_\tau} = 11.9\,\%$, in fair agreement with the measured value of $10.83 \pm 0.06\,\%$.

12 The Weak Interactions of Leptons

12.1 Hint: Don't forget colour.

12.2 Hint: If this were the only Feynman diagram contributing to the process $\nu_e e^- \to \nu_e e^-$, following the derivation of Chapter 12.2.1 would give

$$\sigma_{CC}(\nu_e e^- \to \nu_e e^-) = \frac{G_F^2 s}{\pi}.$$

It should be noted that this neglects the NC Z-exchange diagram and that $\mathcal{M} \to \mathcal{M}_{CC} + \mathcal{M}_{NC}$, which has the effect to reduce the $\nu_e e^- \to \nu_e e^-$ cross section through negative interference.

12.3 Answer: The probability of an interaction is

$$P = \sigma_{CC}(\nu_e e^- \to \nu_e e^-)[\mathrm{cm}^2] \times n_e[\mathrm{cm}^{-2}] < 10^{-15}.$$

12.4 Hint: BEquating (12.34) and (12.20) gives

$$\frac{G_F^2}{\pi} sx\left[d(x) + (1-y)^2\overline{u}(x)\right] = \frac{G_F^2}{2\pi} s\left[(1-y)F_2^{\nu p} + xy^2F_1^{\nu p} + xy\left(1-\frac{y}{2}\right)F_3^{\nu p}\right]$$

$$2xd(x) + 2x(1-y)^2\overline{u}(x) = (1-y)F_2^{\nu p} + xy^2F_1^{\nu p} + xy\left(1-\frac{y}{2}\right)F_3^{\nu p}$$

$$2x(d(x)+\overline{u}(x)) - 4xy\overline{u}(x) + 2xy^2\overline{u}(x) = F_2^{\nu p} + y(xF_3^{\nu p} - F_2^{\nu p}) + y^2(xF_1^{\nu p}).$$

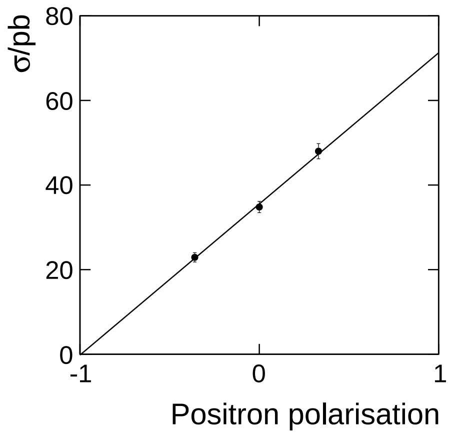
12.5 Answer: You should find that the parton model with $Q_u = +2/3$ and $Q_d = -1/3$ predicts:

$$F_2^{eN}/F_2^{\nu N} = \tfrac{1}{2}(Q_u^2 + Q_d^2) = \tfrac{5}{18} = 0.278,$$

consistent with the measured value of 0.29 ± 0.02.

12.6 Answer: $xs(x) = \frac{5}{6}F_2^{\nu N} - 3F_2^{eN}$.

12.7 Answer: The data are plotted in the figure below, along with a linear fit (χ^2-minimization). The linear fit has $\chi^2 = 0.48$ for one degree of freedom, and therefore the data are consistent with the hypothesis that he cross section depends linearly on the degree of positron polarisation. The fit results indicate that the cross section is expected to be zero for $P(e^+) = -1$ when the positrons are all left-handed. Consequently the data support the hypothesis that the weak charged current only couples to RH antiparticles and thus has the form $V - A$. Add the weak charged current been of the form $V + A$ a negative slope with intercept at $P(e^+) = +1$ would have been observed.



13 Neutrinos and Neutrino Oscillations

13.1 **Hint:** Assuming the mass eigenstates propagate with equal velocity, $\beta_1 = \beta_2 = \beta$, and $T = L/\beta$.

13.2 **Hint:** First note that

$$\Delta m^2[\mathrm{GeV}^2] = 10^{-18}\Delta m^2[\mathrm{eV}^2] \quad \text{and} \quad L[\mathrm{m}] = 10^3 L[\mathrm{km}]\,.$$

To convert from natural units $L[\mathrm{GeV}^{-1}]$ to SI units $L[\mathrm{m}]$ the expression in brackets needs to be multiplied by the factor $\mathrm{GeV}/(\hbar c)$.

13.3 **Hint:** The expression for $P(\nu_e \to \nu_e)$ can be obtained from equation (13.24) by making the replacing in the sub-scripts $\mu \to e$. You will also need to use the complex number identity

$$|z_1 + z_2 + z_3|^2 \equiv |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,\Re\mathrm{e}\{z_1 z_2^* + z_1 z_3^* + z_2 z_3^*\}\,.$$

13.4 **Hint:** In part e) assume $\theta_{12} \approx 35°$, $\theta_{23} \approx 45°$ and $\theta_{13} \approx 10°$ and $\Delta_{13} \approx \Delta_{23}$. For a terrestrial experiment the $\sin\Delta_{12}$ term results in oscillations over very large distances. Consider a beam neutrino experiment, similar to MINOS with a peak beam energy of 3 GeV. The first oscillation maximum from the Δ_{23} term will occur at

$$\Delta_{23} = \frac{\pi}{2} \quad \Rightarrow \quad L = 1500\,\mathrm{km}\,.$$

But at this distance

$$\frac{\Delta_{12}}{\Delta_{23}} = \frac{\Delta m^2_{12}}{\Delta m^2_{32}} \approx 0.033$$

$$\Rightarrow \quad \Delta_{12} = 0.033\frac{\pi}{2} = 0.05\,.$$

Answer: For these assumptions, $P(\nu_e \to \nu_\mu) - P(\overline{\nu}_e \to \overline{\nu}_\mu) \approx 0.03$.

Note: It should be noted that the above treatment uses the vacuum oscillation formula and neglects ”matter effects”.

13.5 **Hint:** b) Under the above redefinition of the phases of the fermion fields:

$$U \to \begin{pmatrix} \cos\theta e^{i(\delta_1+\theta'_e-\theta'_1)} & \sin\theta e^{i\left(\frac{\delta_1+\delta_2}{2}-\delta+\theta'_e-\theta'_2\right)} \\ -\sin\theta e^{i\left(\frac{\delta_1+\delta_2}{2}+\delta+\theta'_\mu-\theta'_1\right)} & \cos\theta e^{i(\delta_2+\theta'_\mu-\theta'_2)} \end{pmatrix}.$$

All complex phases can be eliminated if the following four conditions are satisfied

$$\theta'_1-\theta'_e=\delta_1 \quad\text{and}\quad \theta'_2-\theta'_e=\frac{\delta_1+\delta_2}{2}-\delta\,, \tag{13.1}$$

$$\theta'_1-\theta'_\mu=\frac{\delta_1+\delta_2}{2}+\delta \quad\text{and}\quad \theta'_2-\theta'_\mu=\delta_2\,. \tag{13.2}$$

Now choose $\theta'_e = 0$, which is equivalent to writing all the phases relative to the phase of the electron.

13.6 **Hint:** In both cases the double angle formula $\sin 2\theta = 2\sin\theta\cos\theta$ is used. For the second identify, it is easiest to start from

$$\sin^2 2\theta_{23}\cos^4\theta_{13}+\sin^2 2\theta_{13}\sin^2\theta_{23} = 4\sin^2\theta_{23}\cos^2\theta_{23}\cos^4\theta_{13} + 4\sin^2\theta_{13}\cos^2\theta_{13}\sin^2\theta_{23}\,.$$

13.7 **Hint:** In Figure 13.20 the distance L_0 in $L_0/E_{\bar{\nu}_e}$, is the average distance to many reactors weighted by expected flux. The variety of actual distances, smears out the calculated form of the oscillation probability, with the smearing becoming more notable at small values of E, or equivalently large values of $L_0/E_{\bar{\nu}_e}$. The first oscillation minimum occurs at $L/E < 30\,\text{km}$ but is not clearly resolved. The second oscillation minimum is clearly defined at

$$L_0/E_{\bar{\nu}_e} \approx 50\,\text{km}\,\text{MeV}^{-1} = 50000\,\text{km}\,\text{GeV}^{-1}\,.$$

Determining the angle θ_{12} requires care. The amplitude of the oscillations (estimated from the first oscillation maximum and the well-resolved second oscillation minimum) is about 0.4. Without experimental effects this would be equal to $\cos^4\theta_{13}\sin^2 2\theta_{12}$. However, the sharpness of the oscillation structure is smeared out due to the reactors being at a variety of distances from the experiment. The effect of this smearing can be estimated. According to the survival probability formula, the peak at $L/E = \pi$ should correspond to a survival probability of $\cos^4\theta_{13} \simeq 0.95$. The measured survival probability is about 0.75, due to the smearing out of the peak due to the ranges of L to the different reactors.

13.8 **Hint:** The interpretation of the MINOS data is relatively straightforward as the distance from the source of the beam to the far detector is fixed, $L = 735\,\text{km}$ and the energy of the neutrino is relatively well measured.

13.9 **Note:** Part d) of *question 13.9* should be ignored - it is poorly worded. The intention was to consider the case where the decay products of the pion were close to being perpendicular to the direction of the boost. Close to $\theta^* \sim \pi/2$ the transverse momentum is approximately p* and the longitudinal momentum is primarily due to the Lorentz boost.

Answer: a) $p^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$.

Hint: c) Flipping the sign of β, gives the Lorentz transformation from the laboratory frame to the pion rest frame. Consideration of EE^* gives the desired relation.

Hint: e) Here were are working in the small angle limit where

$$\cos\theta) \approx \left(1 - \frac{\theta^2}{2}\right) .$$

In addition, assume that $E_\nu \gg m_\pi$ such that $\gamma \gg 1$,

$$\beta = \left(1 - \frac{1}{\gamma^2}\right)^{\frac{1}{2}} \approx 1 - \frac{1}{2\gamma^2} .$$

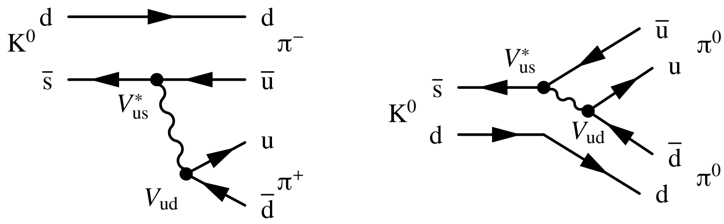
Answer: f) The neutrino energies for a set of pion beam energies are tabulated below for $\theta = 0°$ and $\theta = 2.5°$: The effect of going away from the beam axis is

E_π	E_ν at $\theta = 0°$	E_ν at $\theta = 2.5°$
1.0 GeV	0.43 GeV	0.39 GeV
1.5 GeV	0.65 GeV	0.53 GeV
2.0 GeV	0.86 GeV	0.62 GeV
2.5 GeV	1.08 GeV	0.67 GeV
3.0 GeV	1.29 GeV	0.68 GeV
3.5 GeV	1.50 GeV	0.68 GeV
4.0 GeV	1.72 GeV	0.67 GeV
4.5 GeV	1.93 GeV	0.65 GeV
5.0 GeV	2.15 GeV	0.62 GeV

to produce a "narrow-band" beam, where most the neutrino energy depends only very weakly on the energy of the decaying pion producing the neutrino.

14 CP Violation and Weak Hadronic Interactions

14.1 Answer: The diagrams have the form. The two lowest-order Feynman diagrams for the $K^0 \to \pi^+\pi^-$ and $K^0 \to \pi^0\pi^0$ decays are:



In the two flavour approximation, the matrix elements for all diagrams in this questions are proportional to

$$\mathcal{M} \propto |V_{us}||V_{ud}| \approx \sin\theta_C \cos\theta_C \,.$$

14.2 Answer: From consideration of the CKM matrix alone,

$$\begin{aligned} Br(\mathrm{B}^0 \to \mathrm{D}^- \pi^+) &: Br(\mathrm{B}^0 \to \pi^+ \pi^-) \,:\, Br(\mathrm{B}^0 \to \mathrm{J}/\psi\, \mathrm{K}^0) \\ &= |V_{cb}|^2|V_{ud}|^2 \,:\, |V_{ub}|^2|V_{ud}|^2 \,:\, |V_{cb}|^2|V_{cs}|^2 \\ &= 1.6\times10^{-3} \,:\, 1.5\times10^{-5} \,:\, 1.6\times10^{-3} \,. \end{aligned}$$

14.3 Answer: On the basis of the CKM matrix alone, one would expect

$$\frac{\Gamma(\mathrm{D}^0 \to \mathrm{K}^+\pi^-)}{\Gamma(\mathrm{D}^0 \to \mathrm{K}^-\pi^+)} \approx \frac{|V_{cd}|^2|V_{us}|^2}{|V_{ud}|^2|V_{cs}|^2} = \frac{0.225^2 \cdot 0.225^2}{0.974^2 \cdot 0.973^2} = 3\times10^{-3} \,,$$

explaining most of the difference in the observed decay rates.

14.4 Answer:

$$W = \mathrm{B}^-(\mathrm{b\bar{u}})\,, \quad X = \overline{\mathrm{D}}^0(\mathrm{c\bar{u}})\,, \quad Y = \mathrm{K}^-(\mathrm{s\bar{u}}) \quad \text{and} \quad Z = \pi^0(\mathrm{u\bar{u}})\,.$$

14.5 Answer:
a)

$$N_2^{\text{free}} = 4\,, \quad N_3^{\text{free}} = 9 \quad \text{and} \quad N_4^{\text{free}} = 16\,.$$

b)

$$N_2^{\text{real}} = 1 \ : \ N_2^{\text{phase}} = 3\,, \quad N_3^{\text{real}} = 3 \ : \ N_3^{\text{phase}} = 6 \quad \text{and} \quad N_4^{\text{real}} = 6 \ : \ N_4^{\text{phase}} = 10\,.$$

c)

$$N_2^{\text{real}} = 1 \ : \ N_2^{\text{phase}} = 0\,, \quad N_3^{\text{real}} = 3 \ : \ N_3^{\text{phase}} = 1 \quad \text{and} \quad N_4^{\text{real}} = 6 \ : \ N_4^{\text{phase}} = 3\,.$$

d) CP violation arises from at least one complex phase in the mixing matrix, and therefore CP violation can arise in quark mixing for three or more generations, but not for two generations.

14.6 **Hint:** In both cases the flavour change is $\mathrm{u\bar{u}} \to \mathrm{s\bar{s}}$.

14.7 **Hint:** Remember that ε is a small parameter, which allows certain approximations to be made.

14.8 **Answer:**

$$V_{\text{ud}}V_{\text{us}}m_{\text{u}} \ :: \ V_{\text{cd}}V_{\text{cs}}m_{\text{c}} \ : \ V_{\text{td}}V_{\text{ts}}m_{\text{t}} = 0.07\,\text{GeV} \ : \ 0.33\,\text{GeV} \ : \ 0.06\,\text{GeV}\,.$$

14.9 **Hint:** This is a tricky problem. The first part is more obvious if one starts from the required solution and works backwards. You will also need to remember that

$$\Delta m - \tfrac{i}{2}\Delta\Gamma = \lambda_+ - \lambda_- = \left[(M_{12} - \tfrac{i}{2}\Gamma_{12})(M_{12}^* - \tfrac{i}{2}\Gamma_{12}^*)\right]^{\frac{1}{2}}\ .$$

The second part of the question uses the measured properties of the neutral kaon system to extract information about the effective Hamiltonian. The angle ϕ, defined by $\varepsilon = |\varepsilon|e^{i\phi}$, was measured by CPLEAR

$$\phi = \arg\varepsilon = (43.19 \pm 0.73)^\circ\,,$$

this can be used to infer that $\Im\mathfrak{m}\,\{M_{12}\} \gg \Im\mathfrak{m}\,\{\Gamma_{12}\}$.

14.10 **Hint:** The relation

$$|\varepsilon| \propto \mathcal{A}_{\text{ut}}\,\Im\mathfrak{m}\,(V_{\text{ud}}V_{\text{us}}^*V_{\text{td}}V_{\text{ts}}^*) + \mathcal{A}_{\text{ct}}\,\Im\mathfrak{m}\,(V_{\text{cd}}V_{\text{cs}}^*V_{\text{td}}V_{\text{ts}}^*) + \mathcal{A}_{\text{tt}}\,\Im\mathfrak{m}\,(V_{\text{td}}V_{\text{ts}}^*V_{\text{td}}V_{\text{ts}}^*)\,,$$

can be manipulated into the form

$$|\varepsilon| = a\eta(1-\rho+b+c)$$
$$\Rightarrow \quad \eta(1-\rho+\text{constant}) = \text{constant}\,,$$

which is the equation of a hyperbola in the (ρ,η) plane.

14.11 **Hint:** From the earlier question

$$\Delta m = m(\mathrm{B}_H) - m(\mathrm{B}_L) \approx \sum_{\mathrm{q,q'}} \frac{G_\mathrm{F}^2}{3\pi^2} f_\mathrm{B}^2 m_\mathrm{B} |V_\mathrm{qd} V_\mathrm{qb}^* V_\mathrm{q'd} V_\mathrm{q'b}^*| \, m_\mathrm{q} m_\mathrm{q'} \, .$$

14.12 **Answer:** $\beta^* = 0.063$.

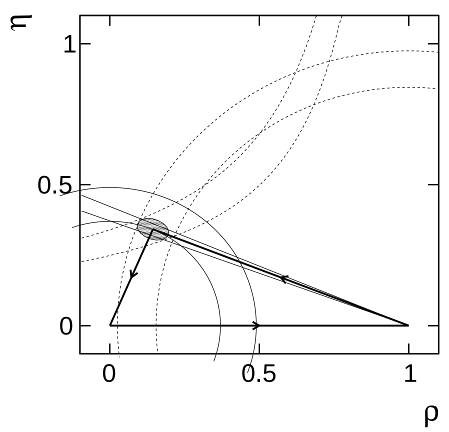
14.13 **Hint:** First convince yourself that the laboratory frame energy of the B mesons does not depend strongly on the decay angle in the centre-of-mass frame. Thus

$$d = \frac{\mathrm{p'}}{m} \tau c = 197 \, \mu\mathrm{m} \, .$$

14.14 **Answer:** The length of the shortest side of the unitarity triangle shown Figure 14.25 is

$$x = 0.43 \pm 0.06 \, ,$$

giving only a weak constraint on ρ and η.



15 Electroweak Unification

15.1 **Hint:** Just consider diagrams involving the exchange of either a γ, Z or W. For there first and last parts of the question there are two diagrams.

15.2 **Hint:** Think about the handedness of the $\nu_\mu \overline{\nu}_\mu$.

15.3 **Answer:** The individual partial decay widths are proportional to:

$$\mu : c_V^2 + c_A^2 = 0.2516\,, \quad d : c_V^2 + c_A^2 = 0.3725 \quad \text{and} \quad u : c_V^2 + c_A^2 = 0.2861\,,$$

and therefore

$$\begin{aligned} R_\mu &= \frac{\Gamma(Z \to \mu^+\mu^-)}{\Gamma(Z \to \text{hadrons})} \\ &= \frac{0.2516}{9 \cdot 0.3725 + 6 \cdot 0.2861} = 0.496 \approx \frac{1}{20}\,. \end{aligned}$$

15.4 **Answer:** The spin-averaged matrix element squared (averaging over the two spin states of the electron since the neutrino is left-handed) for the NC scattering process is

$$\begin{aligned} \langle |\mathcal{M}_{fi}|^2 \rangle &= \frac{1}{2} \frac{g_Z^4 s^2}{m_Z^4} \left[4(c_L^\nu)^2 (c_L^e)^2 + 4(c_L^\nu)^2 (c_R^e)^2 \tfrac{1}{4}(1 + \cos\theta^*)^2 \right] \\ &= \frac{1}{2} \frac{g_Z^4 s^2}{m_Z^4} \left[(c_L^e)^2 + (c_R^e)^2 \tfrac{1}{4}(1 + \cos\theta^*)^2 \right]. \end{aligned}$$

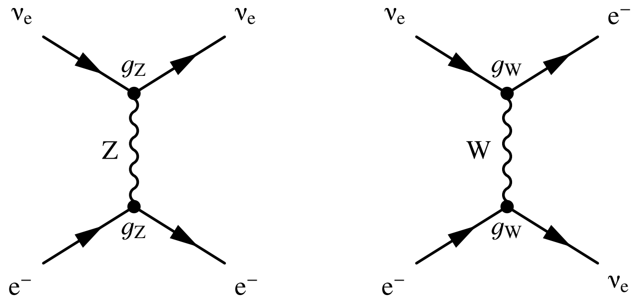
The NC cross section is

$$\sigma \approx \frac{2 m_e E_\nu G_F^2}{\pi} \times 0.09\,.$$

15.5 **Hint:** In the process $\sigma(\nu_e e^- \to \nu_e e^-)$, both charged-current and neutral-current diagrams contribute and can interfere.
Consequently the spin-averaged matrix element for this mixed NC and CC weak interaction is

$$\langle |\mathcal{M}|^2_{\text{NC+CC}} \rangle = \frac{1}{2} \left[(\mathcal{M}_{LL}^{\text{CC}} + \mathcal{M}_{LL}^{\text{NC}})^2 + (\mathcal{M}_{LR}^{\text{NC}})^2 \right].$$

You will also need to use the relation $g_Z/m_Z = g_W/m_W$.



Answer:

$$\sigma(\nu_\mu e^- \to \nu_\mu e^-) : \sigma(\nu_e e^- \to \nu_e e^-) : \sigma(\nu_\mu e^- \to \nu_e \mu^-) = c_L^2 + \tfrac{1}{3}c_R^2 \ : \ (1 + c_L)^2 + \tfrac{1}{3}c_R^2 \ : \ 1$$
$$= 0.09 \ : \ 0.55 \ : \ 1 \, .$$

16 Tests of the Standard Model

16.1 Answer:

a)

$$\Gamma_{ee} = 0.03371\Gamma_Z \quad \text{and} \quad \Gamma_{\text{hadrons}} = 0.6992\,\Gamma_Z\,.$$

b)

$$N_\nu = \frac{498}{167} = 2.98\,,$$

consistent with the claim that there are three light neutrino generations.

16.2 Hint: Start from

$$\frac{d\sigma}{d\Omega} = \kappa\left[a(1+\cos^2\theta) + 2b\cos\theta\right]\,,$$

where a and b are constants related to the couplings to the Z, and κ is a normalisation factor.

16.3 Answer: $\sin^2\theta_W = 0.2317 \pm 0.0012$.

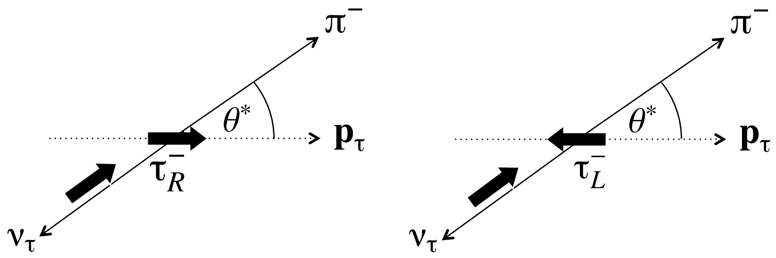
16.4 The e^+e^- Stanford Linear Collider (SLC), operated at $\sqrt{s} = m_Z$ with left- and right-handed longitudinally polarised beams. This enabled the $e^+e^- \to Z \to f\bar{f}$ cross section to be measured separately for left-handed and right-handed electrons.

Assuming that the electron beam is 100 % polarised and that the positron beam is unpolarised, show that the left-right asymmetry A_{LR} is given by

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{(c_L^e)^2 - (c_R^e)^2}{(c_L^e)^2 + (c_R^e)^2} = \mathcal{A}_e\,,$$

where σ_L and σ_R are respectively the measured cross sections at the Z resonance for LH and RH electron beams.

Hint: The matrix-elements for the different helicity combinations in the process



$e^+e^- \rightarrow Z \rightarrow \mu^+\mu^-$ are given by equations (16.9)- (16.12)

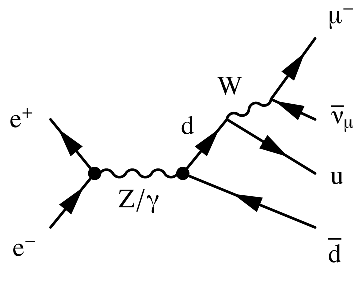
$$|\mathcal{M}_{RL\rightarrow RL}|^2 = |P_Z(s)|^2\, g_Z^4 s^2 (c_R^e)^2 (c_R^\mu)^2 (1+\cos\theta)^2\,,$$
$$|\mathcal{M}_{RL\rightarrow LR}|^2 = |P_Z(s)|^2\, g_Z^4 s^2 (c_R^e)^2 (c_L^\mu)^2 (1-\cos\theta)^2\,,$$
$$|\mathcal{M}_{LR\rightarrow RL}|^2 = |P_Z(s)|^2\, g_Z^4 s^2 (c_L^e)^2 (c_R^\mu)^2 (1-\cos\theta)^2\,,$$
$$|\mathcal{M}_{LR\rightarrow LR}|^2 = |P_Z(s)|^2\, g_Z^4 s^2 (c_L^e)^2 (c_L^\mu)^2 (1+\cos\theta)^2\,,$$

where $|P_Z(s)|^2 = 1/[(s-m_Z^2)^2 + m_Z^2\Gamma_Z^2]$ and $RL \rightarrow LR$ refers to a $e_R^- e_L^+ \rightarrow \mu_L^- \mu_R^+$.

16.5 **Hint:** In the limit $\sqrt{s} \gg m_\tau$, the matrix-elements for the different helicity combinations in the process $e^+e^- \rightarrow Z \rightarrow \tau^+\tau^-$ are given by equations (16.9)-(16.12) as in the previous question (with the replacement $\mu \rightarrow \tau$).

16.6 **Hint:** To obtain the pion energy distributions in the laboratory frame, consider the decay in the tau rest frame (as shown below) and boost to the laboratory frame.
Answer: $\mathcal{A}_\tau = -P_\tau = 0.14$ and $\sin^2\theta_W = 0.233$.

16.7 **Hint:** The first three diagrams (CC03) involve the production of two W bosons The remaining seven diagrams, all arise from pair production of quarks or leptons through Z or γ exchange with a W radiated from one of the final state particles.



16.8 **Hint:** There is a t-channel and a u-channel diagram.

16.9 **Answer:** $BR(W \rightarrow q\overline{q}') = 68.1 \pm 1.2\,\%$.
Note: In calculating the error, you will need to assume that the backgrounds are

relatively small, so that the uncertainties on the event counts are given by Poisson errors.

16.10 **Answer:** The jet pairing most consistent with being from the process $e^+e^- \to W^+W^-$ has

$$(13)(24)\ :\quad m_{13} = (84.8 \pm 9.3)\,\mathrm{GeV} \quad \text{and} \quad m_{24} = (82.6 \pm 5.9)\,\mathrm{GeV}\,.$$

16.11 **Hint:** Either show that

$$\mathrm{p}^* = \frac{1}{2m_\mathrm{t}}\sqrt{\left[(m_\mathrm{t}^2 - (m_\mathrm{W} + m_\mathrm{b})^2\right]\left[m_\mathrm{t}^2 - (m_\mathrm{W} - m_\mathrm{b})^2\right]}\,,$$

and then note that $m_\mathrm{b} \ll m_\mathrm{W}$, or from the outset neglect the b mass.

17 The Higgs Boson

17.1 **Hint:** Show that the matrix element for the t-channel process $e^+e^- \to W_L^+W_L^-$ scales as

$$\mathcal{M}^2 \propto \left(\frac{E_W}{m_W}\right)^4 .$$

17.2 **Hint:** The partial derivatives with respect to each of the four components of the spinor ψ_i are

$$\frac{\partial \mathcal{L}_D}{\partial\partial(\partial_\mu \psi_i)} = i\overline{\psi}\gamma^\mu \quad \text{and} \quad \frac{\partial \mathcal{L}_D}{\partial \psi_i} = -m\overline{\psi} ,$$

17.3 **Hint:** Show that the with the gauge transformation $F^{\mu\nu\prime} = F^{\mu\nu}$.

17.4 **Hint:**

$$\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu \\
&= -\tfrac{1}{4}(\partial^\mu A^\nu)(\partial_\mu A_\nu) - \tfrac{1}{4}(\partial^\nu A^\mu)(\partial_\nu A_\mu) + \tfrac{1}{4}(\partial^\mu A^\nu \partial_\nu A_\mu) + \tfrac{1}{4}(\partial^\nu A^\mu)(\partial_\mu A_\nu) - j^\mu A_\mu \\
&= -\tfrac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu) + \tfrac{1}{2}(\partial^\nu A^\mu)(\partial_\mu A_\nu) - j^\mu A_\mu \\
&= -\tfrac{1}{2}(\partial^\nu A^\mu)(\partial_\nu A_\mu) + \tfrac{1}{2}(\partial^\nu A^\mu)(\partial_\mu A_\nu) - j^\mu A_\mu .
\end{aligned}$$

17.5 **Answer:** Introducing odd powers of the field ϕ into the Higgs potential would break the underlying gauge invariance of the Lagrangian, which is the whole point of introducing the Higgs mechanism in the first place.

17.6 **Hint:** There are no tricks here, just go through the algebra.

17.7 **Hint:** The original Lagrangian is

$$\begin{aligned}
\mathcal{L} &= (D_\mu\phi)^*(D^\mu\phi) = (\partial_\mu\phi_i g B_\mu\phi)^*(\partial^\mu\phi_i g B^\mu\phi) \\
&= (\partial_\mu\phi^*)(\partial^\mu\phi) + ig(\partial_\mu\phi^*)B^\mu\phi - ig(\partial^\mu\phi)B_\mu\phi^* + g^2 B_\mu B^\mu\phi\phi^* .
\end{aligned}$$

Then consider the effect of

$$\phi(x) \to \phi'(x) = e^{ig\chi(x)}\phi(x) \quad \text{and} \quad B_\mu \to B'_\mu = B_\mu - \partial_\mu\chi(x) .$$

17.8 **Hint:** To find the eigenvalues solve:

$$\mathbf{MX} = \begin{pmatrix} g_W^2 & -g_W g' \\ -g_W g' & g'^2 \end{pmatrix} \mathbf{X} = \lambda \mathbf{X}\,,$$

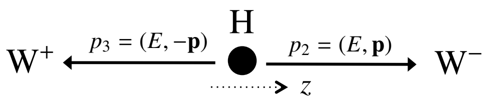
where **M** is the mass matrix.

17.9 **Hint:** The interaction terms in the Lagrangian arise from

$$\begin{aligned}(D_\mu\phi)^\dagger(D^\mu\phi) =& \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{8}g_W^2(W_\mu^{(1)} + iW_\mu^{(2)})(W^{(1)\mu} - iW^{(2)\mu})(v+h)^2 \\ &+ \tfrac{1}{8}(g_W W_\mu^{(3)} - g'B_\mu)(g_W W^{(3)\mu} - g'B^\mu)(v+h)^2 \\ =& \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{8}g_W^2(W_\mu^{(1)} + iW_\mu^{(2)})(W^{(1)\mu} - iW^{(2)\mu})(v+h)^2 \\ &+ \tfrac{1}{8}(g_W^2 + g'^2)Z^\mu Z_\mu(v+h)^2\,.\end{aligned}$$

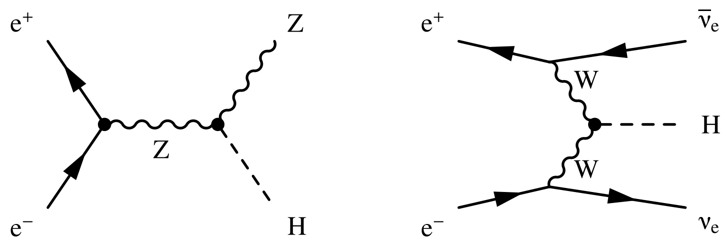
Answer: $g_{HZZ} = \frac{1}{2}\frac{g_W}{\cos\theta_W}m_Z$.

17.10 **Hint:** For the decay $H \to W^+W^-$ the matrix element is



$$\mathcal{M}_{fi} = -g_W m_W g_{\mu\nu}\,\epsilon^\mu(p_2)^*\,\epsilon^\nu(p_3)^*\,.$$

17.11 **Answer:**



17.12 **Answer:** Taking $m_H = 126$ GeV, $v = 246$ GeV, $\Gamma_H = 0.004$ GeV, $\alpha = 1/128$, $m_b = 5$ GeV:

$$\begin{aligned}\sigma^0_{e^+e^-} : \sigma^0_{\mu^+\mu^-} : \sigma_{QED} &= \frac{m_e^2 m_b^2}{16\pi v^4\Gamma_H^2} : \frac{m_\mu^2 m_b^2}{16\pi v^4 \Gamma_H^2} : \frac{16\pi\alpha^2}{27 m_H^2} \\ &= 2.2\times10^{-12}\,\text{GeV}^{-2} : 9.5\times10^{-8}\,\text{GeV}^{-2} : 7.1\times10^{-9}\,\text{GeV}^{-2}\,.\end{aligned}$$

A Appendix A **Errata**

p 56: *Question 2.8*: The reaction should (of course) read:

$$\mathrm{p} + \overline{\mathrm{p}} \to \mathrm{p} + \mathrm{p} + \overline{\mathrm{p}} + \overline{\mathrm{p}}\,.$$

p 57: the factor of $\frac{1}{4}$ in the last line of *Question 2.16* should be removed, *i.e.* Find the eigenvalue(s) of the operator $\hat{\mathbf{S}}^2 = (\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2)$, and deduce that the eigenstates of $\hat{S}_z$ are a suitable representation of a spin-half particle.

p 78: the mass of the pion in *Question 3.1* should be 140 MeV, not 140 GeV.

p146: In the matrix in the *footnote* $B_{22} \to B_{21}$.

p177: *Question 7.2* should be ignored. There was an error in my original solution, whereby finding a closed form was relatively straightforward - it isn't!

p231: there is a typo in the equation at the bottom of the page:

$$\tfrac{1}{2}\left[\mathbf{S}^2 - \mathbf{S_1}^2 - \mathbf{S_1}^2\right] \to \tfrac{1}{2}\left[\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right].$$

p312: In *Figure 12.5*, the arrows on the u and ν_μ are the wrong-way around, only left-handed chiral states participate in the weak charged-current.

p341: There is a typo ($\mathrm{p}_1 \to \mathrm{p}_2$) in Equation (13.13), which should read

$$\Delta\phi_{12} = (E_1 - E_2)\left[T - \left(\frac{E_1 + E_2}{\mathrm{p}_1 + \mathrm{p}_2}\right)L\right] + \left(\frac{m_1^2 - m_2^2}{\mathrm{p}_1 + \mathrm{p}_2}\right)L\,.$$

This typo is repeated in *question 13.1*.

p362: In *question 13.2*, there is a spurious 4 in the denominator of the argument of the $\sin^2(...)$ in the second equation, it should read

$$\sin^2(2\theta)\sin^2\left(\frac{\Delta m^2[\mathrm{GeV}^2]L[\mathrm{GeV}^{-1}]}{4E_\nu[\mathrm{GeV}]}\right) \quad\to\quad \sin^2(2\theta)\sin^2\left(1.27\frac{\Delta m^2[\mathrm{eV}^2]L[\mathrm{km}]}{E_\nu[\mathrm{GeV}]}\right).$$

The expression in the main text is correct.

p363: Part d) of *question 13.9* should be ignored - it is poorly worded. The intention was to get the student to consider the case where the decay products of the pion were close to being perpendicular to the direction of the boost. Close to $\theta^* \sim \pi/2$

the transverse momentum is approximately p* and the longitudinal momentum is primarily due to the Lorentz boost.

p427: The last matrix element should read $\mathcal{M}_{LR}^2$ not $\mathcal{M}_{RR}^2$.

p458: *Question 16.7* should read $\mu^-\overline{\nu}_\mu u\overline{d}$.

p498: *Question 17.8* the expression for the fields should read:

$$A_\mu = \frac{g' W_\mu^{(3)} + g_W B_\mu}{\sqrt{g_W^2 + g'^2}} \quad \text{and} \quad Z_\mu = \frac{g_W W_\mu^{(3)} - g' B_\mu}{\sqrt{g_W^2 + g'^2}},$$