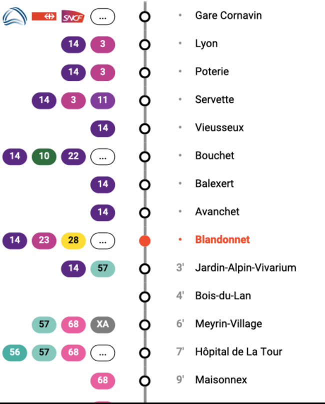
Part III Physics

Particle Physics

Dr C.G.Lester

https://www.youtube.com/watch?v=1e1eLe1ihT0

es Horribles Cernettes", CERN's most famous pop group (and the subjects
the first image to be uploaded to the world wide web) singing "Collider".

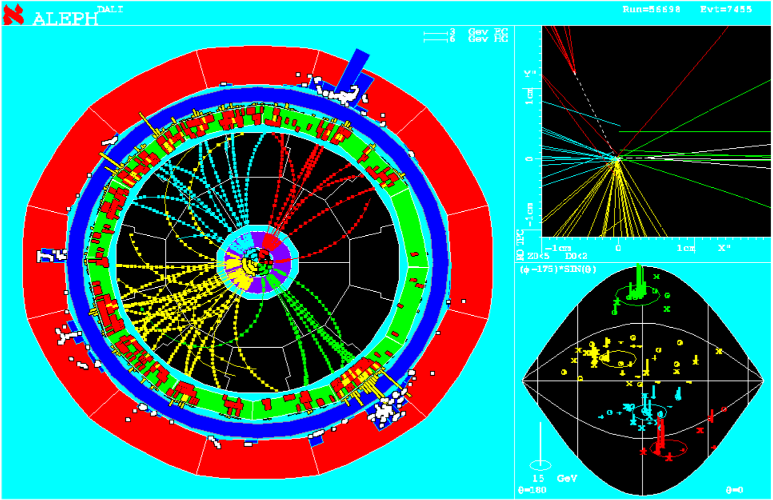


CERN Pizza Recipe:
https://www.hep.phy.cam.ac.uk/~lester/HiggsPizza.pdf

The "Higgs Boson Pizza Day" was held on Monday, 4 July 2016, on the fourth anniversary of the announcement of the discovery of the Higgs boson at CERN. On this

Sub-divisions (Handouts)

Dr C.G. Lester, 2023



H1: Introduction

Preliminaries

Web-page / Classes / Feedback / Correction

`https://www.hep.phy.cam.ac.uk/~lester/teaching/partIIparticles`

- Course web-page above contains examples sheet, old exam papers and solutions, errata, lecturer's contact details, and (when ready) arrangements for examples classes ,*etc.*
- Please send corrections on course and/or feedback to me: `Lester@hep.phy.cam.ac.uk`

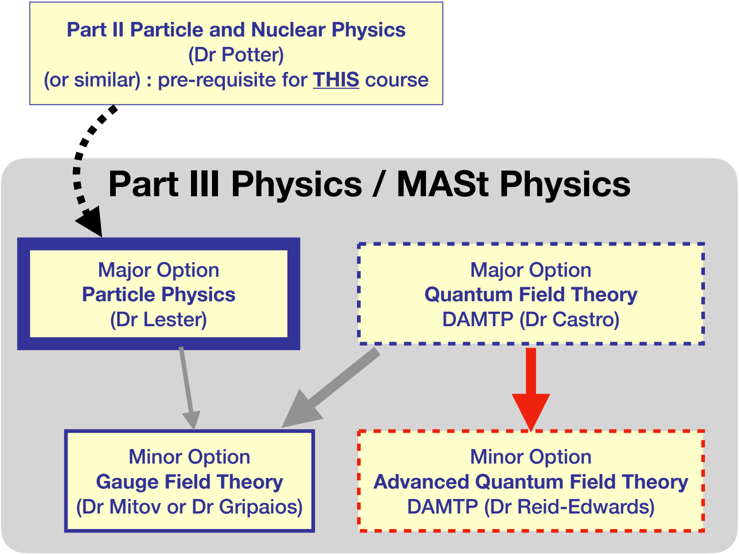
Format

- For historical reasons, the fourteen sections of the course are called 'handouts'. (List on page 2.)
- Some handouts contain additional theoretical background in non-examinable appendices.

Books

- **"Modern Particle Physics"**, Mark Thomson (Cambridge) BASED ON THIS COURSE!
- **"Particle Physics"**, Martin and Shaw (Wiley): fairly basic but good.
- **"Introductory High Energy Physics"**, Perkins (Cambridge): slightly below level of the course but well written.
- **"Introduction to Elementary Physics"**, Griffiths (Wiley): about right level but doesn't cover the more recent material.
- **"Quarks and Leptons"**, Halzen & Martin (Wiley): good graduate level textbook (slightly above level of this course).

Cambridge Particle Physics Courses



Aims of this course

The course is intended as an overview-style course.

It aims to provide:

- a context for the other more rigourous courses (QFT, AQFT, Gauge Field Theory),
- examples of the experiments and types of experimental evidence which have lead to our current underfstanding of The Standard Model, and
- 'just enough' of the theory to understand how/why the experiments constrain theory.

Level of rigour

Since the QFT, AQFT and Gauge Field Theory courses are either not yet lectured or are lectured in parallel, it is necessary for many results in this course to be presented without proof, or with only plausibility arguments, or with outline theoretical motivations. That will be dissatisfying for some taking the course – but are a necessary evil if this course is to complement those other courses.

Past student advice:

This mini-review was taken from `https://www.reddit.com/r/Physics/comments/iatn6o/an_interesting_question_from_my_2020_particle/`

> *"Technically, this is Part III Physics from the Natural Sciences Tripos. You do get to borrow a QFT course from the Part III Mathematical Tripos though.*
>
> **[redacted]** *the lecturer* **[redacted]** *[likes] to point with a great big stick.*
>
> *This book [Thomson] is based on the course; author is a previous lecturer. Perhaps flicking through the preview might help? It's not a formal QFT course, so there's less maths. It tries to explain both theory and experiment. If you want more theory, I'd recommend the Gauge Field Theory courses or the QFT and AQFT courses from Part III Maths.*
>
> *Pre-req: "Students who are not familiar with the overall structure of The Standard Model, the quark model of the hadrons, scattering processes, and wave equations at some level, have found the course hard in the past." You use quite a lot of Einstein notation / tensors like 4-vectors, Bra-Kets and matrices, so perhaps be comfortable with that (if you aren't already).*
>
> *Have fun in Part III!"*

Non-examinable material

What material is deemed to be **non-examinable** ?

- any slide content explicitly labelled as '**non-examinable**', or
- any slide content explicitly labelled as '**not examinable**', or
- any material located in an appendix to a handout or an appendix to the course as a whole, whether or not it carries either of the labels above, unless it is explicitly labelled '**examinable**'.

What does it mean if material is deemed to be **non-examinable** ?

no student is expected or required to read, learn or revise any such material for Tripos; the exam should not *require* prior knowledge of such material.

Terms and conditions, small print, *etc.*

(i) It is not the case that a Tripos question could never have a domain overlapping with or including material deemed to be **non-examinable**. Such overlaps, in the rare cases in which they appear, simply indicate that an examiner has judged that material in the overlap can be reasonably deduced from material which was deemed to be examinable. Therefore, a more specific (though wordier) name for the material could be 'material-which-does-not-need-to-be-learned-or-revised'. (ii) In the event that material has been mis-labelled, a correction would be issued to the class (either by email or verbally in lectures) before the end of Michaelmas Term and then recorded on the course website. (iii) If in doubt about the status of any material, ask the lecturer for clarifications before the end of Michaelmas Term. (iv) Occasionally material from non-examinable sections will be discussed in lectures if the lecturer judges that it could be of benefit to some members of the audience. The discussion of such material in lectures does not change its status unless an official announcement to that effect is given.

Lecture Zero

The course proper begins on Monday!
Before then, here are a few things which fit nowhere else:

- Units.
- Assumed knowledge about Dirac δ-Functions.
- Standard Model - review.
- Special Relativity - things you should be familiar with.
- Why Mandelstam variables matter.

Units in Particle Physics

S.I. Units measure:

mass in kg, length in m, time in s, charge in C.

In principle, particle physicists using 'natural' units measure:

mass in GeV/c^2, length in $\hbar c/GeV$, time in $\hbar/GeV$, charge in $(\varepsilon_0 \hbar c)^{\frac{1}{2}}$.

NB: You could change GeV to MeV, TeV or any other eV-based energy unit without upsetting anyone at CERN.

Heaviside-Lorentz convention:

$c = \hbar = \varepsilon_0 = 1$ (and $\mu_0 = 1$ too since $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$)

In practice, particle physicists using natural units measure:

mass in GeV, length in $1/GeV$, time in $1/GeV$, and charge is *dimensionless* on account of using that Heaviside-Lorentz convention!

$1 = \hbar c = 197$ MeV fm : (the one unit-based fact memorized in CERN)

Lazy particle physicists correct their omissions of $\hbar$ and c factors with the following tricks: They get the "*GeV*"s on the bottom/top correctly they remembering that:

- they like **high** energies and **high** momenta: .... so these are GeV
- they like **small** lengths and **small** times: .... so these are GeV^{-1}

To help them recover c and $\hbar$ factors omitted from energies they (mostly) use the following *aides-mémoire*:

- (for mass): $E \sim mc^2$, ...so energy is $\propto GeV/c^2$,
- (for momentum): $E \sim (mc)(c) \sim pc$, ...so momentum is $\propto GeV/c$,
- (for time): $\Delta E \Delta t \sim \hbar$, ...so time is $\propto \hbar/GeV$,
- (for length): $1 = \hbar c \approx 197\ MeV \cdot fm$, ... so lenth is $\propto \hbar c/GeV$,

and to get proper S.I. units like Joules and meters:

- (energies to Joules): $eV \approx 1.60 \times 10^{-19} J$,
- (lenths to meters): $1 = \hbar c \approx 197\ MeV \cdot fm$.

Your lecturer's height in centimeters may help you remember the boxed fact ...

Assumed knowledge concerning Dirac δ-functions of one real variable:

It is assumed you know this:

$$\int_X g(x)\delta(u(x))dx = \int g(x(u))\delta(u)\left|\frac{dx}{du}\right| du = \sum_{x\in X \text{ s.t. } u(x)=0} \frac{g(x)}{\left|\frac{du}{dx}\right|}, \qquad (1)$$

If the above is not familiar to you, convince yourself of it using other sources.

Examples of the above:

$$\int_{-\infty}^{\infty} \sin x\ \delta(x-a)dx = \sin a, \qquad \text{(trivial example)}$$

$$\int_{-\infty}^{\infty} e^x \delta(x^2-a^2)dx = \sum_{x=\pm a} \frac{e^x}{|2x|} = \frac{e^a}{|2(a)|} + \frac{e^{-a}}{|2(-a)|} = \frac{1}{2|a|}(e^a+e^{-a}) = \frac{1}{|a|}\cosh a.$$

The second example is presumably non-trivial, as many past students of the course have mistakenly produced other answers for it like $e^a + e^{-a}$ or $2\cosh a$.

The generalisation to two or more variables ...

This is not examinable, but the generlisations of the result (1) to Dirac δ-functions of more than one variable would be as follows:

Two variables:

$$\int_X g(x,y)\delta(u(x,y))\delta(v(x,y))dxdy = \int g(x(u,v),y(u,v))\delta(u)\delta(v)\left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| dudv$$

$$= \sum_{(x,y)\in X \text{ s.t. } u(x,y)=v(x,y)=0} \frac{g(x,y)}{\left|\left|\frac{\partial(u,v)}{\partial(x,y)}\right|\right|}.$$

Many variables:

$$\int_X g(\vec{x})\delta^n(\vec{u}(\vec{x}))d^n x = \int g(\vec{x}(\vec{u}))\delta^n(\vec{u})\left|\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)}\right|\right| d^n u = \sum_{\vec{x}\in X \text{ s.t. } \vec{u}(\vec{x})=0} \frac{g(\vec{x})}{\left|\left|\frac{\partial(u_1,\ldots,u_n)}{\partial(x_1,\ldots,x_n)}\right|\right|}.$$

Reminder: Matter in the Standard Model

The Standard Model's point-like spin-1/2 fermions:

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	-1	0.0005	d	$-1/3$	~ 0.3
	ν_1	0	≈ 0	u	$+2/3$	~ 0.3
Second Generation	μ^-	-1	0.106	s	$-1/3$	~ 0.5
	ν_2	0	≈ 0	c	$+2/3$	~ 1.5
Third Generation	τ^-	-1	1.77	b	$-1/3$	~ 4.5
	ν_3	0	≈ 0	t	$+2/3$	173

Open Questions (not addressed in this course!) :

- Why are there three generations, rather than some other number?
- Why are there generations at all?
- Why are the neutrinos so light compared to the others? (e.g. v_1 has a non-zero mass below $\sim 3\mathrm{eV}$)
- Are the neutrinos Majorana?
- Why are there no particles that could explain Dark Matter?

Reminder: Forces in the Standard Model

Spin-1 Gauge Bosons mediate the four forces of the SM:

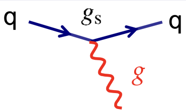
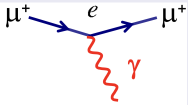
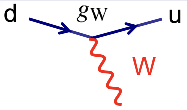
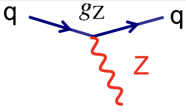
Force	Boson(s)	J^P	m/GeV	Charge "g"	coupling constant "α"	Notes
EM (QED)	Photon γ	1^-	0	e	α	Handout 3
Weak	$W^\pm/Z$	1^-	80/91	g_W, g_Z	α_W, α_Z	Handout 13
Strong (QCD)	8 Gluons g	1^-	0	g_s	α_s	Handout 8
Gravity	Graviton?	2^+ ?	0 ?	?	?	Not in SM!

Charges g vs Coupling Constants α

- Coupling constants α are always dimensionless whereas charges g might have strange units.
- $g = \sqrt{4\pi\alpha}$ is true for all forces, but only in natural units with Heaviside Lorentz convention.
- $g = \sqrt{4\pi\alpha\epsilon_0\hbar c} = e$ for QED without Heaviside Lorentz convention.
- Since always dimensionless, the coupling constant α sets the intrinsic strength of each force and does not depend on choice of units.

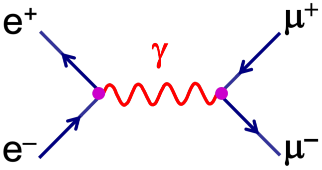
Reminder: Standard Model Vertices

Although much more detail will be provided earlier, it will be assumed that you have some partial knowledge of the following properties of the SM interactions:

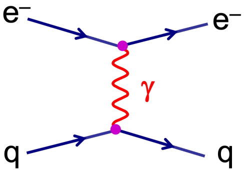
Strong	Electromagnetic	Weak Charged Current (CC)	Weak Neutral Current (NC)
			
Only quarks	All charged fermions	All fermions	All fermions
Never changes flavour	Never changes flavour	Always changes flavour	Never changes flavour
$\alpha_S \sim 1$	$\alpha \simeq 1/137$	$\alpha_W \sim 1/40$	$\alpha_Z \sim 1/40$

Feynman Diagram conventions

electron-positron annihilation example:

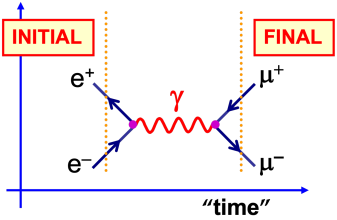


electron-quark scattering example:



Usually in this course: ‘time’ runs from left to right, but only in the sense that:

- LHS of diagram is initial state
- RHS of diagram is final state
- Middle is how process ‘might’ have happened



Anti-particles are indicated by arrows in the negative ‘time’ direction.

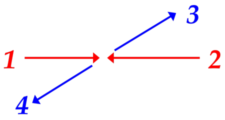
Much more detail appears in Handout 3 and onwards!

Special Relativity and 4-Vector Notation

- Will use 4-vector notation with t as the time-like component, e.g.
  $p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\}$ (contravariant)
  $p_\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\}$ (covariant)
  with
  $$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- In particle physics, usually deal with relativistic particles. Require all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.
  $p^\mu p_\mu = E^2 - p_x^2 - p_y^2 - p_z^2 = E^2 - p^2 = m^2$ (Invariant Mass)
  $x^\mu p_\mu = Et - p_x x - p_y y - p_z z = Et - \vec{p} \cdot \vec{r}$ (Phase)
- A few words on NOTATION: Four vectors are written as either: p^μ or p. Four vector scalar products as either: $p^\mu p_\mu$ or $p \cdot r$. Three vectors are written as: $\vec{p}$. Quantities evaluated in the centre of mass frame are written: $\vec{p}^*$ or p^* etc.

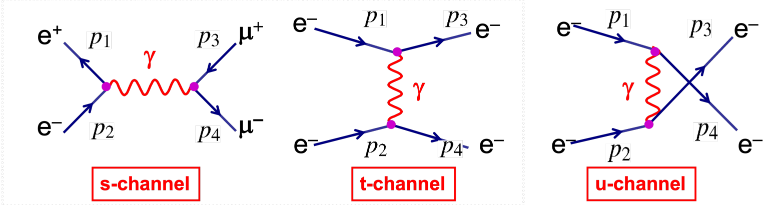
The Mandelstam Variables: s, t, and u

Sattering processes like $1 + 2 \to 3 + 4$ are often characterised by three useful real Lorentz Invariant quantities: s, t, and u. These are built from the squares of particular sums or differences of the Lorents four-vectors p_1, p_2, p_3 and p_4:



$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2$$

Simple Feynman diagrams for two-to-two processes are named s-, t- or u-channel according to the squared foured-momentum of the exchanged particle:



Why care about s, t and u ?

Only two of s, t and u are independent if the masses of the four external particles are known.

Proof is in example sheet question 1 which will ask you to show that

$$s + t + u \equiv m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

Without loss of generality, take s and t to be the pair of independent variables.

s contains everything that is worth knowing about the initial state, and
t contains everything that is worth knowing about the interaction occurring.

The lecturer will show why this is so by counting degrees of freedom in rotationally and boost invariant momenentum conserving theories. Deep Water!

Hardly surprising then that energy in centre-of-mass frame is $\sqrt{s}$

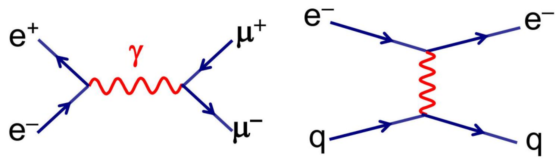
s is Lorentz invariant so takes same value in any frame. In the c.o.m. frame $p_1^* = (E_1, \vec{p}^{\,*})$ and $p_2^* = (E_2, -\vec{p}^{\,*})$ so $s = (E_1 + E_2, \vec{0})^2 = (\text{centre-of-mass energy})^2$.

The first five lectures

Aiming towards proper calculations of decay rates and scattering probabilities

Will concentrate on:

- $e^+e^- \to \mu^+\mu^-$
- $e^-q \to e^-q$ (to probe proton structure)



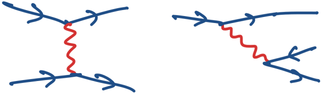
We will need:

- relativistic calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{ phase space })$$

- relativistic treatment of spin-half particles: (see Dirac Equation, etc, Handout 2)
- relativistic calculation of interaction Matrix Element

- We want to precisely calculate decay rates and scattering probabilities



- To do so, we use **FERMI'S GOLDEN RULE**

$$\Gamma_{fi} = 2\pi \left| T_{fi} \right|^2 \rho(E_f) \qquad (A)$$

which calculates:

number of transitions per unit time

Γ_{fi} = Rate at which initial state $|i\rangle$ transitions to final state $|f\rangle$

under various assumptions (see next page).

The assumptions of **FERMI'S GOLDEN RULE.**

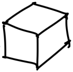
In the form in which it was given ($\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$) :

$\{|i\rangle\}$ should be a basis of the free theory composed of eigenstates of $\hat{H}_{\text{free}}$.

- All basis states are normalised such that

$$\langle i | j \rangle = \int_{\mathcal{V}} \psi_i^* \psi_j \, dV = \delta_{ij}$$

(where $\mathcal{V}$ is a box of volume V inside which the states are confined)



- T_{fi} is the "Transition Matrix Element"

(A2) $$T_{fi} = \langle f | \hat{H}_{int} | i \rangle + \sum_{j \neq i} \frac{\langle f | \hat{H}_{int} | j \rangle \langle j | \hat{H}_{int} | i \rangle}{E_i - E_j} + \dots$$

- $\hat{H}_{int}$ is the "interaction" part of the full Hamiltonian:

$$\hat{H}_{full} = \hat{H}_{free} + \hat{H}_{int} , \qquad \text{and}$$

- $\rho(E_f)$ is the density of final states, defined by

$$\rho(E_f) = \left| \frac{dN}{dE} \right|_{E = E_f}$$

where dN is the number of final states with energy between E_f and $E_f + dE$

(see revision in next slide)

(E_f = Final state energy = Initial State Energy)

ASIDE

A proof of Fermi's Golden Rule in the above form is given in sec 2.3.6 of Mark Thomson's book.

It is assumed that you have seen similar proofs in earlier courses, so the origin of F.G.R. is not discussed here. For an alternative more in-depth proof see section 46 of P.A.M. Dirac's book "The Principles of Quantum Mechanics".

Learning/take home points for Fermi's Golden Rule:

$$\Gamma_{fi} = 2\pi \left| T_{fi} \right|^2 \rho(E_f) \qquad \text{(A)}$$

Reaction rates are controlled by **Matrix Element** and **Phase Space**

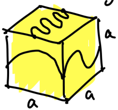
Matrix Element ← Quantum Physics/ Particle Content/ SM/SUSY/Leptoquarks

Phase Space ← Relativity, Mom Cons, Energy Cons.

Revision of Phase Space (non-relativistic, single particle)

When a state is confined to a box, allowed momenta are only able to take values on a lattice.

Suppose V is a cube of side "a". $(a^3 = V)$



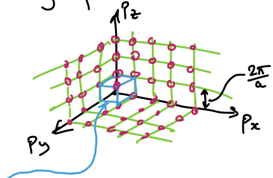
then allowed states $\psi \propto e^{\pm i(\underline{p}\cdot\underline{x} - \omega t)}$ must have $\underline{p} = (p_x, p_y, p_z) = \frac{2\pi}{a}(n_x, n_y, n_z)$

integers

so that periodic B.C.s are satisfied

$$\psi(x+a, y, z) = \psi(x,y,z)$$
$$\psi(x, y+a, z) = \psi(x,y,z)$$
$$\psi(x, y, z+a) = \psi(x,y,z).$$

Hence single particle states live on this lattice



(Volume of SINGLE STATE in mom space) =

$$= \left(\frac{2\pi}{a}\right)\left(\frac{2\pi}{a}\right)\left(\frac{2\pi}{a}\right) = \frac{(2\pi)^3}{V}$$

∴ number of states dn in element $d^3\underline{p}$ is

$$dn = \frac{d^3\underline{p}}{\left(\frac{(2\pi)^3}{V}\right)} = V\frac{d^3\underline{p}}{(2\pi)^3}$$

(B)

(NOT MANIFESTLY LORENTZ COVARIANT)

ASIDE ON YUCKY PHASE-SPACE

We want $\rho(E_f) \equiv \left|\frac{dn}{dE}\right|_{E=E_f}$.

We COULD get $\rho(E_f)$ by using

① $dn = V\frac{d^3p}{(2\pi)^3} = V\frac{4\pi p^2 dp}{(2\pi)^3} \Rightarrow \boxed{\frac{dn}{dp} = V\frac{4\pi p^2}{(2\pi)^3}}$

② $E^2 = m^2 + p^2 \Rightarrow 2E\,dE = 2p\,dp \Rightarrow \boxed{\frac{dp}{dE} = \frac{E}{p} = \frac{1}{\beta}}$

to deduce that

$$\rho(E_f) \equiv \left|\frac{dn}{dE}\right|_{E=E} = \left|\frac{dn}{dp}\frac{dp}{dE}\right|_{E=E_f}$$

$$\therefore \rho(E_f) = V\frac{4\pi p^2}{(2\pi)^3}\frac{1}{\beta}$$

but this is SUPER UGLY!

See nice trick on next slide that avoids β usage!

26 **Writing Golden Rule in nicer form.**

Ideally we'd like to write the Golden Rule in a form that avoids it being polluted by β's and other things that are frame dependent (except where absolutely necessary). Want to aim for explicit "Lorentz Covariance".

Step 1: write density of states more nicely! ☺.

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE}\,\delta(E-E_i)\,dE \qquad \text{(c)}$$

Why is this better?

- Not because $E_f \to E_i$ ... that's just Energy Conservation.
- Better because integral scans "ALL ENERGIES" rather than "just one".
  - "ALL ENERGIES" ⟶ "ALL ENERGIES" in a boost.
  - "ALL ENERGIES" is a "nice set" under Lorentz Tfms.

What does F.G.R. look like now?

$$\Gamma_{if} = 2\pi \, |T_{fi}|^2 \rho(E_f)$$ (A)

using (C)

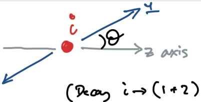
$$\Rightarrow \Gamma_{if} = 2\pi \int |T_{fi}|^2 \frac{dn}{dE} \delta(E-E_i)\, dE$$

$$\Rightarrow \Gamma_{if} = 2\pi \int |T_{fi}|^2 \delta(E-E_i)\, dn$$ (D)

NICER

Integral is now over all allowed final states of any energy. ☺

Let's consider a special case:



(Decay $i \to (1+2)$ in REST FRAME (i))

Momentum conservation says that final state (1+2) can be parameterised by the momentum of just one of the two final state particles (w.l.o.g particle 1) since other one has momentum equal and opposite*.

Thus it suffices to replace $dn \to V \frac{d^3 p_1}{(2\pi)^3}$ (uses Ⓑ) and since also $E \equiv E_f \equiv E_1 + E_2$ we find that:

* in QFT course you will see this is not an ad-hoc additional principle. Comes from way interactions are constrained to be local.

$$\Gamma_{if} = 2\pi \int |T_{fi}|^2 \delta(E - E_i)\, dn \qquad \text{(D)}$$

GENERAL

$$\Rightarrow \quad \Gamma_{if} = 2\pi \int |T_{fi}|^2 \delta(E_1 + E_2 - E_i)\, V \frac{d^3 p_1}{(2\pi)^3} \qquad \text{(uses (B))}$$

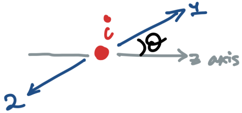
$i \to 1+2$

"Because we can!"

$i \to 1+2$

$$\Rightarrow \quad \Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \delta(E_1 + E_2 - E_i)\, \delta^3(\underline{p}_1 + \underline{p}_2 - \underline{p}_i) \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \qquad \text{(E)}$$

MATHS, PHYSICS, ENERGY CONS, MOM CONS, DENSITY OF STATES



(Decay $i \to (1+2)$ in REST FRAME (i))

$i \to 1+2$

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \, \delta(E_1+E_2-E_i)\, \delta^3(\underline{p}_1+\underline{p}_2-\underline{p}_i)\, \frac{d^3\underline{p}_1}{(2\pi)^3}\, \frac{d^3\underline{p}_2}{(2\pi)^3} \qquad (E)$$

MATHS — PHYSICS — ENERGY CONS — MOM CONS — DENSITY OF STATES

Nicer Still!

Integral is now over all final state momenta, not just some special ones! ☺

Becoming "more" Lorentz Covariant

Meaning is starting to "factorise".

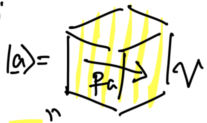
27 Let's try to move to a "better" wave func (or state) normalisation.

("better" = more manifestly Lorentz Covariant)

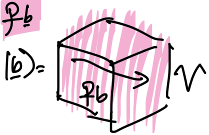
Up to now, we have been using single particle states satisfying

$$\int_V \psi^* \psi \, dV = 1 \quad \text{with} \quad \langle i | j \rangle = \delta_{ij}$$

ie $|a\rangle$ = "The box of volume V contains a single particle with energy E_a and mom p_a"



$|b\rangle$ = "The box of volume V contains a single particle with energy E_b and mom p_b"

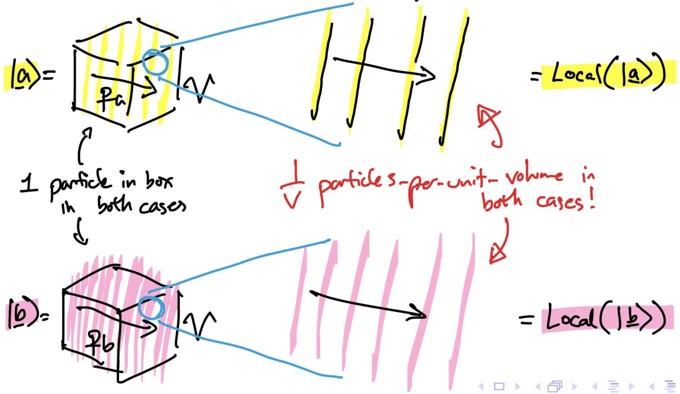


EQUIVALENTLY:

Within a "small bit" of the box, $|a\rangle$ & $|b\rangle$ each look locally like a bit of a

"$\frac{1}{V}$-particles-per-unit-volume soup"

of particles with energy E_a or E_b respectively.



LENGTH CONTRACTION caused by RUNNING PAST

Local regions of $|a\rangle$ changes wavelength and thus also momentum, as expected by Lorentz Tfm:

Alas ☹ ! If we run past $|a\rangle$ at the right rate so as to exactly transform $E_a \to E_a' = E_b$ and $p_a \to p_a' = p_b$

we do **NOT** find that $|a\rangle \to |a\rangle' = |b\rangle$

Instead we find that $|a\rangle' = \sqrt{\gamma}\,|b\rangle$.

↑ γ = gamma-factor associated with the boost belonging to our "run"

Why?

Because the LENGTH CONTRACTION also increases the local particle density by a factor of γ! ☹.

[$\int_V \psi^* \psi \, dV = 1$ and $\langle i | j \rangle = \delta_{ij}$, so $\frac{1}{V}$ particles per unit volume as 1 particle in box of volume V.]

Summary:

- One-particle states which are correctly normalised for one observer (according to our current convention) will not appear correctly normalised to observers moving at other velocities.

Choose, therefore, to adopt a new normalization convention which (unlike the last one) is consistent for all observers.

$\psi''_a \underset{\text{def}}{\equiv} \sqrt{\gamma_a}\, \psi_a$ could work, and might be elegant,

$\gamma_a = \frac{1}{\sqrt{1-\beta_a^2}}$ where $\beta_a = \frac{p_a}{E_a}$

but for a mix of historical and practical reasons we use

$$\psi'_a \underset{\text{def}}{\equiv} \sqrt{2E_a}\sqrt{V}\,\psi_a$$

(F)

(States in new norm convention are notated with a "prime": ψ'_a)

which is called "2E particles per unit volume" since

$$(\#\text{particles in box}) \sim \int_V \psi'^{*}_a \psi'_a \, dV = 2E_a V \int_V \psi^{*}_a \psi_a \, dV = 2E_a V$$ and

$$(\text{vol of box}) = V \quad \therefore \quad (\#\text{particles per unit vol}) \sim \frac{2E_a V}{V} = 2E_a .$$

Where do we use these newly normalised states?

- Recall definition of T_{fi} in (A2) as a sandwich of $\hat{H}_{int}$ with "old" states as the bread on either side.

  Eg: $$T_{fi} = \underbrace{\langle \psi_1, \psi_2, \ldots}_{\langle f|} | \hat{H}_{int} | \underbrace{\ldots \psi_{N-1}, \psi_N \rangle}_{|i\rangle} + \ldots \qquad (G)$$

- For any such T_{fi}, denote by "M_{fi}" the corresponding

  "Lorentz Invariant Matrix Element"

  by replacing $\langle f|$ and $|i\rangle$ with states of the new norm (and by multiplying by one conventional factor of $\frac{1}{V}$).

- i.e. for the T_{fi} in (G) define the corresponding M_{fi} to be:

$$M_{fi} \underset{def}{\equiv} \frac{1}{V} \langle \psi_1', \psi_2', \ldots | \hat{H}_{int} | \ldots \psi_{N-1}', \psi_N' \rangle + \ldots \qquad (H)$$

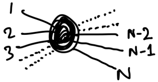
DEFINITION

$$\therefore \quad M_{fi} = \frac{1}{V} \sqrt{2E_1}\sqrt{V} \ldots \sqrt{2E_N}\sqrt{V} \langle \psi_1, \psi_2 \ldots | \hat{H}_{int} | \ldots \psi_{N-1}, \psi_N \rangle + \ldots$$

COROLLARY:

$$\therefore \quad M_{fi} = V^{N/2 - 1} \left(\sqrt{2E_1} \cdot \ldots \cdot \sqrt{2E_N} \right) T_{fi} . \qquad (I)$$

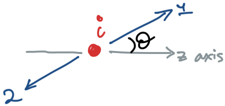
COROLLARY



ASIDE: The lengths of the const factor $\frac{1}{V}$ will be seen later. Likewise, the appropriateness of the name "Lorentz Invariant Matrix Element" is not yet clear, but will later become so.

28

Making (E) nicer still.



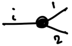
Recall that for the process we had the expression:

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \, \delta(E_1+E_2-E_i)\, \delta^3(p_1+p_2-p_i) \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \qquad \text{(E)}$$

$i \to 1+2$

MATHS — PHYSICS — ENERGY CONS — MOM CONS — DENSITY OF STATES

Does this become nicer with the newly normalised states?

For this process $N=3$ (i.e. three external legs ) and so

(I) $\Rightarrow$ $$M_{fi} = V^{\frac{3}{2}-1}\left(\sqrt{2E_i}\sqrt{2E_1}\sqrt{2E_2}\right) T_{fi}$$

and thus (E) $\Rightarrow$

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |M_{fi}|^2 \frac{V^{2(1-\frac{3}{2})}}{(2E_i)(2E_1)(2E_2)} \delta(E_1+E_2-E_i)\, \delta^3(p_1+p_2-p_i) \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3}$$

$i \to 1+2$

$$\Rightarrow \quad \Gamma_{if} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \, \delta(E_1+E_2-E_i)\, \delta^3(p_1+p_2-p_i) \frac{d^3p_1}{(2\pi)^3\, 2E_1} \frac{d^3p_2}{(2\pi)^3\, 2E_2} \qquad \text{(J)}$$

$i \to 1+2$

$$\Rightarrow \boxed{\Gamma_{if} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_1+E_2-E_i)\, \delta^3(\underline{p}_1+\underline{p}_2-\underline{p}_i) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}} \quad (J)$$

TIME DILATION FACTOR — PHYSICS — ENERGY CONS — MOM CONS — DENSITY OF STATES

SUPER NICE

LHS **SHOULD** time dilate by factor γ of initial state E_i.
But $E_i = \gamma m_i$, so time dilation is perfectly captured by leading $\frac{1}{2E_i}$ term! Great! ☺

Separately:

- $\frac{d^3 p_1}{(2\pi)^3 2E_1}$ is Lorentz **INVARIANT** (see ExSheet Q2).
- $\delta^4(p_1^\mu + p_2^\mu - p_i^\mu)$ is Lorentz **INVARIANT**.

(non-trivial! uses $\mathrm{Det}(\Lambda) = 1$ and $\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$ in Jacobian)

So BY PROCESS OF ELIMINATION $|M_{fi}|^2$ is also **LORENTZ INVARIANT**. *

{* This justifies the name we gave it earlier! ☺

ASIDE

[At this point one might wish to compare/contrast the calculation leading to (J) above for [$i \to 1+2$] with a similar but subtly different calculation (page 47) for the case [$1+2 \to 3+4$]. Differences occur because the number of external particles is then $N=4$ instead of $N=3$.

——— However ———

Instead we try to use (J) to calculate an [$i \to 1+2$] particle decay rate.]

Decay Rate Calculations

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \, \delta\left(E_i - E_1(p_1) - E_2(p_2)\right) \delta^3\left(\vec{p}_i - \vec{p}_1 - \vec{p}_2\right) \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}$$

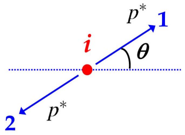
Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient

- In the C.o.M. frame $E_i = m_i$ and $\vec{p}_i = 0 \Rightarrow$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \, \delta\left(m_i - E_1(p_1) - E_2(p_2)\right) \delta^3\left(\vec{p}_1 + \vec{p}_2\right) \frac{\mathrm{d}^3\vec{p}_1}{2E_1(p_1)} \frac{\mathrm{d}^3\vec{p}_2}{2E_2(p_2)}$$

- Integrating over $\vec{p}_2$ using the δ-function:

$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \, \delta\left(m_i - E_1(p_1) - E_2(p_1)\right) \frac{\mathrm{d}^3\vec{p}_1}{4E_1(p_1)E_2(p_2)}$$



(note $E_2(p_1) = \left(m_2^2 + |\vec{p}_1|^2\right)$ since the δ-function imposes $\vec{p}_2 = -\vec{p}_1$) ... and so writing $|\vec{p}_1|$ as p_1:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \, \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 \, \mathrm{d}p_1 \, \mathrm{d}\Omega}{E_1(p_1)E_2(p_1)}.$$

- The last expression for Γ_{fi} can be written in the form

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \, g(p_1)\, \delta(f(p_1))\, \mathrm{d}p_1 \, \mathrm{d}\Omega \tag{2}$$

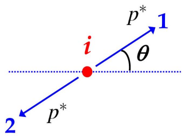
where $g(p_1) = p_1^2/(E_1 E_2) = p_1^2 \left(m_1^2 + p_1^2\right)^{-1/2} \left(m_2^2 + p_1^2\right)^{-1/2}$

and $f(p_1) = m_i - \left(m_1^2 + p_1^2\right)^{1/2} - \left(m_2^2 + p_1^2\right)^{1/2}$

Note:

- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products

i.e. $f(p_1) = 0$ for $p_1 = p^*$



- Eq. (2) can be integrated using the property of δ-function derived earlier (equation (1))

$$\int g(p_1)\, \delta(f(p_1))\, \mathrm{d}p_1 = \frac{1}{|\, \mathrm{d}f/\mathrm{d}p_1|\, p^*} \int g(p_1)\, \delta(p_1 - p^*)\, \mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|} p_{p^*}$$

where p^* is the value for which $f(p^*) = 0$

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{\left(m_1^2 + p_1^2\right)^{1/2}} - \frac{p_1}{\left(m_2^2 + p_1^2\right)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1 \frac{E_1 + E_2}{E_1 E_2}$$

giving
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{E_1 E_2}{p_1 (E_1 + E_2)} \frac{p_1^2}{E_1 E_2} \right|_{p_1 = p^*} \mathrm{d}\Omega$$
$$= \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{E_1 + E_2} \right|_{p_1 = p^*} \mathrm{d}\Omega$$

But from $f(p_1) = 0$, i.e. energy conservation: $E_1 + E_2 = m_i$ so

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i} \int |M_{fi}|^2 \, \mathrm{d}\Omega$$

which, in the particle's rest frame $E_i = m_i$, becomes the important result:

$$\boxed{\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \, \mathrm{d}\Omega} \qquad (3)$$

and this is VALID FOR ALL TWO-BODY DECAYS !

- p^* can be obtained by solving $f(p^*) = 0$. This gives:

$$p^* = \frac{1}{2m_i} \sqrt{\left[(m_1^2 + p^{*2})^{1/2} + (m_1^2 + m_2)^2 \right] \left[m_i^2 - (m_1 - m_2)^2 \right]} \qquad \text{(Question 3)}$$

Now try questions 4 and 5.

Cross section definition

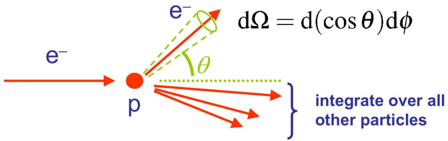
$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the effective crosssectional area of the target particles for the interaction to occur.
- It is a property of the pair of things interacting, rather than of either one singly.
- In general this has nothing to do with the physical size of the target although there are exceptions. (E.g. the cross section for neutron absorption on nucleii is the projective area of the nucleus!)

Differential Cross section:

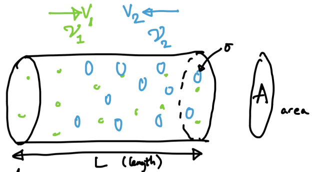
$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$



$$\mathrm{d}\Omega = \mathrm{d}(\cos\theta)\mathrm{d}\phi$$

with $$\boxed{\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega}$$

Classical link between cross section and effective area of interaction.



Suppose that:

- two beams of particle are in collision:
- "Species 1" (scatterers) are:
  - point-like
  - travelling → with speed v_1
  - have number-density ν_1 particles-per-unit volume
- "Species 2" (targets) are:
  - present area "σ" to species 1.
  - travelling ← with speed v_2
  - have number-density ν_2 particles-per-unit volume

∴ In time δt each scatterer "passes" a volume $(v_1+v_2)\delta t A$ of targets.

∴ " " " " " " an effective target area of $(v_1+v_2)\delta t A \nu_2 \sigma$ distributed within a total area A.

The probability of interaction of the scatter is thus $(v_1+v_2)\delta t\, \nu_2 \sigma$.

There are, however, $AL\nu_1$ scatterers in the volume, so the expected number of interactions in the whole region is $(v_1+v_2)\delta t\, \nu_2 \sigma A L \nu_1$.

Therefore the interaction rate in this cylinder is $(v_1+v_2)\nu_2 \sigma A L \nu_1$ (K)

Saw that the interaction rate in this cylinder is: $(v_1+v_2)\nu_2 \sigma A L \nu_1$ (K)

Our form of Fermi's Golden Rule computes the interaction rate in the cylinder only for the special case: $\nu_1 = \frac{1}{V} = \nu_2$ and $AL=V$.

$\therefore$ (K) $\Rightarrow$ $$\Gamma_{if} = (v_1+v_2)\frac{1}{V}\sigma V\frac{1}{V}$$

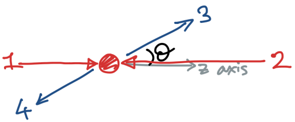
$$\Rightarrow \boxed{\sigma = \frac{\Gamma_{if} V}{v_1+v_2}} \quad \text{(M)}$$

Message:

$\therefore$

- F.G.R. can be used to calculate "σ," the effective area presented by scatterers to each other
- This cross section "σ" is really a property of the pair of species meeting each other – rather than of either species separately.

Let's now compute σ for the process $1+2\to 3+4$:



Aside: this is closely related to what we did on slide 38

We start by adopting (E) with the replacements:

$E_1 \to E_3$

$E_2 \to E_4$

$E_i \to E_1 + E_2$

$\vec{p}_1 \to \vec{p}_3$

$\vec{p}_2 \to \vec{p}_4$

which results in:

$$\Gamma_{fi}^{1+2\to 3+4} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \delta(E_3+E_4-E_1-E_2)\,\delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{d^3\vec{p}_4}{(2\pi)^3} \qquad (N)$$

MATHS — PHYSICS — ENERGY CONS — MOM CONS — DENSITY OF STATES

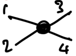
$1+2\to 3+4$

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |T_{fi}|^2 \, \delta(E_3+E_4-E_1-E_2)\, \delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2) \frac{d^3 p_3}{(2\pi)^3} \frac{d^3 p_4}{(2\pi)^3} \quad \text{(N)}$$

MATHS — PHYSICS — ENERGY CONS — MOM CONS — DENSITY OF STATES

Does this become nicer with the newly normalised states?

For this process $N=4$ (i.e. three external legs



) and so

(I) $\Rightarrow$

$$M_{fi} = V^{\frac{4}{2}-1} \left(\sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_3}\sqrt{2E_4}\right) T_{fi}$$

and thus (N) $\Rightarrow$

$$\Gamma_{if} = (2\pi)^4 \cdot V \cdot \int |M_{fi}|^2 \frac{V^{2(1-\frac{4}{2})}}{(2E_1)(2E_2)(2E_3)(2E_4)} \delta(E_3+E_4-E_1-E_2)\, \delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2) \frac{d^3 p_3}{(2\pi)^3} \frac{d^3 p_4}{(2\pi)^3}$$

$1+2\to 3+4$

$$\Rightarrow \Gamma_{if} = \frac{(2\pi)^4}{V(2E_1)(2E_2)} \int |M_{fi}|^2 \delta(E_3+E_4-E_1-E_2)\, \delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2) \frac{d^3 p_3}{(2\pi)^3\, 2E_3} \frac{d^3 p_4}{(2\pi)^3\, 2E_4} \quad \text{(O)}$$

∴ (M) & (O) ⇒

$1+2 \to 3+4$

$$\sigma = \frac{(2\pi)^{-2}}{(2E_1)(2E_2)(v_1+v_2)} \int |M_{fi}|^2 \delta(E_3+E_4-E_1-E_2)\,\delta^3(\underline{p}_3+\underline{p}_4-\underline{p}_1-\underline{p}_2)\,\frac{d^3\underline{p}_3}{2E_3}\,\frac{d^3\underline{p}_4}{2E_4}$$ (P)

- $\frac{(2\pi)^{-2}}{(2E_1)(2E_2)(v_1+v_2)}$: ???? ← INVESTIGATE!
- $|M_{fi}|^2$: PHYSICS
- $\delta(E_3+E_4-E_1-E_2)$: ENERGY CONS
- $\delta^3(\ldots)$: MOM CONS
- $\frac{d^3\underline{p}_3}{2E_3}\frac{d^3\underline{p}_4}{2E_4}$: DENSITY OF STATES

σ should** be **INVARIANT** w.r.t. Longitudinal Lorentz Boosts.

** (No Lorentz contraction ⊥ to Lorentz boost!)

(ENERGY CONS), (MOM CONS) and (PHASE-SPACE) terms are certainly Lorentz **INVARIANT.**

APPENDIX 1 (slide 45) shows that (provided $\underline{p}_1$ & $\underline{p}_2$ are collinear, which is a property preserved by Longitudinal boosts) it is always the case that

$$(2E_1)(2E_2)(v_1+v_2) = F \underset{\text{def}}{\equiv} 4\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2}$$

Hence the "????" term is Longitudinally Lorentz **INVARIANT** too!

(And as a corollary we see that consistency will thus force $|M_{fi}|^2$ to be Longitudinally Lorentz **INVARIANT** too.)*

* $|M_{fi}|^2$ is actually Lorentz INVARIANT, but the corollary on the left only guarantees this Longitudinally.

SUPER NICE

(Longitudinally) Lorentz Invariant Flux

$$F \underset{def}{\equiv} 4\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2} = (2E_1)(2E_2)(v_1 + v_2)$$

is called the Lorentz Invariant Flux.

As it is Lorentz Invariant, it can be evaluated easily in different frames to get different looking (but ultimately identically valued) expressions for it. E.g:

Centre of Mass Frame

$$\begin{aligned} F &= 4E_1^* E_2^* (v_1^* + v_2^*) \\ &= 4E_1^* E_2^* \left(\frac{|\underline{p}_1^*|}{E_1^*} + \frac{|\underline{p}_2^*|}{E_2^*} \right) \\ &= 4(E_2^* + E_1^*)|\underline{p}^*| \\ &= 4|\underline{p}^*|\sqrt{s} \end{aligned}$$

Ⓠ

(↑ Mandlestam s)

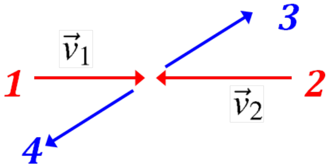
Fixed Target Frame

$$\begin{aligned} F &= 4E_1 E_2 (v_1 + v_2) \\ &= 4E_1 m_2 (v_1 + 0) \\ &= 4E_1 m_2 \frac{|\underline{p}_1|}{E_1} \\ &= 4 m_2 |\underline{p}_1| \end{aligned}$$

Ⓡ

$2 \to 2$ Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider $2 \to 2$ scattering in C.O.M. frame
- Start from



$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \, \delta\left(E_1 + E_2 - E_3 - E_4\right) \delta^3\left(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4\right) \frac{\mathrm{d}^3\vec{p}_3}{2E_3} \frac{\mathrm{d}^3}{2E_4}$$

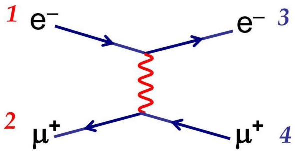
- Here $\vec{p}_1 + \vec{p}_2 = 0$ and $E_1 + E_2 = \sqrt{s}$

$$\Rightarrow \sigma = \frac{(2\pi)^{-2}}{4\,|\vec{p}_i^*|\,\sqrt{s}} \int |M_{fi}|^2 \, \delta\left(\sqrt{s} - E_3 - E_4\right) \delta^3\left(\vec{p}_3 + \vec{p}_4\right) \frac{\mathrm{d}^3\vec{p}_3}{2E_3} \frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

$\star$ The integral above, , is exactly the same as the integral which appeared in the one-to-two particle decay calculation, except that it has m_a replaced by $\sqrt{s}$. So:

$$\sigma = \frac{(2\pi)^{-2}}{4\,|\vec{p}_i^*|\,\sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 \;\mathrm{d}\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 \;\mathrm{d}\Omega^*}$$



- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$

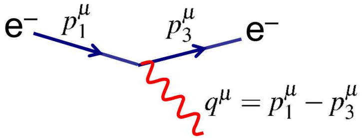
$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 \, \mathrm{d}\Omega^*$$

- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 \, \mathrm{d}\Omega^*$$

because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\,\mathrm{d}\phi^*$ refer to the C.o.M frame

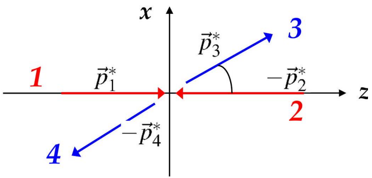
- For the last calculation in this section, we need to find a L.I. expression for $d\sigma$
- Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam $t = (p_1 - p_3)^2$ ... i.e. the square of the four-momentum transfer



- Want to express $\mathrm{d}\Omega^*$ in terms of Lorentz Invariant $\mathrm{d}t$ where
  $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2p_1 \cdot p_3$

- In C.o.M. frame:

$$p_1^{*\mu} = (E_1^*, 0, 0, |\vec{p}_1^*|)$$
$$p_3^{*\mu} = (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*)$$
$$p_1^\mu p_{3\mu} = E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^*$$
$$t = m_1^2 + m_3^3 - 2E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^*$$



giving $\quad \mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\,\mathrm{d}(\cos\theta^*)$

therefore $\quad \mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\,\mathrm{d}\phi^* = \frac{\mathrm{d}t\ \mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$

hence $\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2\ \mathrm{d}\Omega^* = \frac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\ \mathrm{d}\phi^*\ \mathrm{d}t$

- Finally, integrating over $\mathrm{d}\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

Lorentz Invariant differential cross section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^{\,*}|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^{\,*}|^2 = \frac{1}{4s}\left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$

e.g. electron or neutrino scattering

In this limit

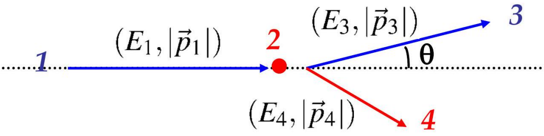
$$|\vec{p}_i^{\,*}|^2 = \frac{\left(s-m_2^2\right)^2}{4s}$$

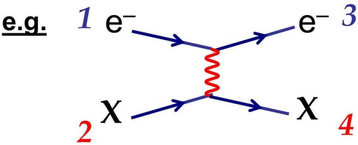
and so

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi\left(s-m_2^2\right)^2}\left|M_{fi}\right|^2} \qquad (m_1=0)\,.$$

$2 \to 2$ Body Scattering in Lab. Frame

- The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, \quad m_2 = m_4 = M$





- Wish to express the cross section in terms of scattering angle of the e^-

$$\mathrm{d}\Omega = 2\pi \mathrm{d}(\cos\theta)$$

$$\text{therefore} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

- The rest is some rather tedious algebra.... start from four-momenta:

$$p_1 = (E_1, 0, 0, E_1)\,, p_2 = (M, 0, 0, 0), p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta)\,, p_4 = (E_4, \vec{p}_4)$$

$$\text{so here } t = (p_1 - p_3)^2 = -2p_1\cdot p_3 = -2E_1E_3(1-\cos\theta)$$

But from (E, p) conservation $p_1 + p_2 = p_3 + p_4$ and so we can also express t in terms of particles 2 and 4.

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2 \cdot p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

Equating the two expressions for t gives $\boxed{E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}}$ and so

$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M \left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

and $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi} \frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi} 2M \frac{E_3^2}{M} \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi} \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi} \frac{1}{16\pi(s - M^2)^2} |M_{fi}|^2 .$$

Thus given that $s = (p_1 + p_2)^2 = M^2 + 2p_1 \cdot p_2 = M^2 + 2ME_1$ we have $(s - M^2) = 2ME_1$ and so finally we have the important result:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2} \qquad \text{(in limit } m_1 \to 0\text{).}$$

In the last equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

so we could instead write the following if we preferred to be explicit about θ dependence:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2}.$$

General form for $2 \rightarrow 2$ Body Scattering in Lab. Frame

$\star$ The calculation of the differential cross section for the case where m_1 cannot be neglected is longer and contains no more 'physics' (see Appendix II). It gives:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{m_2 \left|\vec{p}_1\right|} \cdot \frac{\left|\vec{p}_3\right|^2}{\left|\vec{p}_3\right| (E_1 + m_2) - E_3 \left|\vec{p}_1\right| \cos\theta} \cdot \left|M_{fi}\right|^2}$$

Again, there is only one independent variable, θ, which can be seen from the conservation of energy:

$$E_1 + m_2 = \sqrt{\left|\vec{p}_3\right|^2 + m_3^2} + \sqrt{\underbrace{\left|\vec{p}_1\right|^2 + \left|\vec{p}_3\right|^2 - 2\left|\vec{p}_1\right|\left|\vec{p}_3\right|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e., $\left|\vec{p}_3\right|$ is a function of θ.

Summary

We used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the Lorentz Invariant Matrix Element and wave-functions normalised to $2E$ particles per unit volume. our main results were:

★**Particle decay:**

(4)
$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

★**Scattering cross section in C.o.M. frame:**

(5)
$$\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

★**Invariant differential cross section (valid in all frames):**

(6)
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

and (continued next slide ...)

Summary continued ...

★**Differential cross section in the lab. frame ($m_1=0$)**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \quad (7) \quad \Leftrightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2 |M_{fi}|^2 \quad (8)$$

★**Differential cross section in the lab. frame ($m_1 \neq 0$)**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{|m_2|\vec{p}_1|}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2 \quad (9)$$

with $E_1+m_2 = \sqrt{|\vec{p}_3|^2+m_3^2}+\sqrt{|\vec{p}_1|^2+|\vec{p}_3|^2-2|\vec{p}_1||\vec{p}_3|\cos\theta+m_4^2}$

Summary of the summary:

- We should not need to do any more kinematic calculations, as all the kinematics we need for particle decays and two-to-two scatterings and cross sections are now embedded in the results above.
- Different physical theories change the matrix element, so we can re-use the above for many different sorts of interaction.
- Many later calculations will take one of the above results as a starting point.

Appendix I: Logitudinal invariance of 'Lorentz Invariant Flux'

The argument in this appendix aims to show that the so-called 'Lorentz Invariant Flux', F, defined only for collinear collisions $a \underset{v_a,\vec{p}_a}{\longrightarrow}\underset{v_b,\vec{p}_b}{\longleftarrow} b$ by

$$F = 2E_a 2E_b \left(v_a + v_b\right)$$

may be written in a Lorentz Invariant way, constant across all frames for which the collision is colliner.

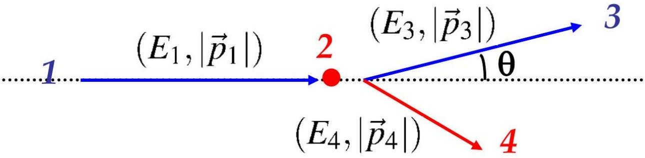
For all such frames: $p_a \cdot p_b = p_a^\mu p_{b\mu} = E_a E_b - \vec{p}_a \cdot \vec{p}_b = E_a E_b + |\vec{p}_a|\,|\vec{p}_b|$. (It is the last step therein which assumes collinearity!) Thus, for all such frames:

$$\begin{aligned} F^2/16 - (p_a^\mu p_{b\mu})^2 &= \frac{1}{16}\left(2E_a 2E_b\left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b}\right)\right)^2 - (p_a \cdot p_b)^2 \\ &= \left(|\vec{p}_a|\,E_b + |\vec{p}_b|\,E_a\right)^2 - \left(E_a E_b + |\vec{p}_a|\,|\vec{p}_b|\right)^2 \\ &= |\vec{p}_a|^2\left(E_b^2 - |\vec{p}_b|^2\right) + E_a^2\left(|\vec{p}_b|^2 - E_b^2\right) \\ &= |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 \\ &= -m_a^2 m_b^2 \end{aligned}$$

and so

$$F = 4\left[(p_a^\mu p_{b\mu})^2 - m_a^2 m_b^2\right]^{1/2} \qquad \square.$$

Appendix II: General 2 → 2 Body Scattering in lab frame I



$p_1 = (E_1, 0, 0, |\vec{p}_1|)$, $p_2 = (M_2, 0, 0, 0)$, $p_3 = (E_3, E_3 \sin\theta, 0, E_3\cos\theta)$, $p_4 = (E_4, \vec{p}_4)$ again

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

But now the invariant quantity t :

$$t = (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2 \cdot p_4 = m_2^2 + m_4^2 - 2m_2E_4$$
$$= m_2^2 + m_4^2 - 2m_2(E_1 + m_2 - E_3)$$
$$\Rightarrow \frac{dt}{d(\cos\theta)} = 2m_2\frac{dE_3}{d(\cos\theta)}$$

Appendix II: General 2 → 2 Body Scattering in lab frame II

Which gives $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$

To determine $\mathrm{d}E_3/\mathrm{d}(\cos\theta$, first differentiate $E_3^2 - |\vec{p}_3|^2 = m_3^2$

$$2E_3\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = 2|\vec{p}_3|\frac{\mathrm{d}|\vec{p}_3|}{\mathrm{d}(\cos\theta)} \tag{10}$$

Then equate

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2 \qquad \text{to give}$$

$$m_1^2 + m_3^2 - 2(E_1E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2)\frac{\mathrm{d}E_3}{\mathrm{d}\cos\theta} - |\vec{p}_1|\cos\theta\frac{\mathrm{d}|\vec{p}_3|}{\mathrm{d}\cos\theta} = |\vec{p}_1||\vec{p}_3|$$

Using (10)

$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{|\vec{p}_1||\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \tag{11}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{m_2}{\pi}\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

Not examinable

Appendix II: General $2 \to 2$ Body Scattering in lab frame III

It is easy to show $|\vec{p}_i^{\,*}|\sqrt{s} = m_2\,|\vec{p}_1|$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}\,\frac{m_2}{64\pi^2 m_2^2\,|\vec{p}_1|^2}\,|M_{fi}|^2$$

and using (11) obtain

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{m_2\,|\vec{p}_1|}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|\,(E_1+m_2) - E_3\,|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}.$$