Particle and Nuclear Physics

Handout #1

Problem Sheet
Introduction
Appendices

Lent/Easter Terms 2026
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Part II Particle and Nuclear Physics
Examples Sheet

==================== **Part 1, Chapters 1-4** ====================

1. Chapter 1: Classification of Particles
   Explain the meaning of the terms *fermion, quark, lepton, hadron, nucleus,* and *boson* as used in the classification of particles.

2. Chapter 2: Natural Units

   (a) Explain what is meant by *natural* units and the *Heaviside-Lorentz* system.

   (b) Calculate a value for the proton mass in natural units, assuming $m_p = 1.67 \times 10^{-27}$ kg.

   (c) The muon decay rate is given by Sargent's Rule
   $$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3},$$
   where m_μ is the muon mass (106 MeV) and G_F is the Fermi constant (1.166×10^{-5} GeV^{-2}). Given $\tau = 1/\Gamma$, calculate the muon lifetime in seconds.

   (d) The cross section for $e^+e^- \to \mu^+\mu^-$ is measured as 0.5 pb. What is this cross-section in natural units?

   [Note that $c = 3.0 \times 10^8\,\mathrm{ms}^{-1}$*;* $\hbar = 6.6 \times 10^{-25}$ GeVs*;* $\hbar c = 197$ MeVfm*; 1 barn* $= 10^{-28}\,\mathrm{m}^2$*]*

3. Chapter 2: Relativistic Kinematics
   Consider the electromagnetic decay of the rho meson, $\rho \to \pi^0\gamma$. Calculate the energies of the photon and pion in the ρ^0 rest frame. The π^0 goes on to rapidly decay to two photons. What range of energies will these two photons take in the ρ rest frame, assuming the ρ decays as above? Draw a rough sketch of the energy distribution of all the photons that might be detected from a ρ decay.

   Discuss how the observation of a range of energies for the electron from neutron decay ($n \to pe^-\bar{\nu}_e$) led to the prediction of the existance of the neutrino.

   [The masses of the ρ *and* π^0 *are* 770 MeV *and* 135 MeV*, respectively.]*

4. Chapter 2: Radioactive Decay
   The decay chain ^{211}Bi→^{207}Tl→^{207}Pb is observed for an initially pure sample of 5×10^{11} Bq of ^{211}Bi. The half life of ^{211}Bi is 2.14 minutes and that of ^{207}Tl is 4.88 minutes; ^{207}Pb is stable. Write down the rate equations for this system, and show that the number of Tl atoms present at time t is given by
   $$N_{\mathrm{Tl}}(t) = X\left(e^{-\lambda_{\mathrm{Bi}}t} - e^{-\lambda_{\mathrm{Tl}}t}\right),$$
   where the λ values represent the corresponding decay rates and X is a constant. What is the maximum ^{207}Tl activity and at what time does it occur?

5. Chapter 2: Cross sections

Define the terms *total cross-section* and *differential cross-section* for scattering processes.

A beam of neutrons with an intensity 10^5 particles per second traverses a thin foil of ^{235}U with a density of $200\,\mathrm{kgm^{-3}}$ and thickness of 0.5 mm. There are three possible outcomes for the neutron-uranium interaction:

i. elastic scattering of the neutron, with a cross-section 0.1 b;
ii. neutron capture followed by the emission of a γ-ray, with a cross-section 70 b;
iii. neutron capture followed by fission, with a cross-section 200 b;

Determine

(a) the intensity of the neutron beam transmitted by the foil;
(b) the rate of fission reactions occurring in the foil induced by the incident beam;
(c) the rate of γ-rays induced by the incident beam;
(d) the flux of neutrons elastically scattered out of the beam at a point 10 m from the foil, assuming that the neutrons are scattered isotropically.

6. Chapter 2: Breit-Wigner Formula

The Breit-Wigner formula for a reaction cross-section is given by

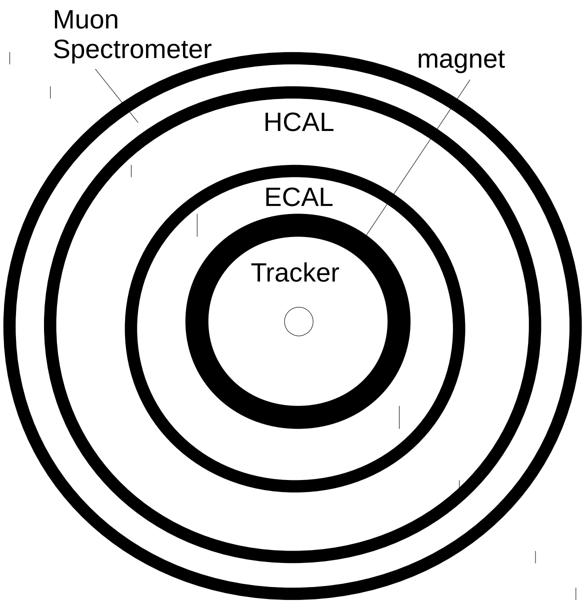
$$\sigma(E) = \frac{\pi g}{p_i^2} \frac{\Gamma_i \Gamma_f}{(E - E_0)^2 + \Gamma^2/4}.$$

Explain the meaning of the symbols in this equation, and outline its derivation.

The maximum value of the cross section for radiative capture of neutrons in ^{113}Cd (i.e. the process $n+{}^{113}\mathrm{Cd} \to {}^{114}\mathrm{Cd}+\gamma$) is 20.6 kb and is reached at a neutron energy of 178 meV, where the elastic width Γ_n is 0.6 meV and the radiative width Γ_γ is 112.4 meV. The spin of ^{113}Cd in its ground state is $J = \frac{1}{2}$. Calculate the elastic cross-section at resonance and find the spin of the compound nucleus formed.

[The mass of the neutron is 939.6 MeV. *]*

7. Chapter 3: Detector Signatures
For each e^+e^- process below, sketch the signature in a typical cylindrical detector e.g.



(a) $e^+e^- \to \mu^+\mu^-$
(b) $e^+e^- \to e^+e^-\mu^+\mu^-$
(c) $e^+e^- \to e^+e^-\gamma$
(d) $e^+e^- \to q\bar{q}g$
(e) $e^+e^- \to \tau^+\tau^-$, where the taus decay as $\tau^+ \to \mu^+\nu_\mu\bar{\nu}_\tau$ and $\tau^- \to \pi^-\pi^0\nu_\tau$
(f) $e^+e^- \to \pi^+n\bar{p}\pi^0K^+K^-$

Calculate the average distance travelled by a tau produced in a $e^+e^- \to \tau^+\tau^-$ collision with $E_{CM} = 10$ GeV. Explain why we don't observe taus directly in typical cylindrical detectors. *[The tau lifetime is* 2.9×10^{-13} *s and the tau mass is* 1.777 GeV*]*

8. Chapter 3: Detector Resolution
In an experiment, the momentum resolution in the tracker for a 10 GeV particle is 10%, while the energy resolution in the electromagnetic calorimeter is 0.16%.

(a) Calculate the momentum and energy resolutions for a 0.5 GeV electron and a 100 GeV electron. Which sub-detector would give the more reliable estimate in each case?
(b) Would a 10 GeV muon or a 100 GeV muon be measured more accurately in a typical cylindrical detector? Explain your reasoning.

9. Chapter 3: Collider Kinematics
   Calculate the centre-of-mass energy for

   i. a fixed target experiment using a 50 GeV electron beam on a proton target,
   ii. a collider experiment using a 50 GeV electron beam and a 50 GeV positron beam.

   Comment on whether a Z boson or a Higgs boson may be produced from the collisions in each case.

   What would be the length of a linear collider used to produce the electron-positron beams in ii), assuming RF cavities capable of providing 16 GeV per km?

   If proton beams are collided instead, why might the centre-of-mass energy be lower than the value calculated in ii)?

10. Chapter 4: Virtual Particles
    Show that the process $\gamma \to e^+e^-$ is kinematically forbidden in a vacuum, but is possible in matter.

    One such possible interaction of a photon in matter is $e^-\gamma \to e^-e^+e^-$. What is the minimum photon energy for this process to occur? How does this change for the photon striking a far more massive object M, $M\gamma \to Me^+e^-$? You may assume $m_M \gg m_e$.

==================== **Part 2, Chapters 5-8** ====================

11. Chapter 5: Feynman Diagrams
Define the terms *scattering amplitude, decay rate*, and *scattering rate*.
Draw all lowest order Feynman diagrams and write down the form of the scattering amplitudes $\mathcal{M}$ for the following processes:

(a) $\mu^- \to e^- \nu_\mu \bar{\nu}_e$
(b) $p \to n e^+ \nu_e$
(c) $\pi^0 \to \gamma\gamma$
(d) $\rho \to \pi^+\pi^-$
(e) $\pi^0 \to \pi^- e^+ \nu_e$
(f) $e^-\gamma \to e^- e^+ e^-$
(g) $e^-\gamma \to \nu\,\bar{\nu}\,e^-$
(h) $e^+e^- \to e^+e^-$
(i) $e^-e^- \to e^-e^-$
(j) $\nu_e\, e^- \to \nu_e\, e^-$
(k) $\nu_e\, e^+ \to \nu_e\, e^+$
(l) $\nu_e\, \mu^- \to \nu_e\, \mu^-$
(m) $\nu_\tau\, p \to \tau^+ n$

(n) $e^+e^- \to \mu^+\mu^-$ at $E_{CM} = 10$ GeV vs $E_{CM} = 90$ GeV.

Are any of these processes forbidden? Where multiple diagrams are possible, note which would dominate (if any).

12. Chapter 6: Electromagnetic Decays
The neutral pion, π^0, is a $J^P = 0^-$, $(u\bar{u} - d\bar{d})/\sqrt{2}$ state and decays via the four possibilities listed below.

Process	Branching fraction
$\pi^0 \to \gamma\gamma$	0.9882
$\pi^0 \to \gamma e^+e^-$	0.0117
$\pi^0 \to e^+e^-e^+e^-$	3×10^{-5}
$\pi^0 \to e^+e^-$	6×10^{-8}

Draw a Feynman diagram for each of the pion decays and use Fermi's Golden Rule $\Gamma = 2\pi|\mathcal{M}|^2\rho$ to *roughly* explain the relative branching fractions. For this question, you may assume the coupling of the photon to a charged particle is $Q\sqrt{\alpha} = Q/\sqrt{137}$ for simplicity.

13. Chapter 6: Drell Yan Production
Draw a typical Feynman diagram for Drell Yan production at a hadron collider. Find the ratio of the Drell Yan production rate for $\pi^-p : \pi^+\pi^- : p\bar{p} : pp$.

14. Chapter 6: Quark Charge and Colour
Estimate $R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ at $E_{CM} = 6$ GeV.
How would R change if

i. the bottom quark mass was 1 GeV?
ii. the electric charge for up-type quarks was $+\frac{3}{4}$ and the down-type quarks was $-\frac{1}{2}$?

In each case, what number of colours would give the best agreement if the measured value of R was $3\frac{1}{3}$ at this E_{CM}?

15. Chapter 7: Quark/Gluon Production

    (a) Draw the lowest order Feynman diagrams for $u\bar{u} \to gg$. What would you expect for $R = \frac{\sigma(u\bar{u} \to gg)}{\sigma(d\bar{d} \to gg)}$ in a $p\bar{p}$ collider?

    (b) Draw the lowest order Feynman diagrams for $gq \to qg$ and $gq \to q\gamma$ and sketch their signatures in a typical detector. How would the energies of the final state particles be determined? What would the ratio of u to d events be for each case in a $p\bar{p}$ collider? What happens to the quarks in the proton and antiproton that do not directly participate in the scattering?

16. Chapter 7: The Strong Coupling Constant
    Sketch the strong coupling constant α_s as a function of the energy scale. Why does this suggest that quarks cannot exist as free particles? Outline two methods for measuring α_s.

17. Chapter 8: Hadron States
    In the lectures we showed the three light quarks (u, d, s) form eight $J^P = \frac{1}{2}^+$ states and ten $J^P = \frac{3}{2}^+$ states. If quarks were spin-0 particles, what baryon states could be formed? Assume all other quark properties remain the same.

18. Chapter 8: Hadron Masses

    (a) What are the quark model mass predictions for the following mesons: K^+, η, ω, η'? Do they agree with the measured masses? If not, can you suggest why this may be?
    [Assume $m_u = m_d = 310$ MeV, $m_s = 483$ MeV and the spin-spin interaction coefficient $A = 0.0615$ GeV^3.]

    (b) What are the quark model mass predictions for the following baryons: Δ, Ξ? Do they agree with the measured masses?
    [Assume in this case $m_u = m_d = 360$ MeV, $m_s = 540$ MeV and the spin-spin interaction coefficient $A = 0.026$ GeV^3.]

    (c) The baryons Λ^0 and Σ^0 have the same quark composition (uds) and both are members of the same $J^P = \frac{1}{2}^+$ baryon octet. Explain why their masses are different (1.116 GeV and 1.193 GeV respectively), and suggest why their lifetimes are very different (2.6×10^{-10} s and 7×10^{-20} s respectively).

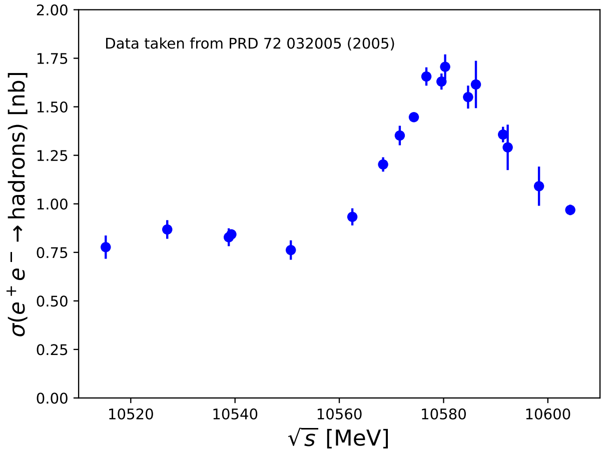
19. Chapter 8: Spin and Parity
    When π^- mesons are stopped in deuterium they form "pionic atoms" (π^-d) which usually undergo transitions to an atomic s-state ($\ell = 0$), whereupon the capture reaction $\pi^-\mathrm{d} \to n\,n$ occurs and destroys them. (The fact that capture normally occurs in an s-state is established from studies of the X-rays emitted in the transitions before capture). Given that the deuteron has spin-parity $J^P = 1^+$ and the pion has $J = 0$, show that these observations imply that the pion has negative intrinsic parity.

20. Chapter 8: Magnetic moments
    The proton has quark content uud and magnetic moment $2.8\,\mu_N$, while the Σ^+ baryon has quark content uus and magnetic moment $2.4\,\mu_N$. Use this information to estimate the magnetic moment of the Σ_b^+ baryon (uub).
    [Assume in this case $m_u = m_d = 0.3$ GeV, $m_s = 0.5$ GeV and $m_b = 5$ GeV.]

21. Chapter 8: The Upsilon Resonance
The BABAR experiment made measurements of $\sigma(e^+e^- \to \text{hadrons})$ for $\sqrt{s}$ in the region of the $\Upsilon(4S)$ $(b\bar{b})$ resonance and some of the results are shown in the figure. The Υ resonance is described by the Breit-Wigner cross section

$$\sigma(i \to f) = \frac{g\pi}{p^2} \frac{\Gamma_i \Gamma_f}{(E - E_0)^2 + \Gamma^2/4}$$

and is known to decay to hadrons close to 100% of the time.



(a) What spin-parity states may be produced in e^+e^- collisions?

(b) Estimate the mass and total width of the $\Upsilon(4S)$ meson.

(c) The detector used to make these cross-section measurements was not fully efficient. Estimate the efficiency by comparing the measured cross section to the theoretical prediction. You may assume the efficiency of the detector was independent of $\sqrt{s}$ and the branching ratio for $\Upsilon(4S) \to e^+e^-$ is 2.5×10^{-5}.

(d) Draw Feynman diagrams for the decays $\Upsilon \to e^+e^-$ and $\Upsilon \to$ hadrons for the $\Upsilon(3S)$ and $\Upsilon(4S)$ resonances. Explain why the $\Upsilon(3S)$ resonance has values of Γ_{ee} similar to that of $\Upsilon(4S)$, but has a total width which is smaller by at least two orders of magnitude.
[B^+ $(\bar{b}u)$ and $B^0(\bar{b}d)$ have masses of about 5280 MeV. *The Υ(3S) resonance has a mass of* 10355 MeV. *]*

22. Chapter 8: Mixed Flavour States
The partial width for the leptonic decay of the ρ^0 meson, $\rho^0 \to e^+e^-$, is 7 keV. Estimate the partial width for the leptonic decay of the ω^0 meson, $\omega^0 \to e^+e^-$.
The partial width for the pionic decay of the ρ^0 meson, $\rho^0 \to \pi^0\gamma$, is 77 keV. Estimate the partial width for the pionic decay of the ω^0 meson, $\omega^0 \to \pi^0\gamma$.
[The π^0 and ρ^0 are both $(u\bar{u} - d\bar{d})/\sqrt{(2)}$ states, while ω^0 is a $(u\bar{u} + d\bar{d})/\sqrt{(2)}$ state. The ρ^0 and ω^0 are both $J^P = 1^-$ states, while the π^0 is 0^-.]

==================== **Part 3, Chapters 9-12** ====================

23. Chapter 9: W boson and the Number of Neutrino Species
The number of neutrino species can be estimated using the total width of the W boson. Using the Standard Model prediction of the partial width for $\mathrm{W}^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}$ decays,

$$\Gamma(\mathrm{W}^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}) = \frac{\mathrm{G_F}}{\sqrt{2}}\frac{\mathrm{M_W^3}}{6\pi},$$

the mass of the W boson, $M_\mathrm{W} = 80.385 \pm 0.015$ GeV and the total width, $\Gamma_\mathrm{W} = 2.085 \pm 0.042$ GeV, estimate the number of light neutrino species. Make clear your assumptions.
[$G_\mathrm{F} = 1.2 \times 10^{-5}$ GeV^{-2}.]

24. Chapter 9: Helicity in the Weak Interaction

    (a) Draw the lowest order Feynman diagrams for $\pi^- \to \mu^-\bar{\nu}_\mu$ and $\pi^- \to e^-\bar{\nu}_e$. Using arguments of lepton universality and density of states only, how would you expect the rates of these two decays to compare?

    (b) Calculate the velocity with which the electron and muon are emitted in the pion rest frame. Note if the velocities you calculate are relativistic or non-relativistic.
    [Assume $m_e = 0.511$ MeV, $m_\mu = 106$ MeV, and $m_\pi = 140$ MeV.]

    (c) The probability for a W-boson to couple to the $\pm$ helicity state of a lepton is equal to $\frac{1}{2}(1 \mp \frac{v}{c})$. What is the consequence for the $\pi^- \to \mu^-\bar{\nu}_\mu$ and $\pi^- \to e^-\bar{\nu}_e$ decay rates?

25. Chapter 9: Weak Force and Conservation
Consider each of the groups of processes given below. In each group, with the aid of Feynman diagrams using the Standard Model vertices, determine which processes are allowed and which are forbidden. By considering the strength of the forces involved, rank the processes in each group in order of expected rate.

    (a) $D_s^+ \to K^+\pi^0$, $D_s^+ \to K^+K^0$, $D_s^+ \to \pi^+\phi$

    (b) $B^0 \to D^-\pi^+$, $B^0 \to \pi^+\pi^-$, $B^0 \to J/\psi K^0$

    (c) $\pi^- \to \mu^-\bar{\nu}_\mu$, $K^- \to \mu^-\bar{\nu}_\mu$, $B^- \to \mu^-\bar{\nu}_\mu$

    (d) $\rho^0 \to \nu\bar{\nu}$, $\pi^0 \to \nu\bar{\nu}$, $\pi^0 \to \mu^+\mu^-$

26. Chapter 10: Z coupling

    (a) In the GWS theory, the couplings of the Z boson to fermions is described by $g_{L,R} \propto (I_3)_{L,R} - Q\sin^2\theta_W$, where L/R denotes a left/right-handed fermion, I_3 denotes weak isospin, Q is the electric charge, and the weak mixing angle is $\theta_W = 29^o$. Assuming $\Gamma(Z \to f\bar{f}) \propto g_L^2 + g_R^2$, predict the branching ratios for the Z boson to decay to hadrons, neutrinos, and $\tau^+\tau^-$.

    (b) In the OPAL experiment at LEP, the cross section for $e^+e^- \to \tau^+\tau^-$ was measured at various centre-of-mass energies. Some of the results are shown below.

$E_{\mathrm{cm}}/\mathrm{GeV}$	$\sigma(e^+e^- \to \tau^+\tau^-)/\mathrm{nb}$
88.481	0.2769 ± 0.0235
89.442	0.4892 ± 0.0091
90.223	0.8331 ± 0.0368
91.283	1.4988 ± 0.0213
91.969	1.1892 ± 0.0235
92.971	0.7089 ± 0.0105
93.717	0.4989 ± 0.0276

Plot these data and make estimates of the Z boson mass, m_Z, the total width of the Z boson, Γ_Z, and the partial decay width to $\tau^+\tau^-$, $\Gamma_{\tau\tau}$. Compare the branching fraction for $Z \to \tau^+\tau^-$ with your theoretical prediction and comment.

Why is the measured resonance curve asymmetric? Indicate what other effects need to be taken into account when accurately determining m_Z, Γ_{Z} and $\Gamma_{\tau\tau}$.

27. Chapter 10: Number of Generations from LEP
Consider a fourth lepton generation exists with $m_L- = 40$ GeV and $m_\nu \sim 0$. Draw Feynman diagrams for possible production mechanisms of this fourth degeneration at a electron-positron collider such as LEP. What might you expect for the possible decays of the charged lepton? Draw Feynman diagrams for the L^- decays and predict the branching ratios for each final state.

Use Sargent's rule for the partial decay rate for $X \to \nu_X e^- \bar{\nu}_e$

$$\Gamma_{X\to e} = \frac{G_F^2 m_X^5}{192\pi^3},$$

to calculate the L^- partial decay rate to electrons and the expected lifetime of L^- in seconds.

Outline how the precision electroweak measurements at LEP ruled out such a fourth generation.

28. Chapter 11: A Higgs Boson Factory
A Higgs factory is being considered to study the properties of the spin-0 Higgs boson with mass 125 GeV. The Higgs boson could be produced through the resonant reaction $\mu^+\mu^- \to H$, with a cross-section described by the Breit-Wigner formula. The partial decay width of the Higgs boson (H) to fermions is proportional to m_f^2.

(a) Explain why a $\mu^+\mu^-$ collider is being considered for a Higgs factory rather than e^+e^- or pp collisions.

(b) Find the ratio of $\Gamma(H \to \tau^+\tau^-)$: $\Gamma(H \to c\bar{c})$: $\Gamma(H \to b\bar{b})$. You may assume $m_\tau = 1.77$ GeV, $m_c = 1.5$ GeV, and $m_b = 4.5$ GeV.

(c) The sum of the branching ratios for Higgs to $\tau^+\tau^-$, $c\bar{c}$, and $b\bar{b}$ is 67%. Calculate the cross section for $\mu^+\mu^- \to H \to b\bar{b}$ at the peak of the resonance. Express your answer in natural units and in barns.

(d) Explain why the branching ratio for $H \to W^+W^-$ is non-zero, despite the fact that $m_H < 2m_W$.

29. Chapter 12: Neutrino Oscillations

A beam of neutrinos can interact with nucleons in a stationary target, either undergoing elastic scattering or producing a charged lepton.

(a) Draw the lowest order Feynman diagrams for the elastic and inelastic scattering of neutrinos with nucleons.

(b) Calculate the minimum energy of the ν which would permit e^- production. How would this threshold energy change for μ^- or τ^- production?

(c) Show that if there are two neutrino mass eigenstates ν_2 and ν_3 with masses m_2 and m_3 and energies E_2 and E_3, mixed so that

$$\begin{aligned}\nu_\mu &= \nu_2 \cos\theta + \nu_3 \sin\theta \\ \nu_\tau &= -\nu_2 \sin\theta + \nu_3 \cos\theta\end{aligned}$$

then the number of muon neutrinos observed at a distance L from the muon source is

$$|\nu_\mu(L)|^2 \approx |\nu_\mu(L=0)|^2 \times \left[1 - \sin^2(2\theta)\,\sin^2\left\{A\left(\frac{(m_2^2 - m_3^2)L}{p}\right)\right\}\right]$$

where A is a constant.

(d) In 2005, the MINOS experiment studied neutrino oscillations by pointing a beam of 1 − 5 GeV muon neutrinos from Fermilab to the MINOS far detector 730 km away. The experiment aimed to make a precise measurement of $m_3^2 - m_2^2$. Sketch the expected energy spectrum of muon neutrinos at the MINOS detector if $\sin^2(2\theta) = 0.90$ and $m_3^2 - m_2^2 = 2.5 \times 10^{-3}$ eV^2. Assume that the energy spectrum of neutrinos produced by the beam at Fermilab was of uniform intensity in the range 1 − 5 GeV and zero elsewhere (i.e. a top-hat function).

(e) If muon neutrinos oscillate into tau neutrinos, will any τ leptons (produced by charged current interactions) be observed in the MINOS far detector? How would your answer change for the future DUNE experiment, which will use a similar ν_μ beam produced at Fermilab aimed at a detector 1300 km away at the Sanford Underground Research Facility?

30. Chapter 12: Grand Unified Theories

Grand Unified Theories predict that protons can decay through the annihilation of two valence quarks to create an antilepton and antiquark via the exchange of a very heavy intermediate boson: $p \to l^+\pi^0$ or $p \to \nu\pi^+$. The non-observation of proton decay can be used to set stringent limits on GUTs.

(a) Assume two new massive bosons exist in nature, $X^{-4/3}$ and $Y^{-1/3}$. Sketch the possible Feynman diagrams for the decay of a proton, indicating the final-state particles. Explain why these Feynman diagrams are non-Standard-Model diagrams.

(b) The Super Kamiokande experiment was designed to search for proton decay as well as neutrino interactions. How might Super K detect the $p \to e^+\pi^0$ decays and distinguish them from neutrino interactions? *Hint: you may want to recall q.12 if you find you are stuck.*

(c) Show the energy of the pion is $E_\pi = (m_p^2 + m_\pi^2 - m_e^2)/2m_p$ in the laboratory frame and the invariant mass of the photons is $m_{\gamma\gamma}^2 = 2E_1E_2(1 - \cos\theta)$ in any frame of reference.

(d) Finally, show the maximum opening angle between the two photons is π and the minimum is $\theta_{min} = 2\sin^{-1}(m_\pi/E_\pi)$ in the laboratory frame.

31. Chapter 13: The Semi-Empirical Mass Formula
The Semi-Empirical mass formula (SEMF) for *nuclear* masses may be written in the form

$$M(A,Z) = Zm_p + (A-Z)m_n - a_{\rm V}A + a_{\rm S}A^{\frac{2}{3}} + a_{\rm C}\frac{Z^2}{A^{\frac{1}{3}}} + a_{\rm A}\frac{(A-2Z)^2}{A} + \delta(A,Z),$$

where m_p and m_n are the masses of the proton and neutron respectively.
[m_p = 938.3 MeV, m_n = 939.6 MeV, m_e = 0.511 MeV, a_V = 15.8 MeV, a_S = 18.0 MeV, a_A = 23.5 MeV, nuclear radius $R = R_0 A^{1/3}$ with $R_0 = 1.2\,\text{fm}$]

(a) Explain the physical significance and functional form of the various terms. Which terms are important for nuclear fission and fusion and why?

(b) Show that the Coulomb term constant a_C can be written as

$$a_{\rm C} = \frac{3e^2}{20\pi\epsilon_0 R_0},$$

assuming the nucleus can be treated as a sphere of uniform charge density. Calculate the value of a_C.

(c) Show that the value of Z of the most stable isobar of mass number A is

$$Z = \frac{m_n - m_p + 4a_{\rm A}}{2a_{\rm C}A^{-\frac{1}{3}} + 8a_{\rm A}/A}.$$

Use this to predict the Z value for the most stable nuclei with $A = 118$ and $A = 201$, and compare with nuclear data, which you can find on the web (e.g. `https://www.nndc.bnl.gov/nudat3/`). Predict the most stable super-heavy nucleus with mass number 302.

(d) On the typical scale of a nucleus, gravitational effects can be safely ignored. However, a neutron star may be considered as nucleus consisting entirely of neutrons and here we can no longer ignore gravity. By treating a nucleus as a sphere of uniform mass density, show the effect of gravitational forces on the binding energy may be accounted for by adding a term $-a_{\rm G}A^{\frac{5}{3}}$ to the SEMF (neglecting the proton-neutron mass difference). Calculate a value for a_G and estimate the *lightest* mass for a neutron star.

32. Chapter 13: The SEMF Asymmetry and Pairing Terms
The form and magnitude of the asymmetry term can be estimated using the Fermi Gas model. This involves treating the N neutrons and Z protons as free fermions of mass m moving in a box of volume $V = \frac{4}{3}\pi R_0^3 A$. The model therefore only accounts for the kinetic energy of the nucleons, and not their potential energy.

A standard calculation, which you have done before (at least for a cubic box), gives the density of states for each species (including spin degeneracy) as

$$g(\epsilon) = BA\epsilon^{\frac{1}{2}} \quad \text{where} \quad B = \frac{4\sqrt{2}m^{\frac{3}{2}}R_0^3}{3\pi\hbar^3}\,.$$

You need not prove this unless you want to practice.

(a) Show that the Fermi energy for the neutrons is $\epsilon_{\rm F} = \left(\frac{3N}{2BA}\right)^{\frac{2}{3}}$

(b) Calculate the Fermi energy $\bar{\epsilon}_{\rm F}$ and the corresponding nucleon momentum for the symmetric case $N = Z = \frac{1}{2}A$.

(c) Show that the total kinetic energy of the nucleons is given by $\frac{3}{5}\left(\frac{3}{2BA}\right)^{\frac{2}{3}}\left(N^{\frac{5}{3}} + Z^{\frac{5}{3}}\right)$.

(d) Expand about the symmetric point $N = Z = \frac{1}{2}A$ by writing $N = \frac{1}{2}A(1+\alpha)$ and $Z = \frac{1}{2}A(1-\alpha)$ to show that the asymmetry energy has the form $a_{\rm A}\frac{(N-Z)^2}{A}$, where $a_{\rm A} = \frac{1}{3}\bar{\epsilon}_{\rm F}$.

(e) One contribution to the pairing energy can also be estimated from this model, reflecting the stepwise increase of the kinetic energy resulting from the exclusion principle. This would be expected to be approximately equal to the energy spacing of levels at the Fermi level, i.e. $1/g(\epsilon_{\rm F})$.

Show that this is, for the $N = Z = \frac{1}{2}A$ case $\frac{4\bar{\epsilon}_{\rm F}}{3A}$. Evaluate and compare with the fitted value in the SEMF for a typical value of $A \sim 100$.

33. Chapter 13: Nuclear Size
The ground state of a ^{17}F nucleus sits 2.25 MeV above the ground state of a ^{17}O nucleus. What is the maximum energy of the positron emitted in the β^+ decay of ^{17}F? Estimate the charge radius of a nucleus with 17 nucleons.
[Consider the nuclear (or atomic) mass differences in terms of the maximum positron energy, and again in terms of the change in the SEMF. $m_p = 938.272$ MeV, $m_n = 939.566$ MeV.*]*

34. Chapter 14: The Nuclear Shell Model
Outline the basis of the Nuclear Shell Model and show how it accounts for magic numbers. How can the shell model be used to predict the spins and parities of nuclear ground states?

Use the shell model to predict the spins and parities of the ground stats of the nuclides listed below and compare to the experimental values given. Comment on any discrepancies you find.

$^{3}_{2}$He	$^{9}_{4}$Be	$^{7}_{3}$Li	$^{12}_{6}$C	$^{13}_{6}$C	$^{15}_{7}$N	$^{17}_{8}$O	$^{23}_{11}$Na	$^{131}_{54}$Xe	$^{207}_{82}$Pb
$\frac{1}{2}^+$	$\frac{3}{2}^-$	$\frac{3}{2}^-$	0^+	$\frac{1}{2}^-$	$\frac{1}{2}^-$	$\frac{5}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^+$	$\frac{1}{2}^-$

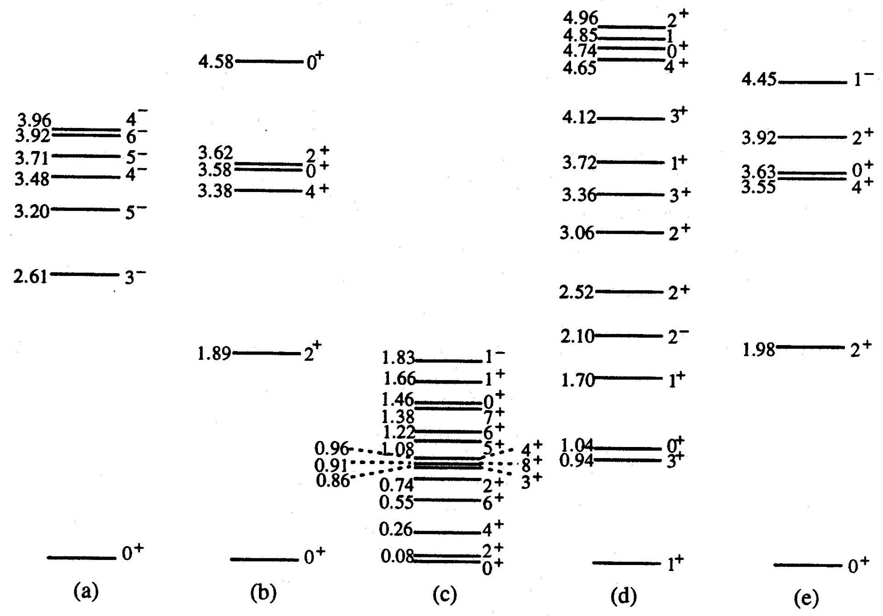
Assume the following ordering of levels:

$1s_{\frac{1}{2}}\ 1p_{\frac{3}{2}}\ 1p_{\frac{1}{2}}\ 1d_{\frac{5}{2}}\ 1d_{\frac{3}{2}}\ 2s_{\frac{1}{2}}\ 1f_{\frac{7}{2}}\ 1f_{\frac{5}{2}}\ 2p_{\frac{3}{2}}\ 2p_{\frac{1}{2}}\ 1g_{\frac{9}{2}}\ 1g_{\frac{7}{2}}\ 2d_{\frac{5}{2}}\ 2d_{\frac{3}{2}}\ 1h_{\frac{11}{2}}\ 3s_{\frac{1}{2}}\ 1h_{\frac{9}{2}}\ 2f_{\frac{7}{2}}\ 3p_{\frac{3}{2}}\ 1i_{\frac{13}{2}}\ 3p_{\frac{1}{2}}\ 2f_{\frac{5}{2}}\cdots$

35. Chapter 14: Energy Levels
The diagram below shows the low-lying energy levels for the nuclides:

$$^{18}_{10}\mathrm{Ne} \quad ^{166}_{68}\mathrm{Er} \quad ^{18}_{9}\mathrm{F} \quad ^{208}_{82}\mathrm{Pb} \quad ^{18}_{8}\mathrm{O}$$

The schemes are drawn to the same scale, with energies (in MeV) with respect to the ground state and the spin and parity (J^P) values given for each level. Identify which scheme corresponds to each nuclide and explain as fully as you can which features of the levels support your choices.



36. Chapter 14: Rotational Excitations
The spin-parity and excitation energies of the five lowest-energy states of $^{174}_{72}\mathrm{Hf}$ are

J^P	0^+	2^+	4^+	6^+	8^+
$E/$ keV	0	91	297	608	1009

Show that these states are consistent with being rotational excitations and obtain a value for the moment of inertia of the $^{174}_{72}\mathrm{Hf}$ nucleus. Compare your result to the expectation if $^{174}_{72}\mathrm{Hf}$ is assumed to be a rigid spherical rotator.
*[A solid sphere of mass m and radius R has moment of inertia $U = \frac{2}{5}mR^2$.
Hint: remember $\hbar = 6.66 \times 10^{-25}$ GeV s or $\hbar = 1.05 \times 10^{-4}$ fm^2 kg s^{-1}]*

37. Chapter 15: Carbon Dating

(a) The $^{14}_{6}\mathrm{C}$ half-life is 5730 years. What is its average lifetime?

(b) An organic artefact has been discovered in an Egyptian tomb and carbon dating shows it to have an activity of 0.13 Bq per gram. What is the age of the artefact?

(c) Use reasonable assumptions to estimate the oldest organic artefact we may age using Carbon dating.

38. Chapter 15: Alpha Decay
An isotope of plutonium, $^{239}\mathrm{Pu}$, is an alpha-emitter with a half-life of 24,120 years. What is the initial activity of 1 kg of $^{239}\mathrm{Pu}$?

39. Chapter 15: Beta Decay
What are the conditions under which the three types of β-decay are kinematically allowed? Use these conditions to determine which of the following $A = 142$ isobars would you expect to be stable, and how would you expect the others to decay. Use Sargent's rule to estimate which of the unstable ones should have the shortest lifetimes, and which the longest. Does this match with your expectation from the classification of the β^+ and β^- decays?

Nuclide	Atomic Mass / m_u	J^P
$^{142}_{57}$La	141.9141	2^-
$^{142}_{58}$Ce	141.9092	0^+
$^{142}_{59}$Pr	141.9100	2^-
$^{142}_{60}$Nd	141.9077	0^+
$^{142}_{61}$Pm	141.9130	1^+
$^{142}_{62}$Sm	141.9152	0^+

[The mass of the electron is 0.00055 m_u.]

40. Chapter 15: Fermi Theory and Sargent's Rule
Show that the electron momentum spectrum in β-decay using Fermi theory can be written as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}p_\mathrm{e}} = \frac{G_\mathrm{F}^2}{2\pi^3} \left(E_0 - E_\mathrm{e}\right)^2 p_\mathrm{e}^2,$$

where G_F is the Fermi constant, E_e and p_e are the energy and momentum of the electron and E_0 is the total energy released. You may treat the electron and neutrino as massless.

Show that the average kinetic energy carried off by the electron in β decay is $E_0/2$ when the electron is highly relativistic, and $E_0/3$ when the electron is non-relativistic.

When the electron is highly relativistic, show that the total decay rate is given approximately by

$$\Gamma = \frac{G_\mathrm{F}^2 E_0^5}{60\pi^3}$$

The E_0^5 dependence is sometimes known as Sargent's Rule.

41. Chapter 15: Gamma Decay
In an experiment, the first three excited states of $^{17}_{9}$F were studied and the following gamma transitions were observed

E1 :	2.6 MeV, 4.2 MeV, 4.7 MeV
M1 :	1.6 MeV
E2 :	0.5 MeV, 1.6 MeV

In addition, a weaker transition of energy 3.1 MeV was seen. The 0.5 MeV gamma-ray corresponds to a transition between the first excited state and the ground state. Use the nuclear shell model to predict the spin-parity of the ground state of $^{17}_{9}$F. Assuming that the first excited state is a single particle excitation of a nucleon to a *nearby (not necessarily closest)* higher energy level, suggest the likely spin-parity assignment for this excited state and discuss whether this is consistent with the observed gamma transition (and lack of any others).

Draw a possible decay scheme for $^{17}_{9}$F showing the energy levels of the first three excited states, the spin-parity assignments and the gamma-ray transitions given above. Explain your reasoning clearly. What might be the likely nature of the 3.1 MeV gamma transition?

42. Chapter 16: Induced Fission
Using the SEMF, estimate the excitation energies of the $^{236}_{92}U^*$ and $^{239}_{92}U^*$ nuclear states formed when $^{235}_{92}U$ and $^{238}_{92}U$ nuclei, respectively, capture a neutron of negligible kinetic energy. Identify the term in the SEMF which is primarily responsible for the difference in the predicted excitation energies and the ground state for both these cases.

The observed excitation energies following low energy (thermal) neutron capture by $^{235}_{92}U$ and $^{238}_{92}U$ are approximately 6.5 MeV and 4.8 MeV, respectively. The fission activation energies for $^{235}_{92}U$ and $^{238}_{92}U$ are approximately 6.2 MeV and 6.6 MeV, respectively. Explain why thermal neutrons can induce rapid fission of $^{235}_{92}U$ but not of $^{238}_{92}U$. Discuss the implications of the energy dependence of the cross sections for neutron induced fission for the design of nuclear reactors which use uranium as a fuel.

43. Chapter 16: Moderating Neutrons for Fission
Compute the maximum fractional energy loss which a non-relativistic neutron can undergo in a single elastic collision with

i. a $^{12}_{6}C$ nucleus
ii. a $^{10}_{5}B$ nucleus.

For each case, calculate the minimum number of collisions which would be required in order to bring a 2.5 MeV fission neutron down to a thermal energy of 0.025 eV. What are the advantages and disadvantages of using each material as a moderator in nuclear reactions?

44. Chapter 16: Fusion
Estimate the size of the Coulomb barrier between two $^{13}_{6}C$ nuclei which needs to be overcome before they can undergo fusion, and thus estimate the temperature needed to bring about fusion in this case.

Numerical answers

1) ; 2) b) 938 MeV, c) 2.2 μs, d) 1.3×10^{-9} GeV^{-2} ; 3) $E_\pi = 397$ MeV, $E_\gamma = 373$ MeV, from π $E_\gamma = 67.5$ MeV, in lab $11.8 - 385$ MeV; 4) 4.5 min, 1.2×10^{11} Bq ; 5) a) 99308 s^{-1}, b) 513 s^{-1}, c) 179 s^{-1}, d) 2×10^{-4} s^{-1} ; 6) 2.8×10^{-13} eV^{-2}, 0; 7) 0.23 mm; 8) a) 0.5% & 0.7%, 100% & 0.05%; 9) i) 10 GeV, ii) 100 GeV, 6.25 km; 10) 2 MeV, 1 MeV; 11) ; 12) ; 13) 8:5:17:∼0; 14) $\frac{10}{3}$, i) $\frac{11}{3}$ & 3, ii) 4.9 & 2; 15) a) 4, b) gq 2, $q\gamma$ 8; 16) ; 17) ; 18) a) K^+ 485 MeV, η 559 MeV, ω 780 MeV, η' 349 MeV, b) Δ 1230 MeV, Ξ 1329 MeV; 19) −1; 20) $2.38 \pm 0.14\,\mu_N$; 21) a) 1^-, b) 10580 MeV& 25 MeV, c) 27%; 22) 0.8 keV, 693 keV; 23) ∼3; 24) b) $\beta_e = 1$, $\beta_\mu = 0.27$; 25) ; 26) a) 69.1%, 20.5%, 3.5%, b) m_Z 91.2 GeV, Γ_Z 2.6 GeV, $\Gamma_{\tau\tau}$ 75 MeV, BR 2.9%; 27) 3×10^{-20} s; 28) b) 1:2.2:19.4, c) 20 pb ; 29) b) 0 GeV, 0.11 GeV, 3.5 GeV; 30) ; 31) b) 0.72 MeV, c) 50, 80, 114, d) a_G 5.8×10^{-37} MeV, ∼5% solar mass; 32) b) 33.3 MeV, 250 MeV, d) a_A 11 MeV; 33) 1.7 MeV, 4.1 fm; 34) ; 35) ; 36) 2.3×10^{-24} kg fm^2, 5.19×10^{-24} kg fm^2; 37) a) 8267 yrs, b) 3215 yrs, c) ∼60,000 yrs; 38) 2.27×10^{12} Bq; 39) ; 40) ; 41) 1st exc. $\frac{1}{2}^+$; 42) 6.7 MeV& 5.2 MeV; 43) i) 28% & 55, ii) 33% & 46; 44) 1.1×10^{11} K.

Suggested Tripos Questions

Relativistic Kinematics: 2016 1(c), 2014 3, 2004 (3) C12(b) (not last part)

Breit-Wigner resonances, production and decay rates: 2020 A3, 2018 A1(a), 2005 (3) A3

Feynman Diagrams: 2016 3(a), 2009 (3) A1(b), 2008 (3) A4

QCD: 2018 B3 last part, 2014 3 last part, 2013 1(b)

Hadron physics and quark model: 2018 B3, 2016 1(b), 2010 A4

Weak interaction: 2020 2, 2018 B4, 2011 3

Electroweak unification: 2018 A1(b), 2015 3, 2013 1(a)

Neutrino Oscillation: 2009 (3) A4

Semi-Empirical Mass Formula: 2018 B2, 2010 A1(a)

Nuclear Forces & Scattering: 2016 4, 2012 1(a)

Shell Model: 2020 5(a)(b), 2017 1(b), 2016 1(a)

Nuclear excitations: 2017 3 (last part), 2015 1(a)

Nuclear decay: 2004 (3) A1

α-decay: 2017 3, 2010 A1(b), 2007 A3

β-decay: 2019 3, 2016 3(b)(c)(d), 2015 4

γ-decay: 2020 5(c), 2018 A1(b), 2014 4, 2010 A3 last part

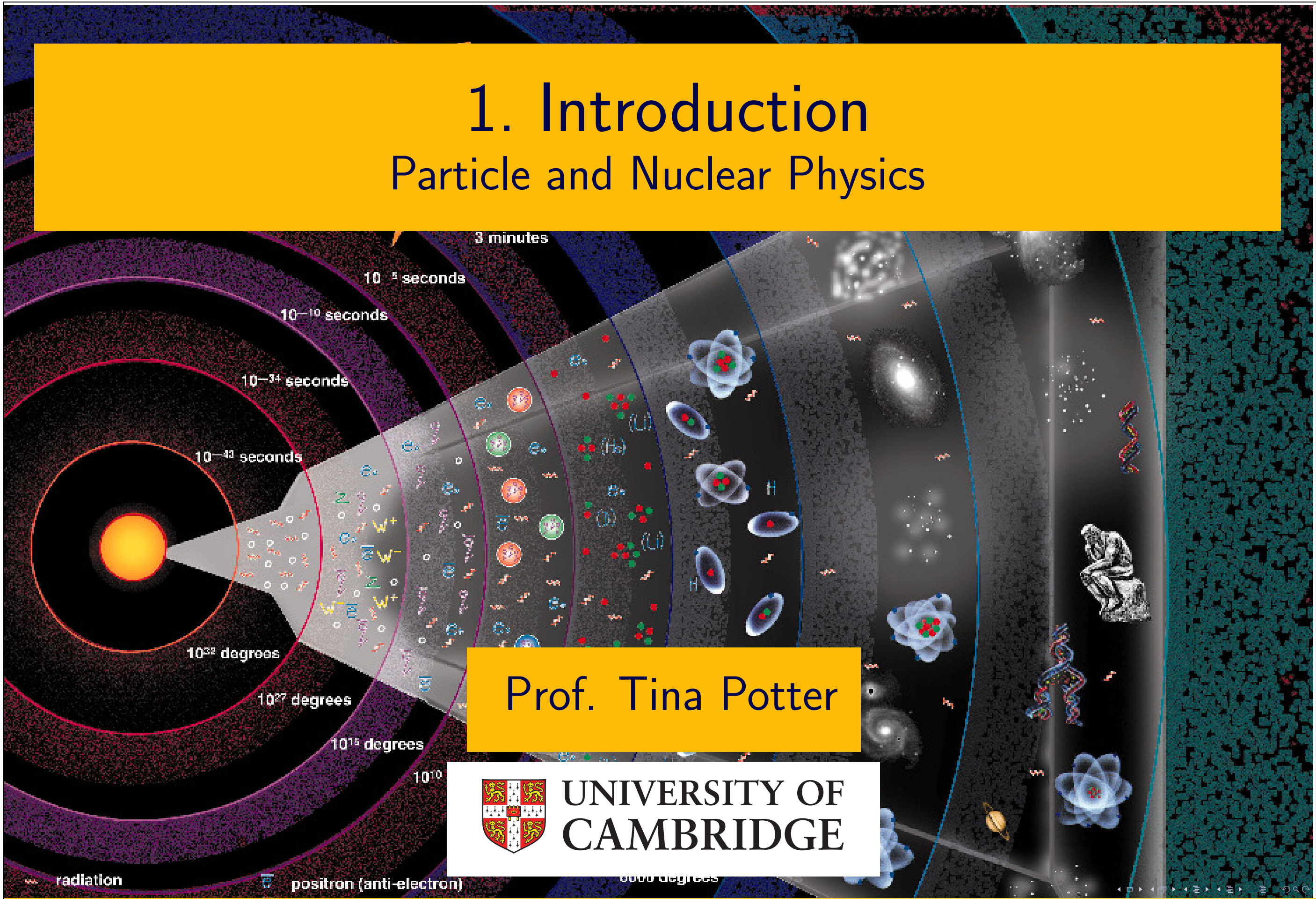
Fission and Fusion: 2011 4

Supervisions

Supervisions might follow this pattern
Supo 1: Q1-10, Chapters 1-4, covered by week 2.5 LT
Supo 2: Q11-22, Chapters 5-8, covered by week 5.5 LT
Supo 3: Q23-30, Chapters 9-12, covered by week 8 LT
Supo 4: Q31-44, Chapters 13-16, covered by week 2 ET

1. Introduction

Particle and Nuclear Physics



Prof. Tina Potter

UNIVERSITY OF CAMBRIDGE

In this section...

- Course content
- Practical information
- Matter
- Forces

Course content

These lectures will cover the core topics of Particle and Nuclear physics.

Particle Physics is the study of
Matter: Elementary particles

Forces: Basic forces in nature
Electroweak (EM & weak)
Strong

Current understanding is embodied in the
Standard Model
which successfully describes all current data*.

Nuclear Physics is the study of
Matter: Complex nuclei
(protons & neutrons)

Forces: Strong "nuclear" force
(underlying strong force)
+ weak & EM decays

Complex many-body problem, requires semi-empirical approach.

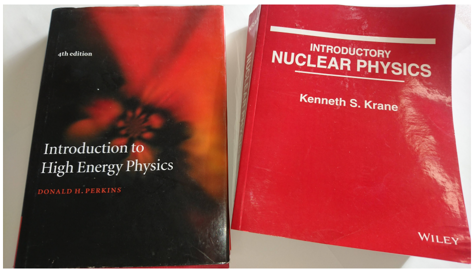
Many models of Nuclear Physics.

Historically, Nuclear Physics preceded and led to Particle Physics.
Our course will discuss Particle Physics first, and then Nuclear Physics.

* *with some interesting exceptions!*

Practical information

Website holds course information, notes, appendices and problem sheets
www.hep.phy.cam.ac.uk/~chpotter/particleandnuclearphysics/mainpage.html



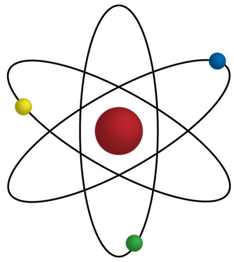
Books
Introduction to High Energy Physics, Perkins
Introductory Nuclear Physics, Krane

Lecturing material provided as **three handouts**.
Lectures will cover additional examples – please attend!!

Problem sets in 4 parts
Part 1, q. 1-10: Chapters 1-4
Part 2, q.11-22: Chapters 5-8
Part 3, q.23-30: Chapters 9-12
Part 4, q.31-44: Chapters 13-16

My availability: before/after lectures, via email (cp594@cam.ac.uk), in-person chats are always welcome

Zooming into matter



Atom *Binding energy* $\sim$ *Rydberg* ~ 10 eV

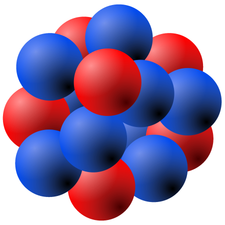
Electrons bound to atoms by EM force

Size: Atom $\sim 10^{-10}$m, $e^- < 10^{-19}$m

Charge: Atom is neutral, electron $-e$

Mass: Atom mass $\sim$ nucleus, $m_e = 0.511 MeV/c^2$

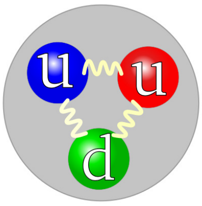
Chemical properties depend of Atomic Number, Z



Nucleus *Binding energy* ~ 10 MeV/*nucleon*

Nuclei held together by strong "nuclear" force

Size: Nucleus (medium Z) ~ 5 fm (1 fm$= 10^{-15}$ m)



Nucleon *Binding energy* ~ 1 GeV

Protons & neutrons held together by the strong force

Size: p, n ~ 1 fm

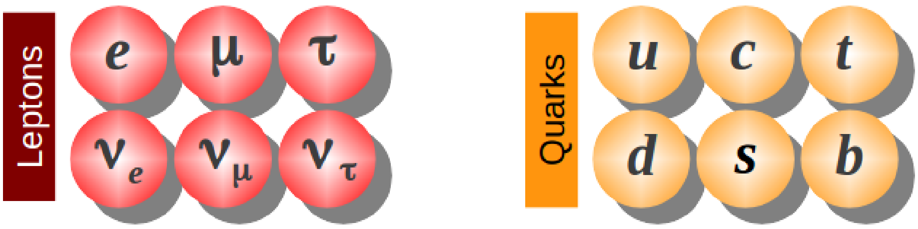
Charge: proton $+e$, neutron is neutral

Mass: p, n $= 939.57$ MeV$/c^2 \sim 1836\, m_e$

Matter

In the Standard Model, all matter is made of spin $\frac{1}{2}$ fundamental particles.

There are two types, each with 3 generations:



Consequence of relativity and quantum mechanics (Dirac equation)

Antiparticle for every existing particle: identical mass, spin, energy, momentum, **but** has the opposite sign of interaction (e.g. electric charge).

Particles and antiparticles

electron e^- & positron e^+

up quark u ($Q = +\frac{2}{3}$) & antiup $\bar{u}$ ($Q = -\frac{2}{3}$)

proton udu & antiproton $\bar{u}\bar{d}\bar{u}$

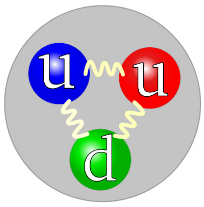
Matter *The first generation*

Almost all the matter in the universe is made up from just four of the fermions.

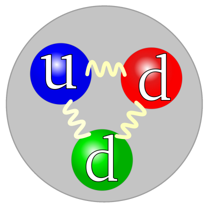
Particle	Symbol	Type	Charge [e]
Electron	e^-	lepton	-1
Neutrino	ν_e	lepton	0
Up quark	u	quark	$+\frac{2}{3}$
Down quark	d	quark	$-\frac{1}{3}$

The proton and neutron are simply the lowest energy bound states of a system of three quarks: essentially all an atomic or nuclear physicist needs.

Proton (p)



Neutron (n)



Matter *Three generations*

Nature is not so simple.
There are 3 generations/families of fundamental fermions (and only 3).

1^{st} generation		2^{nd} generation		3^{rd} generation	
Electron	e^-	Muon	μ^-	Tau	τ^-
Electron Neutrino	ν_e	Muon Neutrino	ν_μ	Tau Neutrino	ν_τ
Up quark	u	Charm quark	c	Top quark	t
Down quark	d	Strange quark	s	Bottom quark	b

- Each generation is a replica of (e^-, ν_e, u, d).
- The mass of the particles increases with each generation:
  the first generation is lightest and the third generation is the heaviest.
- The generations are distinct
  i.e. μ is not an excited e, or $\mu^- \to e^-\gamma$ would be allowed – this is not seen.
- There is a symmetry between the generations,
  but the origin of 3 generations is not understood!

Matter *Leptons*

Leptons are fermions which do not interact via the strong interaction.

Flavour	Charge [e]	Mass	Strong	Weak	EM
1st generation					
e^-	-1	$0.511\ \mathrm{MeV}/c^2$	✗	✓	✓
ν_e	0	$< 2\ \mathrm{eV}/c^2$	✗	✓	✗
2nd generation					
μ^-	-1	$105.7\ \mathrm{MeV}/c^2$	✗	✓	✓
ν_μ	0	$< 0.19\ \mathrm{MeV}/c^2$	✗	✓	✗
3rd generation					
τ^-	-1	$1777.0\ \mathrm{MeV}/c^2$	✗	✓	✓
ν_τ	0	$< 18.2\ \mathrm{MeV}/c^2$	✗	✓	✗

- Spin $\frac{1}{2}$ fermions
- 6 distinct flavours
- 3 charged leptons: e^-, μ^-, τ^-.
  3 neutral leptons: ν_e, ν_μ, ν_τ.
- Antimatter particles $e^+, \bar{\nu}_e$ etc
- e is stable,
  μ and τ are unstable.

- Neutrinos are stable and almost massless. Only know limits on ν masses, but have measured mass differences to be $< 1\ \mathrm{eV}/c^2$. *Not completely true, see later...*
- Charged leptons experience only the electromagnetic & weak forces.
- Neutrinos experience only the weak force.

Matter *Quarks*

Quarks experience all the forces (strong, electromagnetic, weak).

Flavour	Charge [e]	Mass	Strong	Weak	EM
1st generation					
u	$+\frac{2}{3}$	$2.3\ \mathrm{MeV}/c^2$	✓	✓	✓
d	$-\frac{1}{3}$	$4.8\ \mathrm{MeV}/c^2$	✓	✓	✓
2nd generation					
c	$+\frac{2}{3}$	$1.3\ \mathrm{GeV}/c^2$	✓	✓	✓
s	$-\frac{1}{3}$	$95\ \mathrm{MeV}/c^2$	✓	✓	✓
3rd generation					
t	$+\frac{2}{3}$	$173\ \mathrm{GeV}/c^2$	✓	✓	✓
b	$-\frac{1}{3}$	$4.7\ \mathrm{GeV}/c^2$	✓	✓	✓

- Spin $\frac{1}{2}$ fermions
- 6 distinct flavours
- Fractional charge:

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \quad \begin{pmatrix} +\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

- Antiquarks $\bar{u}, \bar{d}$ etc
- Quarks are confined within hadrons, e.g. p=(uud), π^+=($u\bar{d}$)

- Quarks come in three colours (colour charge) Red, Green, Blue.
  Colour is a label for the charge of the strong interaction.
  Unlike the electric charge ($+-$), the strong charge has three orthogonal colours (RGB).

Matter *Hadrons*

Single, free quarks have never been observed. They are always confined in bound states called hadrons.
Macroscopically, hadrons behave as almost point-like composite particles.

Hadrons have two types:

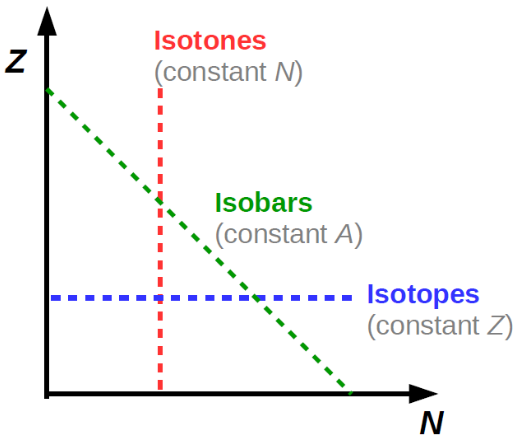
- **Mesons ($q\bar{q}$)**: Bound states of a quark and an antiquark.
  Mesons have integer spin 0, 1, 2... bosons.
  e.g. $\pi^+ \equiv (u\bar{d})$, charge $= (+\frac{2}{3} + +\frac{1}{3})e = +1e$
  $\pi^- \equiv (\bar{u}d)$, charge $= (-\frac{2}{3} + -\frac{1}{3})e = -1e$; antiparticle of π^+
  $\pi^0 \equiv (u\bar{u} - d\bar{d})/\sqrt{2}$, charge $= 0$; is its own antiparticle.
- **Baryons (qqq)**: Bound states of three quarks.
  Baryons have half-integer spin $\frac{1}{2}$, $\frac{3}{2}$... fermions.
  e.g. $p \equiv (udu)$, charge $= (+\frac{2}{3} + -\frac{1}{3} + +\frac{2}{3})e = +1e$
  $n \equiv (dud)$, charge $= (-\frac{1}{3} + +\frac{2}{3} + -\frac{1}{3})e = 0$
  Antibaryons e.g. $\bar{p} \equiv (\bar{u}\bar{d}\bar{u})$, $\bar{n} \equiv (\bar{d}\bar{u}\bar{d})$

Matter *Nuclei*

A nucleus is a bound state of Z protons and N neutrons.
Protons and neutrons are generically referred to as nucleons.
A (mass number) $= Z$ (atomic number) $+ N$ (neutron number).
A nuclide is a specific nucleus, characterised by Z, N.

Notation: Nuclide ^A_ZX.

e.g. $^1_1$H or p:	Z=1,	N=0,	A=1
$^2_1$H or d:	Z=1,	N=1,	A=2
$^4_2$He or α:	Z=2,	N=2,	A=4
$^{208}_{82}$Pb:	Z=82,	N=126,	A=208



In principle, antinuclei and antiatoms can be made from antiprotons, antineutrons and positrons – experimentally challenging!

Matter *The Periodic Table*

Periodic table classifies elements according to their chemical properties.

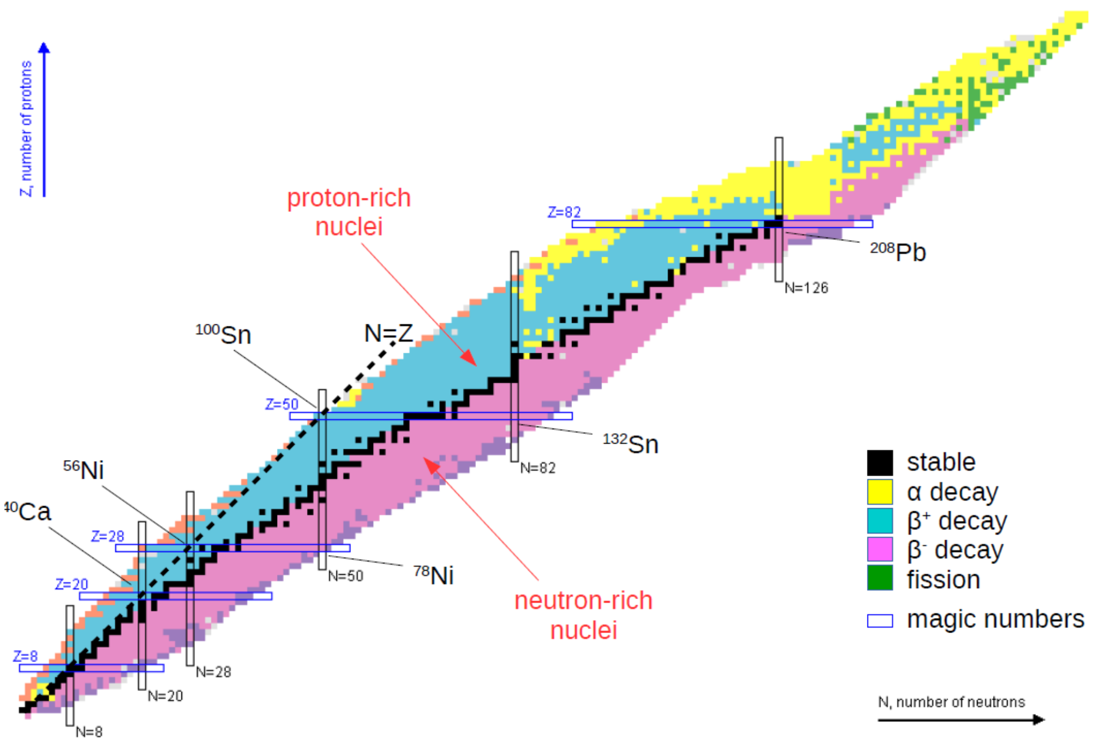
1 H																	2 He
3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba	*	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra	**	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo
	*	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
	**	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

Only hydrogen, helium and lithium were formed in the Big Bang.
All other elements are formed in stars.
Natural elements, H(Z=1) to U(Z=92).

Matter *Chart of the nuclides*

Many more nuclides than elements.

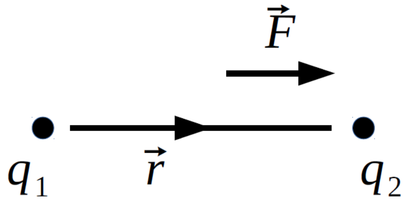
Colour coded according to decay mode.



Forces *Classical Picture*

A force is 'something' which pushes matter around and causes objects to change their motion.

In classical physics, the electromagnetic forces arise via action at a distance through the electric and magnetic fields, $\vec{E}$ and $\vec{B}$.



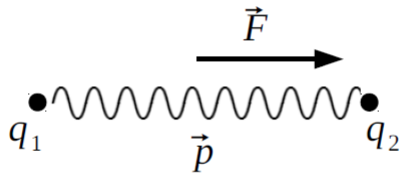
$$\vec{F} = \frac{q_1 q_2 \vec{r}}{r^2}$$

Newton: "...that one body should act upon another at a distance, through a vacuum, without the mediation of anything else, by and through which their force may be conveyed from one to another, is to me so great an absurdity that I believe no man who has, in philosophical matters, a competent faculty of thinking, can ever fall into it. Gravity must be caused by an agent, acting constantly according to certain laws, but whether this agent be material or immaterial, I leave to the consideration of my reader."

Forces *Quantum Mechanics*

Matter particles are quantised in QM, and the electromagnetic field should also be quantised (as photons).

Forces arise through the exchange of virtual field quanta called **Gauge Bosons**.



This process is called "second quantisation".

This process violates energy/momentum conservation *(more later)*.

However, this is permissible for sufficiently short times owing to the Uncertainty Principle

The exchanged particle is "virtual" – meaning it doesn't satisfy $E^2 = p^2c^2 + m^2c^4$.

Uncertainty principle: $\Delta E \Delta t \sim \hbar \quad \Rightarrow \quad$ range $R \sim c\Delta t \sim \hbar c/\Delta E$

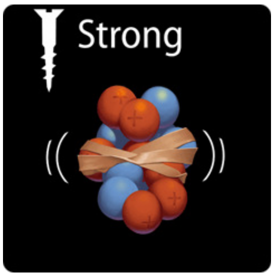
i.e. larger energy transfer (larger force) $\leftrightarrow$ smaller range.

Prob(emission of a quantum)$\propto q_1$, Prob(absorption of a quanta)$\propto q_2$

Coulomb's law can be regarded as the resultant effect of all virtual exchanges.

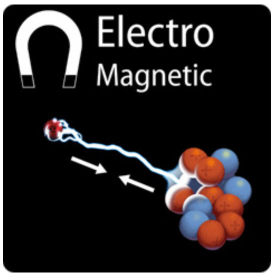
Forces *The four forces*

All known particle interactions can be explained by four fundamental forces.



Carried by the gluon. Holds atomic nuclei together.

Carried by the W and Z bosons. Responsible for radioactive decay.



Carried by the photon. Acts between charged particles.

Carried by the graviton. Acts between massive particles.

Forces *Gauge bosons*

Gauge bosons mediate the fundamental forces

- Spin 1 particles i.e. Vector Bosons
- Interact in a similar way with all fermion generations
- The exact way in which the Gauge Bosons interact with each type of lepton or quark determines the nature of the fundamental forces. This defines the Standard Model.

Force	Boson		Spin	Strength	Mass
Strong	8 gluons	g	1	1	massless
Electromagnetic	photon	γ	1	10^{-2}	massless
Weak	W and Z	W^+, W^-, Z	1	10^{-7}	80, 91 GeV
Gravity	graviton	?	2	10^{-39}	massless

- Gravity is not included in the Standard Model. The others are.

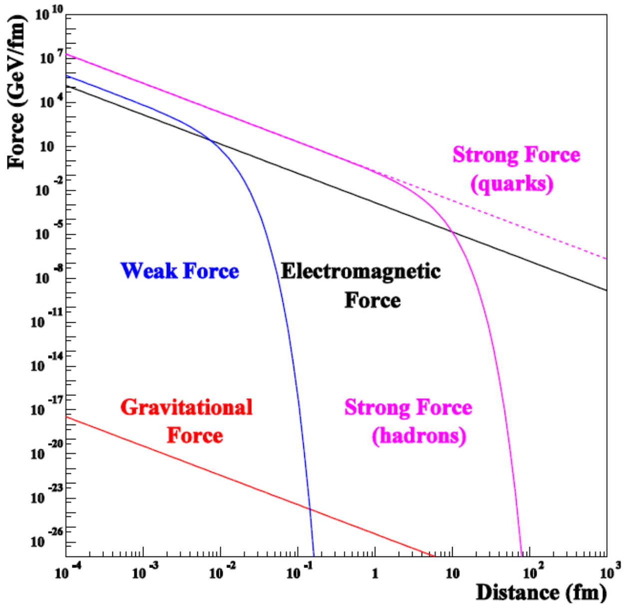
Forces *Range of forces*

The maximum range of a force is inversely related to the mass of the exchanged bosons.

$$\Delta E \Delta t \sim \hbar, \qquad E = mc^2$$

$$\Rightarrow mc^2 \sim \frac{\hbar}{\Delta t} \sim \frac{\hbar c}{r} \quad \Rightarrow r \sim \frac{\hbar}{mc}$$

Force	Range [m]
Strong	inf
Strong (nuclear)	10^{-15}
Electromagnetic	inf
Weak	10^{-18}
Gravity	inf



Due to quark confinement, nucleons start to experience the strong interaction at ∼2 fm.

Summary

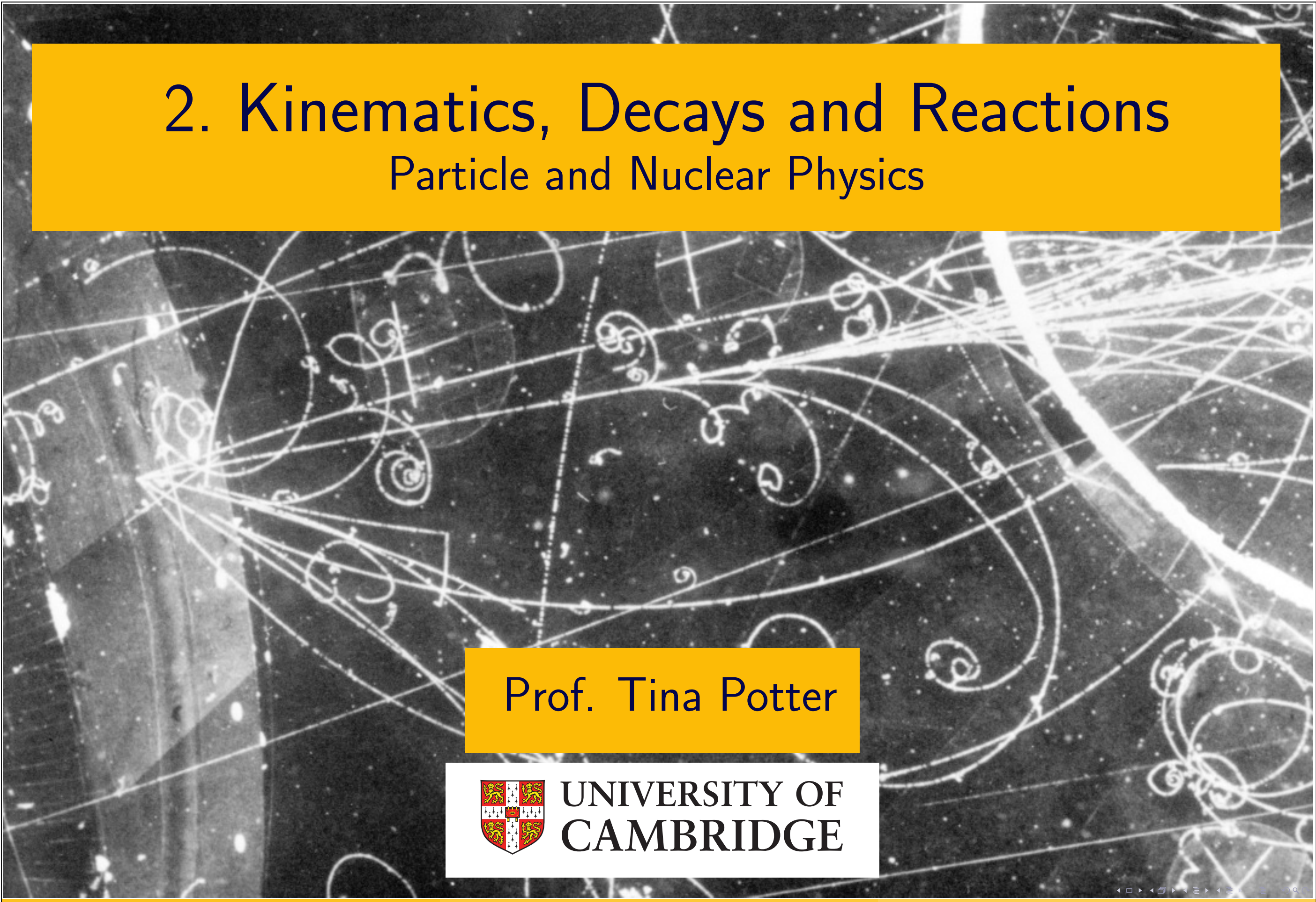
- Particle vs nuclear physics
- Matter: generations, quarks, leptons, hadrons, nuclei
- Forces: classical vs QM, fundamental forces, gauge bosons, range

Problem Sheet: q.1

Up next...
Section 2: Kinematics, Decays and Reactions.

Glossary

- **Strong force** - force which binds quarks into hadrons; mediated by gluons.
- **Electromagnetic Force** - force between charged particles, mediated by photons.
- **Weak force** - force responsible for β-decay. Mediated by W and Z bosons.
- **Gauge boson** - particle which mediates a force.
- **Lepton** - fermion which does not feel the strong interaction.
- **Neutrino** - uncharged lepton which experiences only weak interactions.
- **Quark** - fundamental fermion which experiences all forces.
- **Hadron** - bound state of quarks and/or antiquarks.
- **Baryon** - hadron formed from three quarks.
- **Meson** - hadron formed from quark+antiquark.
- **Generations/Families** - three replicas of the fundamental fermions.
- **Nucleus** - massive bound state of neutrons and protons at centre of an atom.
- **Strong nuclear force** - strong force between nucleons which binds atomic nucleus. Mediated by mesons, such as the pion.
- **Nucleon** - proton or neutron.
- **Nuclide** - specific nuclear species with N neutrons and Z protons.
- **Mass number** - total number of nucleons in nucleus, A.
- **Atomic Number** - number of protons in nucleus, Z.
- **Neutron Number** - number of neutrons in nucleus, N.
- **Isobars** - nuclides with the same Mass Number A.
- **Isotopes** - nuclides with the same Atomic Number Z.
- **Isotones** - nuclides with the same Neutron Number N.



2. Kinematics, Decays and Reactions

Particle and Nuclear Physics

Prof. Tina Potter

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In this section...

- Natural units
- Symmetries and conservation laws
- Relativistic kinematics
- Particle properties
- Decays
- Cross-sections
- Scattering
- Resonances

Units

The usual practice in particle and nuclear physics is to use Natural Units.

- Energies are measured in units of eV:

  Nuclear $\mathrm{keV}(10^3\ \mathrm{eV})$, $\mathrm{MeV}(10^6\ \mathrm{eV})$

  Particle $\mathrm{GeV}(10^9\ \mathrm{eV})$, $\mathrm{TeV}(10^{12}\ \mathrm{eV})$

- Masses are quoted in units of MeV/c^2 or GeV/c^2 (using $E = mc^2$)

  e.g. electron mass $m_e = 9.11 \times 10^{-31}\,\mathrm{kg} = (9.11 \times 10^{-31})(3 \times 10^8)^2\,\mathrm{J}/c^2$

  $= 8.20 \times 10^{-14}/1.602 \times 10^{-19}\ \mathrm{eV}/c^2 = 5.11 \times 10^5\ \mathrm{eV}/c^2 = 0.511\ \mathrm{MeV}/c^2$

- Atomic/nuclear masses are often quoted in unified (or atomic) mass units

  1 unified mass unit (u) = (mass of a $^{12}_{6}$C atom) / 12

  $1\,\mathrm{u} = 1\,\mathrm{g}/\mathrm{N}_A = 1.66 \times 10^{-27}\mathrm{kg} = 931.5\ \mathrm{MeV}/c^2$

- Cross-sections are usually quoted in barns: $1\mathrm{b} = 10^{-28}\,\mathrm{m}^2$.

Units *Natural Units*

Choose energy as the basic unit of measurement...		...and simplify by choosing $\hbar = c = 1$
Energy	GeV	GeV
Momentum	GeV/c	GeV
Mass	GeV/c^2	GeV
Time	$(\mathrm{GeV}/\hbar)^{-1}$	GeV^{-1}
Length	$(\mathrm{GeV}/\hbar c)^{-1}$	GeV^{-1}
Cross-section	$(\mathrm{GeV}/\hbar c)^{-2}$	GeV^{-2}

Reintroduce "missing" factors of $\hbar$ and c to convert back to SI units.

$\hbar c = 0.197\ \mathrm{GeV\,fm} = 1$ Energy $\longleftrightarrow$ Length

$\hbar = 6.6 \times 10^{-25}\ \mathrm{GeV\,s} = 1$ Energy $\longleftrightarrow$ Time

$c = 3.0 \times 10^8\,\mathrm{ms}^{-1} = 1$ Length $\longleftrightarrow$ Time

Units *Examples*

1. **cross-section $\sigma = 2 \times 10^{-6}$ GeV^{-2} change into standard units**
Need to change units of energy to length. Use $\hbar c = 0.197$ GeVfm $= 1$.

$\mathrm{GeV}^{-1} = 0.197\,\mathrm{fm}$
$\mathrm{GeV}^{-1} = 0.197 \times 10^{-15}\,\mathrm{m}$
$\mathrm{GeV}^{-2} = 3.89 \times 10^{-32}\,\mathrm{m}^2$

$$\sigma = 2 \times 10^{-6} \times (3.89 \times 10^{-32}\,\mathrm{m}^2)$$
$$= 7.76 \times 10^{-38}\,\mathrm{m}^2$$

And using $1\,\mathrm{b} = 10^{-28}\,\mathrm{m}^2$, $\sigma = 0.776\,\mathrm{nb}$

2. **lifetime $\tau = 1/\Gamma = 0.5$ GeV^{-1} change into standard units**
Need to change units of energy^{-1} to time. Use $\hbar = 6.6 \times 10^{-25}$ GeV s $= 1$.

$\mathrm{GeV}^{-1} = 6.6 \times 10^{-25}\,\mathrm{s}$

$$\tau = 0.5 \times (6.6 \times 10^{-25}\,\mathrm{s}) = 3.3 \times 10^{-25}\,\mathrm{s}$$

Also, can have Natural Units involving electric charge: $\epsilon_0 = \mu_0 = \hbar = c = 1$

3. **Fine structure constant** (dimensionless)
$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137}$ becomes $\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$ i.e. $e \sim 0.30(n.u.)$

Symmetries and conservation laws

The most elegant and powerful idea in physics
Noether's theorem:
every differentiable symmetry of the action of a physical system has a corresponding conservation law.

Symmetry	**Conserved current**
Time, t	Energy, E
Translational, x	Linear momentum, p
Rotational, θ	Angular momentum, L
Probability	Total probability always 1
Lorentz invariance	Charge Parity Time (CPT)
Gauge	charge (e.g. electric, colour, weak)

Lorentz invariance: laws of physics stay the same for all frames moving with a uniform velocity.
Gauge invariance: observable quantities unchanged (charge, E, v) when a field is transformed.

Relativistic Kinematics *Special Relativity*

Nuclear reactions
Low energy, typically K.E. $\mathcal{O}(10\ \mathrm{MeV}) \ll$ nucleon rest energies.
$\Rightarrow$ non-relativistic formulae ok
Exception: always treat β-decay relativistically
$(m_e \sim 0.5\ \mathrm{MeV} < 1.3\ \mathrm{MeV} \sim m_n - m_p)$

Particle physics
High energy, typically K.E. $\mathcal{O}(100\ \mathrm{GeV}) \gg$ rest mass energies.
$\Rightarrow$ relativistic formulae usually essential.

Relativistic Kinematics *Special Relativity*

Recall the energy E and momentum p of a particle with mass m

$$E = \gamma m, \qquad |\vec{p}| = \gamma\beta m \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c} = v$$

or $\gamma = \frac{E}{m}, \qquad \beta = \frac{|\vec{p}|}{E}$ **and these are related by** $E^2 = \vec{p}^2 + m^2$

Interesting cases

- when a particle is at rest, $\vec{p} = 0$, $E = m$,
- when a particle is massless, $m = 0$, $E = |\vec{p}|$,
- when a particle is ultra-relativistic $E \gg m$, $E \sim |\vec{p}|$.

Kinetic energy (K.E., or T) is the extra energy due to motion

$$T = E - m = (\gamma - 1)m$$

in the non-relativistic limit $\beta \ll 1$, $T = \frac{1}{2}mv^2$

Relativistic Kinematics *Four-Vectors*

The kinematics of a particle can be expressed as a four-vector, e.g.

$$p_\mu = (E, -\vec{p}), \quad p^\mu = (E, \vec{p}) \qquad \text{and} \qquad x_\mu = (t, -\vec{x}), \quad x^\mu = (t, \vec{x})$$

$\mu : 0 \to 3$

multiply by a metric tensor to raise/lower indices

$$p_\mu = g_{\mu\nu} p^\nu, \qquad p^\mu = g^{\mu\nu} p_\nu \qquad\qquad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Scalar product of two four-vectors $A^\mu = (A^0, \vec{A})$, $B^\mu = (B^0, \vec{B})$ is invariant:

$$A^\mu B_\mu = A.B = A^0 B^0 - \vec{A}.\vec{B}$$

or

$$p^\mu p_\mu = p^\mu g_{\mu\nu} p^\nu = \sum_{\mu=0,3} \sum_{\nu=0,3} p^\mu g_{\mu\nu} p^\nu = g_{00} p_0^2 + g_{11} p_1^2 + g_{22} p_2^2 + g_{33} p_3^2$$

$$= E^2 - |\vec{p}|^2 = m^2 \quad \text{invariant mass}$$

$(t, \vec{x})$ and $(E, \vec{p})$ transform between frames of reference, but

$d^2 = t^2 - \vec{x}^2$ Invariant interval is constant

$m^2 = E^2 - \vec{p}^2$ Invariant mass is constant

Relativistic Kinematics *Invariant Mass*

A common technique to identify particles is to form the invariant mass from their decay products.

Remember, for a single particle $m^2 = E^2 - \vec{p}^2$.

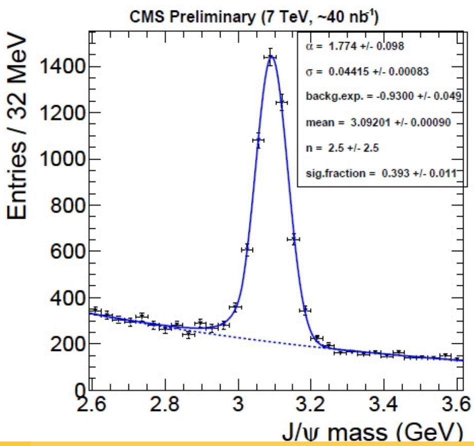
For a system of particles, where $X \to 1 + 2 + 3 \ldots n$:

$$M_X^2 = ((E_1, \vec{p}_1) + (E_2, \vec{p}_2) + \ldots)^2 = \left(\sum_{i=1}^{n} E_i\right)^2 - \left(\sum_{i=1}^{n} \vec{p}_i\right)^2$$

In the specific (and common) case of a two-body decay, $X \to 1 + 2$, this reduces to

$$M_X^2 = m_1^2 + m_2^2 + 2\left(E_1 E_2 - |\vec{p}_1||\vec{p}_2| \cos\theta\right)$$

n.b. sometimes invariant mass M is called "centre-of-mass energy" E_{CM}, or $\sqrt{s}$



Relativistic Kinematics *Decay Example*

Consider a charged pion decaying at rest in the lab frame $\pi^- \to \mu^- \bar{\nu}_\mu$

Find the momenta of the decay products

How do we study particles and forces?

- **Static Properties**
  What particles/states exist?
  Mass, spin and parity (J^P), magnetic moments, bound states
- **Particle Decays**
  Most particles and nuclei are unstable.
  Allowed/forbidden decays $\to$ Conservation Laws.
- **Particle Scattering**
  Direct production of new massive particles in matter-antimatter annihilation.
  Study of particle interaction cross-sections.
  Use high-energies to study forces at short distances.

Force	Typical Lifetime [s]	Typical cross-section [mb]
Strong	10^{-23}	10
Electromagnetic	10^{-20}	10^{-2}
Weak	10^{-8}	10^{-13}

Particle Decays *Reminder*

Most particles are transient states – only a few live forever (e^-, p, ν, γ...).

- **Number** of particles remaining at time t

$$N(t) = N(0)p(t) = N(0)e^{-\lambda t}$$

where $N(0)$ is the number at time $t = 0$.

- **Rate of decays** $\frac{dN}{dt} = -\lambda N(0)e^{-\lambda t} = -\lambda N(t)$

Assuming the nuclei only decay. More complicated if they are also being created.

- **Activity** $A(t) = \left|\frac{dN}{dt}\right| = \lambda N(t)$

- It's rather common in nuclear physics to use the **half-life** (i.e. the time over which 50% of the particles decay). In particle physics, we usually quote the mean life. They are simply related:

$$N(\tau_{1/2}) = \frac{N(0)}{2} = N(0)e^{-\lambda\tau_{1/2}} \quad \Rightarrow \quad \tau_{1/2} = \frac{\ln 2}{\lambda} = 0.693\tau$$

Particle Decays *Multiple Particle Decay*

Decay Chains frequently occur in nuclear physics

$$N_1 \xrightarrow{\lambda_1} N_2 \xrightarrow{\lambda_2} N_3 \rightarrow \cdots$$

Parent *Daughter* *Granddaughter*

e.g. $^{235}U \rightarrow {}^{231}Th \rightarrow {}^{231}Pa$
$\tau_{1/2}(^{235}U) = 7.1 \times 10^8$ years
$\tau_{1/2}(^{231}Th) = 26$ hours

Activity (i.e. rate of decay) of the daughter is $\lambda_2 N_2(t)$.
Rate of change of population of the daughter

$$\frac{dN_2(t)}{dt} = \lambda_1 N_1(t) - \lambda_2 N_2(t)$$

Units of Radioactivity are defined as the number of decays per unit time.

Becquerel (Bq) = 1 decay per second
Curie (Ci) = 3.7×10^{10} decays per second.

Particle Decays

A decay is the transition from one quantum state (initial state) to another (final or daughter).

The transition rate is given by Fermi's Golden Rule:

$$\Gamma(i \to f) = \lambda = 2\pi \left|M_{fi}\right|^2 \rho(E_f) \qquad \hbar = 1$$

where λ is the number of transitions per unit time
M_{fi} is the transition matrix element
$\rho(E_f)$ is the density of final states.

$\Rightarrow$ $\lambda\,\mathrm{d}t$ is the (constant) probability a particle will decay in time $\mathrm{d}t$.

Particle Decays *Single Particle Decay*

Let $p(t)$ be the probability that a particle still exists at time t, given that it was known to exist at $t = 0$.

Probability for particle decay in the next time interval $\mathrm{d}t$ is $= p(t)\lambda\,\mathrm{d}t$
Probability that particle survives the next is $= p(t + \mathrm{d}t) = p(t)(1 - \lambda\,\mathrm{d}t)$

$$p(t)(1 - \lambda\,\mathrm{d}t) = p(t + \mathrm{d}t) = p(t) + \frac{\mathrm{d}p}{\mathrm{d}t}\mathrm{d}t$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p(t)\lambda$$

$$\int_1^p \frac{\mathrm{d}p}{p} = -\int_0^t \lambda\,\mathrm{d}t$$

$$\Rightarrow p(t) = \mathrm{e}^{-\lambda t} \qquad \text{Exponential Decay Law}$$

Probability that a particle lives until time t and then decays in time $\mathrm{d}t$ is

$$p(t)\lambda\,\mathrm{d}t = \lambda \mathrm{e}^{-\lambda t}\,\mathrm{d}t$$

Particle Decays *Single Particle Decay*

- The average lifetime of the particle

$$\tau = \langle t \rangle = \int_0^\infty t\lambda e^{-\lambda t}\,dt = \left[-te^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}\,dt = \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda}$$

$$\tau = \frac{1}{\lambda} \qquad p(t) = e^{-t/\tau}$$

- Finite lifetime $\Rightarrow$ uncertain energy ΔE, (c.f. Resonances, Breit-Wigner)
- Decaying states do not correspond to a single energy – they have a width ΔE

$$\Delta E.\tau \sim \hbar \quad \Rightarrow \quad \Delta E \sim \frac{\hbar}{\tau} = \hbar\lambda \qquad \hbar = 1 \text{ (n.u.)}$$

- The width, ΔE, of a particle state is therefore
  - Inversely proportional to the lifetime τ
  - Proportional to the decay rate λ (or equal in natural units)

Decay of Resonances

QM description of decaying states

Consider a state formed at $t = 0$ with energy E_0 and mean lifetime τ

$$\psi(t) = \psi(0)e^{-iE_0 t}e^{-t/2\tau} \qquad |\psi(t)|^2 = |\psi(0)|^2 e^{-t/\tau}$$

i.e. the probability density decays exponentially (as required).

The frequencies (i.e. energies, since $E = \omega$ if $\hbar = 1$) present in the wavefunction are given by the Fourier transform of $\psi(t)$, i.e.

$$f(\omega) = f(E) = \int_0^\infty \psi(t)e^{iEt}\,dt = \int_0^\infty \psi(0)e^{-t(iE_0+\frac{1}{2\tau})}e^{iEt}\,dt$$

$$= \int_0^\infty \psi(0)e^{-t(i(E_0-E)+\frac{1}{2\tau})}\,dt = \frac{i\psi(0)}{(E_0-E)-\frac{i}{2\tau}}$$

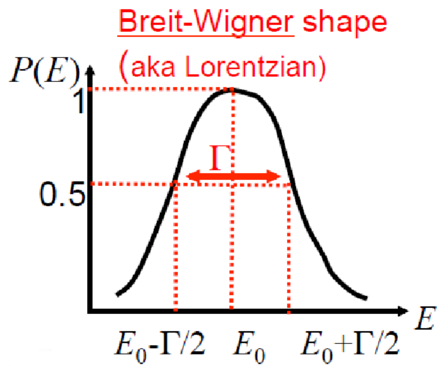
Probability of finding state with energy $E = f(E) * f(E)$ is

$$P(E) = |f(E)|^2 = \frac{|\psi(0)|^2}{(E_0-E)^2+\frac{1}{4\tau^2}}$$

Decay of Resonances *Breit-Wigner*

Probability for producing the decaying state has this energy dependence, i.e. resonant when $E = E_0$

$$P(E) \propto \frac{1}{(E_0 - E)^2 + 1/4\tau^2}$$



Consider full-width at half-maximum Γ

$$P(E = E_0) \propto 4\tau^2$$

$$P(E = E_0 \pm \frac{1}{2}\Gamma) \propto \frac{1}{(E_0 - E_0 \mp \frac{1}{2}\Gamma)^2 + 1/4\tau^2} = \frac{1}{\frac{\Gamma^2}{4} + \frac{1}{4\tau^2}}$$

$$P(E = E_0 \pm \frac{1}{2}\Gamma) = \frac{1}{2}P(E = E_0), \quad \Rightarrow \frac{1}{\frac{\Gamma^2}{4} + \frac{1}{4\tau^2}} = 2\tau^2$$

Total width (using natural units) $\Gamma = \frac{1}{\tau} = \lambda$

Partial Decay Widths

Particles can often decay with more than one decay mode
e.g. $Z \to e^+e^-$, or $\mu^+\mu^-$, or $q\bar{q}$ etc, each with its own transition rate,
i.e. from initial state i to final state f: $\lambda_f = 2\pi |M_{fi}|^2 \rho(E_f)$

The total decay rate is given by $\lambda = \sum_f \lambda_f$

This determines the average lifetime $\tau = \frac{1}{\lambda}$

The total width of a particle state $\Gamma = \lambda = \sum_f \lambda_f$

is defined by the partial widths $\Gamma_f = \lambda_f$

The proportion of decays to a particular decay mode is called the branching fraction or branching ratio $B_f = \frac{\Gamma_f}{\Gamma}, \quad \sum_f B_f = 1$

Reactions and Cross-sections

The strength of a particular reaction between two particles is specified by the interaction cross-section.

Cross-section σ – the effective target area presented to the incoming particle for it to cause the reaction.

Units: σ 1 barn (b) $= 10^{-28} m^2$ Area

σ is defined as the reaction rate per target particle Γ, per unit incident flux Φ

$$\Gamma = \Phi\sigma$$

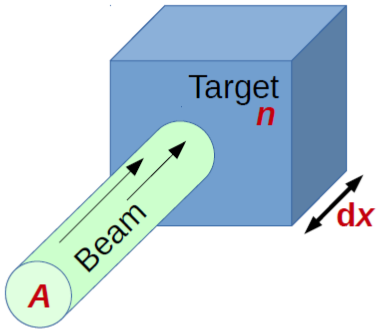
where the flux Φ is the number of beam particles passing through unit area per second.

Γ is given by Fermi's Golden Rule (previously used λ).

Scattering with a beam

Consider a beam of particles incident upon a target:

Beam of N particles per unit time in an area A



Target of n nuclei per unit volume

Target thickness $\mathrm{d}x$ is small

Number of target particles in area A, $N_T = nA\,\mathrm{d}x$
Effective area for absorption $= \sigma N_T = \sigma nA\,\mathrm{d}x$
Incident flux $\Phi = N/A$
Number of particles scattered per unit time

$$= -dN = \Phi\sigma N_T = \frac{N}{A}\sigma nA\,\mathrm{d}x$$

$$\sigma = \frac{-dN}{nN\,\mathrm{d}x}$$

Attenuation of a beam

Beam attenuation in a target of thickness L:

- Thick target $\sigma n L \gg 1$:

$$\int_{N_0}^{N} -\frac{dN}{N} = \int_0^L \sigma n \, \mathrm{d}x$$

$$N = N_0 \mathrm{e}^{-\sigma n L}$$

This is exact.

i.e. the beam attenuates *exponentially.*

- Thin target $\sigma n L << 1$, $\mathrm{e}^{-\sigma n L} \sim 1 - \sigma n L$

$$N = N_0(1 - \sigma n L)$$

Useful approximation for thin targets.

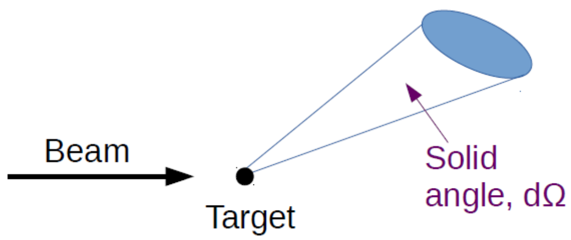
Or, the number scattered $= N_0 - N = N_0 \sigma n L$

Mean free path between interactions $= 1/n\sigma$
often referred to as "interaction length".

Differential Cross-section

The angular distribution of the scattered particles is not necessarily uniform

** n.b. $\mathrm{d}\Omega$ can be considered in position space, or momentum space **



Number of particles scattered per unit time into $\mathrm{d}\Omega$ is $\mathrm{d}N_{\mathrm{d}\Omega} = \mathrm{d}\sigma \, \Phi N_T$

Differential cross-section

units: area/steradian

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}N_{\mathrm{d}\Omega}}{(\Phi \times N_T \times \mathrm{d}\Omega)}$$

The differential cross-section is the number of particles scattered per unit time and solid angle, divided by the incident flux and by the number of target nuclei, N_T, defined by the beam area.

Most experiments do not cover 4π solid angle, and in general we measure $\mathrm{d}\sigma/\mathrm{d}\Omega$.

Angular distributions provide more information than the total cross-section about the mechanism of the interaction, e.g. angular momentum.

Partial Cross-section

Different types of interaction can occur between particles
e.g. $e^+e^- \to \gamma$, or $e^+e^- \to Z$...

$$\sigma_{\text{tot}} = \sum_i \sigma_i$$

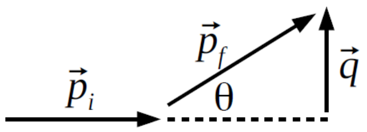
where the σ_i are called partial cross-sections for different final states.

Types of interaction

- Elastic scattering: $a + b \to a + b$
  only the momenta of a and b change
- Inelastic scattering: $a + b \to c + d$
  final state is not the same as initial state

Scattering in QM

Consider a beam of particles scattering from a fixed potential $V(r)$:



$\vec{q} = \vec{p}_f - \vec{p}_i$
"momentum transfer"

NOTE: using natural units $\vec{p} = \hbar\vec{k} \quad \to \quad \vec{p} = \vec{k}$ etc

The scattering rate is characterised by the interaction cross-section

$$\sigma = \frac{\Gamma}{\Phi} = \frac{\text{Number of particles scattered per unit time}}{\text{Incident flux}}$$

How can we calculate the cross-section?

Use Fermi's Golden Rule to get the transition rate

$$\Gamma = 2\pi |M_{fi}|^2 \rho(E_f)$$

where M_{fi} is the matrix element and $\rho(E_f)$ is the density of final states.

Scattering in QM

1[st] order Perturbation Theory using plane wave solutions of form

$$\psi = N\mathrm{e}^{-i(Et-\vec{p}.\vec{r})}$$

Require:

1. Wave-function normalisation
2. Matrix element in perturbation theory M_{fi}
3. Expression for incident flux Φ
4. Expression for density of states $\rho(E_f)$

1. **Normalisation**
   Normalise wave-functions to one particle in a box of side L:

$$|\psi|^2 = N^2 = 1/L^3$$

$$N = (1/L)^{3/2}$$

Scattering in QM

2. **Matrix Element**
   This contains the interesting physics of the interaction:

$$M_{fi} = \langle \psi_f | \hat{H} | \psi_i \rangle = \int \psi_f^* \hat{H} \psi_i \, \mathrm{d}^3\vec{r} \quad = \int N\mathrm{e}^{-i\vec{p}_f.\vec{r}} V(\vec{r}) N\mathrm{e}^{i\vec{p}_i.\vec{r}} \, \mathrm{d}^3\vec{r}$$

$$M_{fi} = \frac{1}{L^3} \int \mathrm{e}^{-i\vec{q}.\vec{r}} V(\vec{r}) \, \mathrm{d}^3\vec{r} \qquad \text{where} \quad \vec{q} = \vec{p}_f - \vec{p}_i$$

3. **Incident Flux**
   Consider a "target" of area A and a beam of particles travelling at velocity v_i towards the target. Any incident particle within a volume $v_i A$ will cross the target area every second.

$$\Phi = \frac{v_i A}{A} n = v_i n$$

where n is the number density of incident particles $= 1$ per L^3

Flux $=$ number of incident particles crossing unit area per second

$$\Phi = v_i / L^3$$

Scattering in QM

4. **Density of States** *also known as "phase space"*

For a box of side L, states are given by the periodic boundary conditions:

$$\vec{p} = (p_x, p_y, p_z) = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Each state occupies a volume $(2\pi/L)^3$ in p space (neglecting spin).

Number of states between p and $p + \mathrm{d}p$ in solid angle $\mathrm{d}\Omega$

$$\mathrm{d}N = \left(\frac{L}{2\pi}\right)^3 \mathrm{d}^3\vec{p} = \left(\frac{L}{2\pi}\right)^3 p^2\,\mathrm{d}p\,\mathrm{d}\Omega \qquad (\mathrm{d}^3\vec{p} = p^2\,\mathrm{d}p\,\mathrm{d}\Omega)$$

$$\therefore \rho(p) = \frac{\mathrm{d}N}{\mathrm{d}p} = \left(\frac{L}{2\pi}\right)^3 p^2\,\mathrm{d}\Omega$$

Density of states in energy $E^2 = p^2 + m^2 \Rightarrow 2E\,\mathrm{d}E = 2p\,\mathrm{d}p \Rightarrow \frac{\mathrm{d}E}{\mathrm{d}p} = \frac{p}{E}$

$$\rho(E) = \frac{\mathrm{d}N}{\mathrm{d}E} = \frac{\mathrm{d}N}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}E} = \left(\frac{L}{2\pi}\right)^3 p^2\frac{E}{p}\,\mathrm{d}\Omega$$

For relativistic scattering $(E \sim p)$ $\quad \rho(E) = \left(\frac{L}{2\pi}\right)^3 E^2\,\mathrm{d}\Omega$

Scattering in QM

Putting all the parts together:

$$\mathrm{d}\sigma = \frac{1}{\Phi}2\pi\,|M_{fi}|^2\,\rho(E_f) \quad = \frac{L^3}{v_i}2\pi\left|\frac{1}{L^3}\int \mathrm{e}^{-i\vec{q}.\vec{r}}V(\vec{r})\,\mathrm{d}^3\vec{r}\right|^2\left(\frac{L}{2\pi}\right)^3 p_f E_f\,\mathrm{d}\Omega$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(2\pi)^2 v_i}\left|\int \mathrm{e}^{-i\vec{q}.\vec{r}}V(\vec{r})\,\mathrm{d}^3\vec{r}\right|^2 p_f E_f$$

For relativistic scattering, $v_i = c = 1$ and $p \sim E$

Born approximation for the differential cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{E^2}{(2\pi)^2}\left|\int \mathrm{e}^{-i\vec{q}.\vec{r}}V(\vec{r})\,\mathrm{d}^3\vec{r}\right|^2$$

n.b. may have seen the *non-relativistic* version, using m^2 instead of E^2

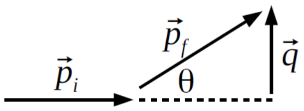
Rutherford Scattering

Consider relativistic elastic scattering in a Coulomb potential

$$V(\vec{r}) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Z\alpha}{r}$$

Special case of Yukawa potential $V = ge^{-mr}/r$ with $g = Z\alpha$ and $m = 0$ (see Appendix C)

$$|M_{if}|^2 = \frac{16\pi^2 Z^2\alpha^2}{q^4}$$



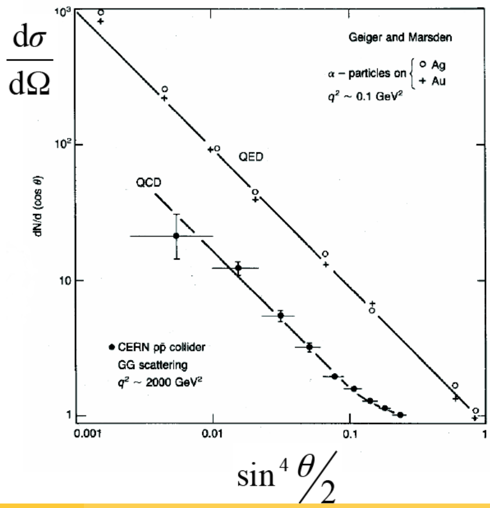
$$\vec{q} = \vec{p}_f - \vec{p}_i$$

$$|\vec{q}|^2 = |\vec{p}_i|^2 + |\vec{p}_f|^2 - 2\vec{p}_i.\vec{p}_f$$

elastic scattering, $|\vec{p}_i| = |\vec{p}_f| = |\vec{p}|$

$$= 2|\vec{p}|^2(1-\cos\theta) = 4E^2\sin^2\frac{\theta}{2}$$

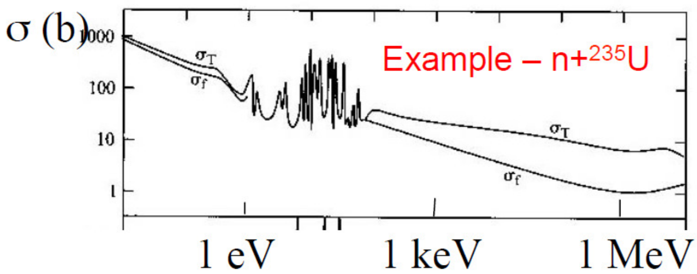
$$\frac{d\sigma}{d\Omega} = \frac{4E^2Z^2\alpha^2}{q^4} = \frac{4E^2Z^2\alpha^2}{16E^4\sin^4\frac{\theta}{2}} = \frac{Z^2\alpha^2}{4E^2\sin^4\frac{\theta}{2}}$$



Cross-section for Resonant Scattering

Some particle interactions take place via an intermediate resonant state which then decays

$$a + b \to Z^* \to c + d$$



Two-stage picture: (Bohr Model)

Formation $a + b \to Z^*$

Occurs when the collision energy $E_{CM} \sim$ the natural frequency (i.e. mass) of a resonant state.

Decay $Z^* \to c + d$

The decay of the resonance Z^* is independent of the mode of formation and depends only on the properties of the Z^*.
May be multiple decay modes.

Resonance Cross-Section

The resonance cross-section is given by

$$\sigma = \frac{\Gamma}{\Phi} \quad \text{with} \quad \Gamma = 2\pi \left|M_{fi}\right|^2 \rho(E_f)$$

$$d\sigma = \frac{1}{\Phi} 2\pi \left|M_{fi}\right|^2 \rho(E_f) \quad ^{**}$$

$$= \frac{L^3}{v_i} 2\pi \left|M_{fi}\right|^2 \frac{p_f^2 L^3}{v_f (2\pi)^3} d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{p_f^2}{(2\pi)^2 v_i v_f} \left|M_{fi}\right|^2$$

factors of L cancel as before, $M \propto 1/L^3$

** same as Born Approx.

incident flux $\Phi = \frac{v_i}{L^3}$

density of states $\rho(p) = \frac{dN}{dp} = \left(\frac{L}{2\pi}\right)^3 p^2 \, d\Omega$

Only need to account for $\rho(E)$ of one particle. Energy conservation fixes the other.

$$\rightarrow \rho(E) = \frac{dN}{dp}\frac{dp}{dE} = \left(\frac{L}{2\pi}\right)^3 p^2 \frac{E}{p} d\Omega$$

$$= \left(\frac{L}{2\pi}\right)^3 p^2 \frac{1}{v} d\Omega$$

using $\beta = v/c = p/E$

The matrix element M_{fi} is given by 2^{nd} order Perturbation Theory

$$M_{fi} = \sum_Z \frac{M_{iZ} M_{Zf}}{E - E_Z}$$

n.b. 2^{nd} order effects are large since $E - E_Z$ is small $\rightarrow$ large perturbation

where the sum runs over all intermediate states.

Near resonance, effectively only one state Z contributes.

Resonance Cross-Section

Consider one intermediate state described by

$$\psi(t) = \psi(0) e^{-iE_0 t} e^{-t/2\tau} = \psi(0) e^{-i\left(E_0 - i\frac{\Gamma}{2}\right)t}$$

this describes a states with energy $= E_0 - i\Gamma/2$

$$\left|M_{fi}\right|^2 = \frac{\left|M_{iZ}\right|^2 \left|M_{Zf}\right|^2}{(E - E_0)^2 + \frac{\Gamma^2}{4}}$$

Rate of decay of Z:

$$\Gamma_{Z\rightarrow f} = 2\pi \left|M_{Zf}\right|^2 \rho(E_f) = 2\pi \left|M_{Zf}\right|^2 \frac{4\pi p_f^2}{(2\pi)^3 v_f} = \left|M_{Zf}\right|^2 \frac{p_f^2}{\pi v_f}$$

Rate of formation of Z:

$$\Gamma_{i\rightarrow Z} = 2\pi \left|M_{iZ}\right|^2 \rho(E_i) = 2\pi \left|M_{iZ}\right|^2 \frac{4\pi p_i^2}{(2\pi)^3 v_i} = \left|M_{iZ}\right|^2 \frac{p_i^2}{\pi v_i}$$

nb. $\left|M_{Zi}\right|^2 = \left|M_{iZ}\right|^2$.

Hence M_{iZ} and M_{Zf} can be expressed in terms of partial widths.

Resonance Cross-Section

Putting everything together: $$\frac{d\sigma}{d\Omega} = \frac{p_f^2}{(2\pi)^2 v_i v_f} |M_{fi}|^2$$

$$\Rightarrow \sigma = \frac{4\pi p_f^2}{(2\pi)^2 v_i v_f} \frac{\pi v_f}{p_f^2} \frac{\pi v_i}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E-E_0)^2 + \frac{\Gamma^2}{4}} \quad = \frac{\pi}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

We need to include one more piece of information to account for spin...

Resonance Cross-Section

Breit-Wigner Cross-Section $$\sigma = \frac{\pi g}{p_i^2} . \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

The g factor takes into account the spin

$$\mathrm{a} + \mathrm{b} \to \mathrm{Z}^* \to \mathrm{c} + \mathrm{d}, \qquad g = \frac{2J_Z + 1}{(2J_\mathrm{a} + 1)(2J_\mathrm{b} + 1)}$$

and is the ratio of the number of spin states for the resonant state to the total number of spin states for the a+b system,
i.e. the probability that a+b collide in the correct spin state to form Z^*.

Useful points to remember:

- p_i is calculated in the centre-of-mass frame (σ is independent of frame of reference!)
- $p_i \sim$ lab momentum if the target is heavy (often true in nuclear physics, but not in particle physics).
- E is the total energy (if two particles colliding, $E = E_1 + E_2$)
- Γ is the total decay rate
- $\Gamma_{Z\to i}$ and $\Gamma_{Z\to f}$ are the partial decay rates

Resonance Cross-Section *Notes*

- Total cross-section $\sigma_{\text{tot}} = \sum_f \sigma(i \to f)$

  $\sigma = \frac{\pi g}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$

  Replace Γ_f by Γ in the Breit-Wigner formula

- Elastic cross-section $\sigma_{\text{el}} = \sigma(i \to i)$

  so, $\Gamma_f = \Gamma_i$

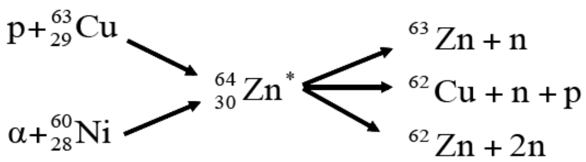
- On peak of resonance $(E = E_0)$ $\sigma_{\text{peak}} = \frac{4\pi g \Gamma_i \Gamma_f}{p_i^2 \Gamma^2}$

  Thus $\sigma_{\text{el}} = \frac{4\pi g B_i^2}{p_i^2}, \quad \sigma_{\text{tot}} = \frac{4\pi g B_i}{p_i^2}, \quad B_i = \frac{\Gamma_i}{\Gamma} = \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}}$

By measuring σ_{tot} and σ_{el}, can cancel B_i and infer g and hence the spin of the resonant state.

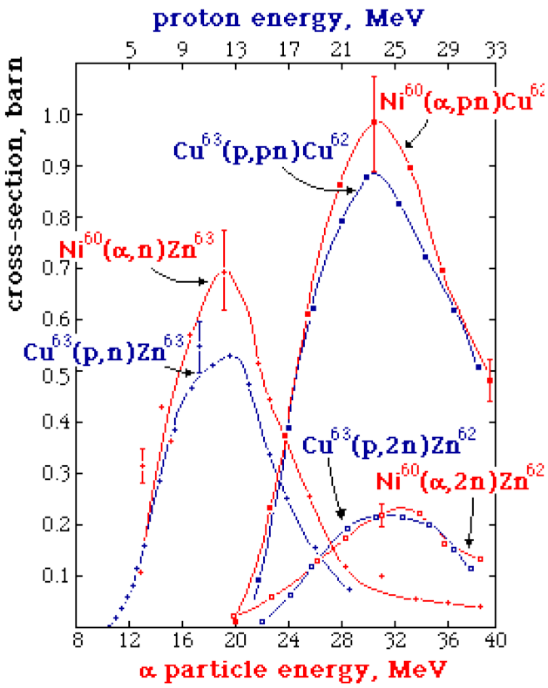
Resonances *Nuclear Physics Example*

Can produce the same resonance from different initial states, decaying into various final states, e.g.



$\sigma[^{60}\text{Ni}(\alpha, n)^{63}Zn] \sim \sigma[^{63}\text{Cu}(p, n)^{63}Zn]$

n.b. common notation for nuclear reactions:
a+A → b+B ≡ A(a,b)B



Energy of p selected to give same c.m. energy as for α interaction.

Resonances *Particle Physics Example*

The Z boson

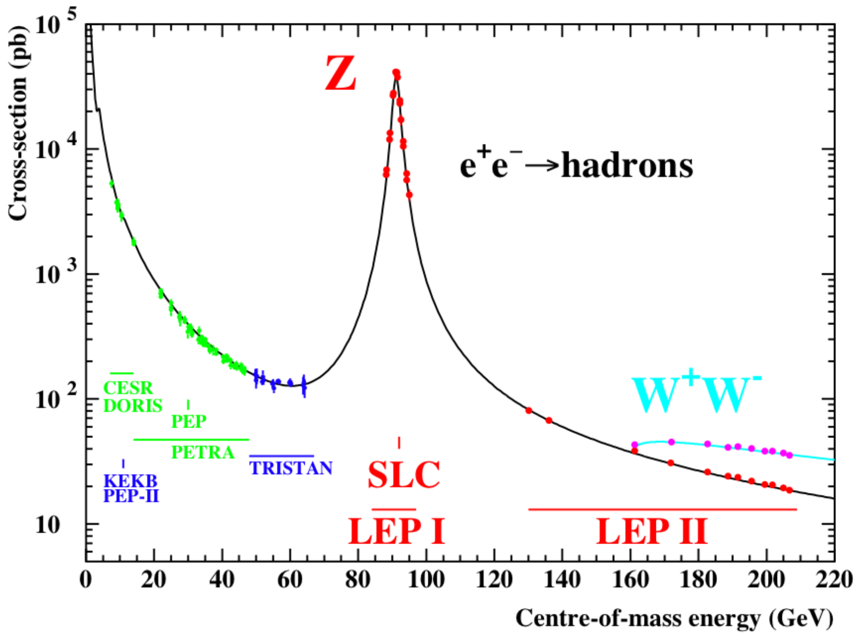
$$\Gamma_Z \sim 2.5 \text{ GeV}$$

$$\tau = \frac{1}{\Gamma_Z} = 0.4 \text{ GeV}^{-1}$$

$$= 0.4 \times \hbar$$

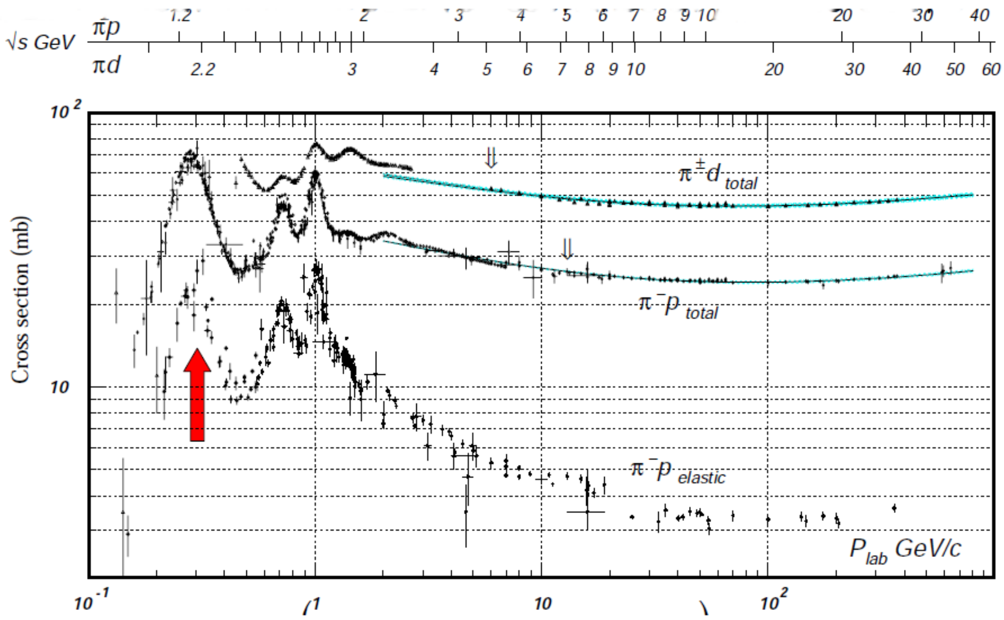
$$= 2.5 \times 10^{-25} \text{ s}$$

$(\hbar = 6.6 \times 10^{-25} \text{ GeV s})$



Resonances *$\pi^- p$ scattering example*

Resonance observed at $p_\pi \sim 0.3$ GeV, $E_{CM} \sim 1.25$ GeV



$$\sigma_{\text{total}} = \sigma(\pi^- p \to R \to \text{anything}) \sim 72 \,\text{mb}$$

$$\sigma_{\text{elastic}} = \sigma(\pi^- p \to R \to \pi^- p) \sim 28 \,\text{mb}$$

Resonances $\pi^- p$ *scattering example*

Summary

- Units: MeV, GeV, barns
- Natural units: $\hbar = c = 1$
- Relativistic kinematics: most particle physics calculations require this!
- Revision of scattering theory: cross-section, Born Approximation.
- Resonant scattering
- Breit-Wigner formula (important in both nuclear and particle physics):

$$\sigma = \frac{\pi g}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E - E_0)^2 + \frac{\Gamma^2}{4}}$$

- Measure total and elastic σ to measure spin of resonance.

Problem Sheet: q.2-6

Up next...
Section 3: Colliders and Detectors

APPENDIX A: PHYSICAL CONSTANTS

Summary of the physical constants and conversion factors used in this course:

Electron charge, $e = 1.602 \times 10^{-19}$ C
$\hbar c = 0.197$ GeV fm
$\hbar = 6.58 \times 10^{-25}$ GeV s
Fine structure constant, $\alpha = 1/137.04$
Bohr magneton, $\mu_B = 9.3 \times 10^{-24}$ JT^{-1}
Nuclear magneton, $\mu_N = 5.1 \times 10^{-27}$ JT^{-1}

1 eV $= 1.602 \times 10^{-19}$ J, 1 MeV $= 10^6$ eV, 1 GeV $= 10^9$ eV
1 fermi(fm) $= 10^{-15}$ m
1 barn(b) $= 10^{-28}$ m^2
1 Curie(Ci) $= 3.7 \times 10^{10}$ decays s^{-1}

Atomic masses are often given in unified (or atomic) mass units:
1 unified mass unit(u) = Mass of an atom of $^{12}_{6}$C/12
1u = 1g/N$_A$ $= 1.66 \times 10^{-27}$ kg $= 931.5$ MeV/c^2

APPENDIX B: PARTICLE PROPERTIES

From the *Review of Particle Physics*, C. Amsler *et al.*, Phys. Lett. **B667** 1 (2008)
http://pdg.lbl.gov/

Quarks (spin 1/2)			
Name	Flavour	Mass (GeV/c^2)	Charge (e)
up	u	≈ 0.35	$+2/3$
down	d	$m_d \approx m_u$	$-1/3$
charm	c	1.5	$+2/3$
strange	s	0.5	$-1/3$
top	t	171.2(2.1)	$+2/3$
bottom	b	4.5	$-1/3$

Leptons (spin 1/2)					
Lepton	Charge	Mass (MeV/c^2)	Mean life (s)	Lepton Decay Mode	Branching Fraction (%)
ν_e	0	$< 2\,eV/c^2$	stable		
ν_μ	0	< 0.19	stable		
ν_τ	0	< 18.2	stable		
e	± 1	0.511^a	stable		
μ	± 1	105.658^b	$2.197 \times 10^{-6^c}$	$e^-\bar{\nu}_e\nu_\mu$	≈ 100
τ	± 1	1776.8(2)	$291(1) \times 10^{-15}$	$\mu^-\bar{\nu}_\mu\nu_\tau$	17.36(5)
				$e^-\bar{\nu}_e\nu_\tau$	17.85(5)
				hadrons $+\nu_\tau$	≈ 65

[a] The error on the e mass is $1.3 \times 10^{-8}\,MeV/c^2$.
[b] The error on the μ mass is $4 \times 10^{-6}\,MeV/c^2$.
[c] The error on the μ lifetime is $2 \times 10^{-11}\,s$.

N.B. Numbers given in brackets correspond to the error in the last digit(s).
For example, $m_\tau = 1776.8(2) MeV/c^2 \equiv (1776.8 \pm 0.2) MeV/c^2$.

Gauge Bosons ($J^{\mathrm{P}} = 1^-$)						
Force	Gauge Boson	Charge (e)	Mass (GeV/c^2)	Full Width (GeV)	Decay Mode	Branching Fraction (%)
E-M	γ	$< 5 \times 10^{-30}$	$< 10^{-18}$ eV/c^2	stable		
Weak (Charged)	$\mathrm{W}^{\pm}$	± 1	80.40(3)	2.14(4)	$e\nu_e$	10.7(2)
					$\mu\nu_\mu$	10.6(2)
					$\tau\nu_\tau$	11.2(2)
					hadrons	67.6(3)
Weak (Neutral)	Z^0	0	91.188(2)	2.495(2)	ee	3.363(4)
					$\mu\mu$	3.366(7)
					$\tau\tau$	3.370(8)
					$\nu\nu$	20.00(6)
					hadrons	69.91(6)
Strong	g	0	0	stable		

Pseudoscalar Mesons ($J^{\mathrm{P}} = 0^-$)					
Particle	Quark Content	Mass (MeV/c^2)	Mean Life (s) or Width (keV)	Decay Mode	Branching Fraction (%)
$\pi^\pm$	$\mathrm{u\bar{d}, d\bar{u}}$	139.5702(4)	$2.6033(5) \times 10^{-8}$ s	$\mu^-\bar{\nu}_\mu$	≈ 100
π^0	$(\mathrm{u\bar{u} - d\bar{d}})/\sqrt{2}$	134.9766(6)	$8.4(6) \times 10^{-17}$ s	$\gamma\gamma$	98.80(3)
η	see note a	547.85(2)	1.30(7) keV	$\gamma\gamma$	39.3(2)
				$\pi^0\pi^0\pi^0$	32.6(2)
				$\pi^+\pi^-\pi^0$	22.7(3)
				$\pi^+\pi^-\gamma$	4.6(2)
η'	see note a	957.7(2)	0.20(2) MeV	$\pi^+\pi^-\eta$	45(2)
				$\rho^0\gamma$	29(1)
				$\pi^0\pi^0\eta$	21(1)
$\mathrm{K}^\pm$	$\mathrm{u\bar{s}, s\bar{u}}$	493.677(16)	$1.238(2) \times 10^{-8}$ s	$\mu^-\bar{\nu}_\mu$	63.5(1)
				$\pi^-\pi^0$	20.7(1)
				$\pi^+\pi^-\pi^-$	5.59(4)
				$\pi^0\mu^-\bar{\nu}_\mu$	3.35(4)
				$\pi^0e^-\bar{\nu}_e$	5.08(5)
$\mathrm{K}^0, \overline{\mathrm{K}^0}$	$\mathrm{d\bar{s}, s\bar{d}}$	497.61(2)	K^0_S: $0.8953(5) \times 10^{-10}$ s	$\pi^+\pi^-$	69.2(1)
				$\pi^0\pi^0$	30.7(1)
			K^0_L: $5.12(2) \times 10^{-8}$ s	$\pi^0\pi^0\pi^0$	19.5(1)
				$\pi^+\pi^-\pi^0$	12.5(1)
				$\pi^\pm\mu^\mp\nu_\mu$	27.0(1)
				$\pi^\pm e^\mp\nu_e$	40.5(1)
$\mathrm{D}^\pm$	$\mathrm{c\bar{d}, d\bar{c}}$	1869.3(4)	$1.040(7) \times 10^{-12}$ s	e^- + any[b]	16.0(4)
				K^- + any	26(1)
				K^+ + any	5.9(8)
				K^0 + any	
				plus	
				$\overline{\mathrm{K}^0}$ + any	61(5)
$\mathrm{D}^0, \overline{\mathrm{D}^0}$	$\mathrm{u\bar{c}, c\bar{u}}$	1864.8(2)	$0.410(2) \times 10^{-12}$ s	K^- + any[c]	55(3)
				K^+ + any	3.4(4)
				e^+ + any	6.5(2)
				μ^+ + any	6.7(6)
				$\overline{\mathrm{K}^0}$ + any	
				plus	
				K^0 + any	47(4)
$\mathrm{D}^\pm_\mathrm{s}$	$\mathrm{c\bar{s}, s\bar{c}}$	1968.5(3)	$0.500(7) \times 10^{-12}$ s	seen	
$\mathrm{B}^\pm$	$\mathrm{u\bar{b}, b\bar{u}}$	5279.1(3)	$1.64(1) \times 10^{-12}$ s	many	
$\mathrm{B}^0, \overline{\mathrm{B}^0}$	$\mathrm{d\bar{b}, b\bar{d}}$	5279.5(3)	$1.53(1) \times 10^{-12}$ s	many	
$\mathrm{B}^0_\mathrm{s}, \overline{\mathrm{B}^0_\mathrm{s}}$	$\mathrm{s\bar{b}, b\bar{s}}$	5366.3(6)	$1.47(3) \times 10^{-12}$ s	many	
$\mathrm{B}^\pm_\mathrm{c}$	$\mathrm{c\bar{b}, b\bar{c}}$	6276(4)	$0.46(18) \times 10^{-12}$ s	many	
η_c	$\mathrm{c\bar{c}}$	2980(1)	27(3) MeV	hadrons	

[a] η and η' are linear combinations of the quark state $(\mathrm{u\bar{u} + d\bar{d}})/\sqrt{2}$ and $\mathrm{s\bar{s}}$ (see lectures).
[b] D^- decay modes; [c] D^0 decay modes.

Vector Mesons ($J^{\mathrm{P}} = 1^-$)					
Particle	Quark Content	Mass (MeV/c^2)	Full Width (MeV)	Decay Mode	Branching Fraction (%)
$\rho^{\pm}$	$\mathrm{u\bar{d}, d\bar{u}}$	775.5(4)	149(1)	$\pi\pi$	100
ρ^0	$(\mathrm{u\bar{u} - d\bar{d}})/\sqrt{2}$				
ω	$(\mathrm{u\bar{u} + d\bar{d}})/\sqrt{2}$	782.6(1)	8.49(8)	$\pi^+\pi^-\pi^0$	89.2(7)
				$\pi^0\gamma$	8.9(2)
				$\pi^+\pi^-$	1.5(1)
ϕ	$\mathrm{s\bar{s}}$	1019.46(2)	4.26(4)	$\mathrm{K^+K^-}$	49.2(6)
				$\mathrm{K^0_L K^0_S}$	34.0(5)
$\mathrm{K}^{*\pm}$	$\mathrm{u\bar{s}, s\bar{u}}$	891.7(3)	50.8(9)	$\mathrm{K}\pi$	≈ 100
$\mathrm{K}^{*0}, \overline{\mathrm{K}^{*0}}$	$\mathrm{d\bar{s}, s\bar{d}}$	896.0(3)	50.3(6)	$\mathrm{K}\pi$	≈ 100
$\mathrm{D}^{*\pm}$	$\mathrm{c\bar{d}, d\bar{c}}$	2010.3(2)	0.096(22)	$\mathrm{D}^0\pi^{-a}$	67.7(5)
				$\mathrm{D}^-\pi^0$	30.7(5)
$\mathrm{D}^{*0}, \overline{\mathrm{D}^{*0}}$	$\mathrm{u\bar{c}, c\bar{u}}$	2007.0(2)	< 2.1	$\mathrm{D}^0\pi^{0b}$	62(3)
				$\mathrm{D}^0\gamma$	38(3)
$\mathrm{D}_{\mathrm{s}}^{*\pm}$	$\mathrm{c\bar{s}, s\bar{c}}$	2112.3(5)	< 1.9	$\mathrm{D}_{\mathrm{s}}^{\pm}\gamma$	94(1)
				$\mathrm{D}_{\mathrm{s}}^{\pm}\pi^0$	6(1)
B^*	$\mathrm{u\bar{b}, b\bar{u}, d\bar{b}, b\bar{d}, s\bar{b}, b\bar{s}}$	5325.1(5)		$\mathrm{B}\gamma$ dominant	
J/ψ	$\mathrm{c\bar{c}}$	3096.92(1)	93(2) keV	hadrons	87.7(5)
				e^+e^-	5.9(1)
				$\mu^+\mu^-$	5.9(1)
$\Upsilon(1\mathrm{s})$	$\mathrm{b\bar{b}}$	9460.3(3)	54(1) keV	$\tau^+\tau^-$	2.6(1)
				e^+e^-	2.4(1)
				$\mu^+\mu^-$	2.48(5)

[a] D^{*-} decay modes; [b] D^{*0} decay modes.

Baryons ($J^{\mathrm{P}} = 1/2^+$)					
Particle	Quark Content	Mass (MeV/c^2)	Mean Life (s) or Full Width (MeV)	Decay Mode	Branching Fraction (%)
p	uud	938.27203(8)	$> 2.1 \times 10^{29}$ years		
n	udd	939.56536(8)	885.7(8) s	$pe^-\bar{\nu}_e$	100
Λ^0	uds	1115.683(6)	$2.63(2) \times 10^{-10}$ s	$p\pi^-$	63.9(5)
				$n\pi^0$	35.8(5)
Σ^+	uus	1189.37(7)	$0.802(3) \times 10^{-10}$ s	$p\pi^0$	51.6(3)
				$n\pi^+$	48.3(3)
Σ^0	uds	1192.64(2)	$7.4(7) \times 10^{-20}$ s	$\Lambda^0\gamma$	100
Σ^-	dds	1197.45(3)	$1.48(1) \times 10^{-10}$ s	$n\pi^-$	99.848(5)
Ξ^0	uss	1314.8(2)	$2.90(9) \times 10^{-10}$ s	$\Lambda^0\pi^0$	99.52(1)
Ξ^-	dss	1321.7(1)	$1.64(2) \times 10^{-10}$ s	$\Lambda^0\pi^-$	99.89(4)
Λ_c^+	udc	2286.5(1)	$2.00(6) \times 10^{-13}$ s	many	
Λ_b	udb	5620(2)	$1.38(5) \times 10^{-12}$ s	many	
Baryons ($J^{\mathrm{P}} = 3/2^+$)					
Δ	uuu, uud, udd, ddd	≈ 1232	≈ 118 MeV	Nπ	> 99
Σ^*	uus, uds, dds	≈ 1385	≈ 36 MeV	$\Lambda^0\pi$	87(2)
				$\Sigma\pi$	12(2)
Ξ^*	uss, dss	≈ 1533	≈ 9 MeV	$\Xi\pi$	100
Ω^-	sss	1672.5(3)	$0.82(1) \times 10^{-10}$ s	Λ^0K$^-$	67.8(7)
				$\Xi^0\pi^-$	23.6(7)
				$\Xi^-\pi^0$	8.6(4)

APPENDIX C: Scattering from a Yukawa potential

Consider relativistc elastic scattering from a Yukawa potential

$$V(\vec{r}) = \frac{g\,\mathrm{e}^{-mr}}{r}$$

The matrix element is given by

$$|M_{if}|^2 = \left| \int \mathrm{e}^{-\mathrm{i}\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r} \right|^2$$

In order to perform the integral, choose the z axis to lie along $\vec{r}$. Then $\vec{q}.\vec{r} = -qr\cos\theta$ and

$$\begin{aligned}
\int \mathrm{e}^{-\mathrm{i}\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r} &= \int_0^\infty \int_0^{2\pi} \int_0^\pi V(r)\, \mathrm{e}^{iqr\cos\theta}\, r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}r \\
&= \int_0^\infty \int_{-1}^{+1} 2\pi V(r)\, \mathrm{e}^{iqr\cos\theta}\, r^2\, \mathrm{d}(\cos\theta)\, \mathrm{d}r \\
&= \int_0^\infty 2\pi V(r) \left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iqr} \right) r^2\, \mathrm{d}r \\
&= \int_0^\infty 2\pi g \frac{\mathrm{e}^{-mr}}{r} \left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iqr} \right) r^2\, \mathrm{d}r \\
&= \int_0^\infty 2\pi g\, \mathrm{e}^{-mr} \left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iq} \right) \mathrm{d}r \\
&= \int_0^\infty \frac{2\pi g}{iq} \left(\mathrm{e}^{-r(m-iq)} - \mathrm{e}^{-r(m+iq)} \right) \mathrm{d}r \\
&= \frac{2\pi g}{iq} \left(\frac{1}{m - \mathrm{i}q} - \frac{1}{m + \mathrm{i}q} \right) = \frac{2\pi g}{iq} \frac{2iq}{m^2 + q^2} \\
&= \frac{4\pi g}{m^2 + q^2}
\end{aligned}$$

The matrix element is then

$$|M_{if}|^2 = \frac{16\pi^2 g^2}{(m^2 + q^2)^2}$$

The Yukawa potential is a general potential, and can be extended to other potentials, e.g. for the Coulomb potential

$$V(\vec{r}) = -\frac{Z\alpha}{r}$$

using $g = Z\alpha$ and $m = 0$, the matrix element for Rutherford Scattering is

$$|M_{if}|^2 = \frac{16\pi^2 Z^2 \alpha^2}{q^4}$$

Appendix D: Interaction via Particle Exchange

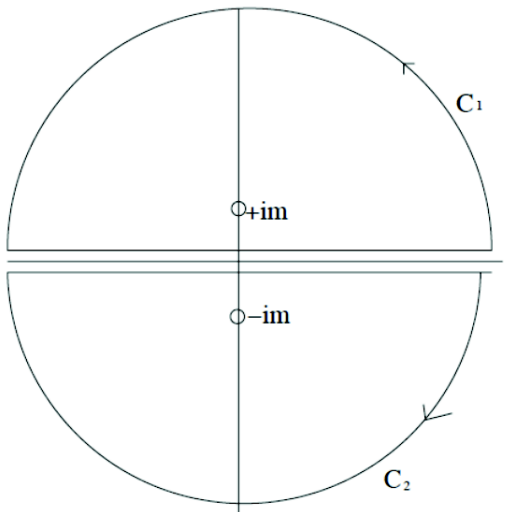
We need to evaluate the following integral in order to determine the energy shift when in state i when a particle of mass m is exchanged between particle 1 and particle 2,

$$\Delta E_i^{1\to 2} = -\frac{g^2}{2(2\pi)^2}\int_0^\infty \frac{p^2}{p^2+m^2}\frac{\mathrm{e}^{\mathrm{i}pr}-\mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}pr}dp$$

Start by rewriting

$$\Delta E_i^{1\to 2} = -\frac{1}{2}\frac{g^2}{2(2\pi)^2}\int_{-\infty}^\infty \frac{p}{p^2+m^2}\frac{\mathrm{e}^{\mathrm{i}pr}-\mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}r}dp$$

using the fact that the integrand is even in p. The integrand has poles at $p = \pm\mathrm{i}m$ (see the figure). The integrals with the $\mathrm{e}^{\mathrm{i}pr}$ and $\mathrm{e}^{-\mathrm{i}pr}$ terms are performed separately. This is because one chooses an infinite semi-circular contour to do the integration over, in such a way that on the circular piece the contribution from infinity vanishes. This happens if the integrand contains a decaying exponential in $|p|$. For $\mathrm{e}^{\mathrm{i}pr}$, this happens for $p = +\mathrm{i}|p|$ and so one closes the contour in the upper half plane (C_1 in the figure). For $\mathrm{e}^{-\mathrm{i}pr}$, we want $p = -\mathrm{i}|p|$, and so close the contour in the lower half plane (C_2 in the figure).



The whole integral is thus:

$$-\frac{g^2}{2(2\pi)^2}\left[\oint_{C_1}\frac{p}{p^2+m^2}\frac{\mathrm{e}^{\mathrm{i}pr}}{\mathrm{i}r}dp - \oint_{C_2}\frac{p}{p^2+m^2}\frac{\mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}r}dp\right].$$

The residue of the pole at $p = \mathrm{i}m$ in the first integrand is:

$$\lim_{p\to \mathrm{i}m}\frac{(p-\mathrm{i}m)}{(p-\mathrm{i}m)(p+\mathrm{i}m)}\frac{p}{\mathrm{i}r}\mathrm{e}^{\mathrm{i}pr} = \frac{1}{2\mathrm{i}r}\,\mathrm{e}^{-mr}$$

and the residue of the pole at $p = -\mathrm{i}m$ in the second integrand is:

$$\lim_{p\to -\mathrm{i}m}\frac{(p+\mathrm{i}m)}{(p-\mathrm{i}m)(p+\mathrm{i}m)}\frac{-p\,\mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}r} = -\frac{1}{2\mathrm{i}r}\,\mathrm{e}^{-mr}\,.$$

Cauchy's residue theorem tells us that the contour integral over an anti-clockwise contour is $2\pi\mathrm{i}$ multiplied by the sum of the residues of the poles enclosed by the contour. For a clockwise contour, there is an additional minus sign. Noting that C_1 is anti-clockwise, and C_2 is clockwise, one has:

$$\begin{aligned} \Delta E_i^{1\to 2} &= -\frac{g^2}{2(2\pi)^2} 2\pi\mathrm{i} \left[\frac{\mathrm{e}^{-mr}}{2\mathrm{i}r} + \frac{\mathrm{e}^{-mr}}{2\mathrm{i}r} \right] \\ &= \underline{-\frac{g^2}{8\pi}\frac{\mathrm{e}^{-mr}}{r}} \end{aligned}$$

as given in the notes.

APPENDIX E: LOCAL GAUGE INVARIANCE IN QED

Consider a non-relativistic charged particle in an electromagnetic field:

$$\underline{F} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

where $\underline{E}$ and $\underline{B}$ can be written in terms of the vector and scalar potentials, $\underline{A}$ and ϕ:

$$\underline{B} = \underline{\nabla} \times \underline{A} \;\text{ and }\; \underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}.$$

The classical Hamiltonian,

$$\underline{H} = \frac{1}{2m}\left(\underline{p} - q\underline{A}\right)^2 + q\phi,$$

can be used along with Schrödinger's equation to obtain

$$H\psi = \left[\frac{1}{2m}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2 + q\phi\right]\psi(\underline{x},t) = \mathrm{i}\frac{\partial\psi}{\partial t}(\underline{x},t). \tag{1}$$

where we have substituted $\underline{p} \to -\mathrm{i}\underline{\nabla}$. We now need to show that Schrödinger's equation is invariant under the local guage transformation

$$\begin{aligned} \psi \to \psi' &= e^{\mathrm{i}q\alpha(\underline{x},t)}\psi \\ \underline{A} \to \underline{A}' &= \underline{A} + \underline{\nabla}\alpha \\ \phi \to \phi' &= \phi - \frac{\partial\alpha}{\partial t} \end{aligned}$$

Substituting for ψ', $\underline{A}'$ and ϕ' in equation (1):

$$\left[\frac{1}{2m}\left(-\mathrm{i}\underline{\nabla} - q(\underline{A} + \underline{\nabla}\alpha)\right)^2 + q(\phi - \frac{\partial\alpha}{\partial t})\right]e^{\mathrm{i}q\alpha}\psi = \mathrm{i}\frac{\partial}{\partial t}(e^{\mathrm{i}q\alpha}\psi)$$

$$\left[\frac{1}{2m}\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2 + q\phi - q\frac{\partial\alpha}{\partial t}\right]e^{\mathrm{i}q\alpha}\psi = \mathrm{i}\left(e^{\mathrm{i}q\alpha}\frac{\partial\psi}{\partial t} + \mathrm{i}q\psi\frac{\partial\alpha}{\partial t}e^{\mathrm{i}q\alpha}\right).$$

The last terms on either side of the above equation cancel.
Now consider the $\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2$ term. In order to show local gauge invariance, we need to show that

$$\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2 e^{\mathrm{i}q\alpha}\psi = \left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2 e^{\mathrm{i}q\alpha}\psi$$

or, equivalently,

$$\left(-\mathrm{i}\underline{\nabla} - q\underline{A}'\right)^2\psi' = \left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2 e^{\mathrm{i}q\alpha}\psi.$$

Now,

$$\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)^2 e^{\mathrm{i}q\alpha}\psi = \left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)\cdot\left(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha\right)e^{\mathrm{i}q\alpha}\psi$$

and

$$\underline{\nabla}\left(e^{\mathrm{i}q\alpha}\psi\right) = e^{\mathrm{i}q\alpha}\left(\underline{\nabla} + \mathrm{i}q\underline{\nabla}\alpha\right)\psi.$$

Therefore,

$$\begin{aligned}(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha)\, e^{\mathrm{i}q\alpha}\psi &= e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} + q\underline{\nabla}\alpha - q\underline{A} - q\underline{\nabla}\alpha\right)\psi \\ &= e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)\psi\end{aligned}$$

and

$$\begin{aligned}(-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha)^2\,\psi' &= (-\mathrm{i}\underline{\nabla} - q\underline{A} - q\underline{\nabla}\alpha)\, e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)\psi \\ &= e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2\psi.\end{aligned}$$

Hence,

$$\underline{\left(-\mathrm{i}\underline{\nabla} - q\underline{A}'\right)^2\psi' = e^{\mathrm{i}q\alpha}\left(-\mathrm{i}\underline{\nabla} - q\underline{A}\right)^2\psi}$$

and Schrödinger's equation is invariant under a local gauge transformation.

APPENDIX F: NEUTRINO SCATTERING IN FERMI THEORY

Calculation of the cross-section for $\nu_e + n \rightarrow p + e^-$ using Fermi theory. The cross-section is given by Fermi's Golden Rule

$$\Gamma = 2\pi \left|M_{fi}\right|^2 \rho\left(E_f\right)$$

where the matrix element, M_{fi}, is given by the 4-point interaction with a strength equal to the Fermi constant, G_F;

$$\left|M_{fi}\right|^2 \approx G_F^2.$$

There are a total of 4 possible spin states for the spin-$\frac{1}{2}$ e and ν. These correspond to a singlet state $S = 0$ (Fermi transition) and three triplet states $S = 1$ (Gamow-Teller transition). Therefore, the matrix element becomes

$$\left|M_{fi}\right|^2 \approx 4G_F^2.$$

The differential cross-section is then given by,

$$d\sigma = 2\pi 4G_F^2 \frac{E_e^2}{(2\pi)^3} d\Omega$$

where E_e is the energy of the electron in the zero-momentum frame. It follows that

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 E_e^2}{\pi^2}.$$

The total energy in the zero-momentum frame, $\sqrt{s} = 2E_e$. Hence, the total cross-section can be written as

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{4G_F^2 E_e^2}{\pi} = \underline{\frac{G_F^2 s}{\pi}}.$$

APPENDIX G: NEUTRINO SCATTERING WITH A MASSIVE W BOSON

From Appendix F, the differential cross-section in Fermi theory is

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 E_e^2}{\pi^2} \ .$$

The correct theory involves exchange of a massive vector boson of mass M_W, which leads to a propagator in the matrix element

$$\frac{1}{q^2 - M_W^2} \ .$$

Fermi theory is equivalent to neglecting the q^2 term in the denominator. Hence, treating W-boson exchange correctly, we have

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 E_e^2}{\pi^2} \left(\frac{M_W^2}{M_W^2 - q^2} \right)^2 \ .$$

Now, for elastic scattering, $q^2 = 0 - |\underline{q}|^2$, where

$$|\underline{q}|^2 = \left(2E_e \sin\frac{\theta}{2} \right)^2 = \frac{1}{2}s(1 - \cos\theta) \equiv u$$

and so

$$du = \frac{1}{2} s \sin\theta \ d\theta = \frac{s}{4\pi} d\Omega \ .$$

We can thus integrate the differential cross-section in terms of u:

$$\begin{aligned}
\sigma &= \int d\Omega \frac{G_F^2 s}{4\pi^2} \left(\frac{M_W^2}{M_W^2 + u} \right)^2 \\
&= \frac{G_F^2 M_W^4}{\pi} \int_0^s du \frac{1}{(M_W^2 + u)^2} \\
&= \frac{G_F^2 M_W^4}{\pi} \left[\frac{-1}{M_W^2 + u} \right]_0^s \\
&= \frac{G_F^2 M_W^4}{\pi} \left(\frac{1}{M_W^2} - \frac{1}{M_W^2 + s} \right) \\
&= \frac{G_F^2 M_W^2 s}{\pi (M_W^2 + s)}
\end{aligned}$$

At small values of s this reduces to the Fermi theory result, while for $s \gg M_W^2$ the cross-section tends towards the constant value

$$\sigma = \underline{\underline{\frac{G_F^2 M_W^2}{\pi}}}$$

and is no longer divergent.

APPENDIX H: GAMOW FACTOR IN ALPHA DECAY

The probability for an α particle to tunnel through the Coulomb barrier can be written as

$$P = \prod_i \exp\{-2G\}$$

where G is the Gamow Factor,

$$G = \int_R^{R'} \frac{\left[2m\left(V(r) - E_0\right)\right]^{1/2}}{\hbar} \, dr.$$

R is the radius of the nucleus of mass number Z, R' is the radius at which the α particle escapes, m is the mass of the α particle, $V(r) = 2(Z-2)e^2/4\pi\epsilon_0 r \equiv B/r$ is the Coulomb potential, and E_0 is the energy release in the decay.
The α particle escapes the nucleus when $r = R'$. Hence, the potential $V(R') = E_0$ and $R' = B/E_0$. Therefore,

$$\begin{aligned} G &= \left(\frac{2m}{\hbar^2}\right)^{1/2} \int_R^{R'} \left[\frac{B}{r} - E_0\right]^{1/2} dr \\ &= \left(\frac{2mB}{\hbar^2}\right)^{1/2} \int_R^{R'} \left[\frac{1}{r} - \frac{1}{R'}\right]^{1/2} dr \end{aligned}$$

In order to perform the integration, let $r = R'\cos^2\theta$ and $dr = -2R'\cos\theta\sin\theta \, d\theta$. Then

$$\begin{aligned} \int \left[\frac{1}{r} - \frac{1}{R'}\right]^{1/2} dr &= \int \left[\frac{1}{R'\cos^2\theta} - \frac{1}{R'}\right]^{1/2} (-2R'\cos\theta\sin\theta)\, d\theta \\ &= \int -2R'^{1/2}\sin^2\theta \, d\theta \\ &= R'^{1/2}\left[\sin 2\theta - \theta\right] \end{aligned}$$

Now, using $\cos\theta = (r/R')^{1/2}$, $\sin\theta = (1 - r/R')^{1/2}$ and $\sin 2\theta = 2\sin\theta\cos\theta$, then

$$\begin{aligned} R'^{1/2}\left[\sin 2\theta - \theta\right]_R^{R'} &= R'^{1/2}\left[2\left(1 - r/R'\right)^{1/2}\left(r/R'\right)^{1/2} - \cos^{-1}\left(r/R'\right)^{1/2}\right]_R^{R'} \\ &= R'^{1/2}\left[\cos^{-1}\left(R/R'\right)^{1/2} - 2\{\left(1 - R/R'\right)\left(R/R'\right)\}^{1/2}\right] \end{aligned}$$

Hence, the Gamow factor

$$G = \left(\frac{2mB}{\hbar^2}\right)^{1/2} R'^{1/2}\left[\cos^{-1}\left(R/R'\right)^{1/2} - 2\{\left(1 - R/R'\right)\left(R/R'\right)\}^{1/2}\right]$$

or

$$\underline{G = \left(\frac{2m}{E_0}\right)^{1/2} \frac{B}{\hbar}\left[\cos^{-1}\left(R/R'\right)^{1/2} - 2\{\left(1 - R/R'\right)\left(R/R'\right)\}^{1/2}\right]}$$

For most practical cases, $R \ll R'$, so the term in square brackets is $\approx \pi/2$ and G becomes

$$\underline{G \approx \left(\frac{2m}{E_0}\right)^{1/2} \frac{B}{\hbar}\frac{\pi}{2}}$$