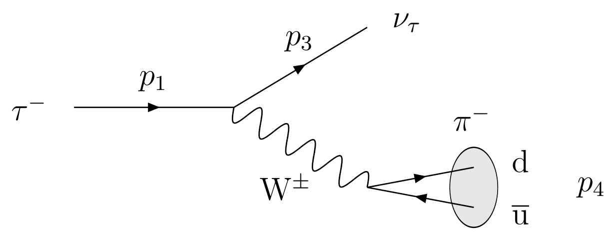
NST Part III Experimental and Theoretical Physics Dr J R Batley

Particle Physics Major Option Exam, January 2006

SOLUTIONS

2. **Tau lepton decay:**

Feynman diagram for $\tau^- \to \pi^- \nu_\tau$ decay:



The matrix element is

$$M_{\rm fi} = \sqrt{2} f_\pi V_{\rm ud} G_{\rm F} (j.p_4) \ ,$$

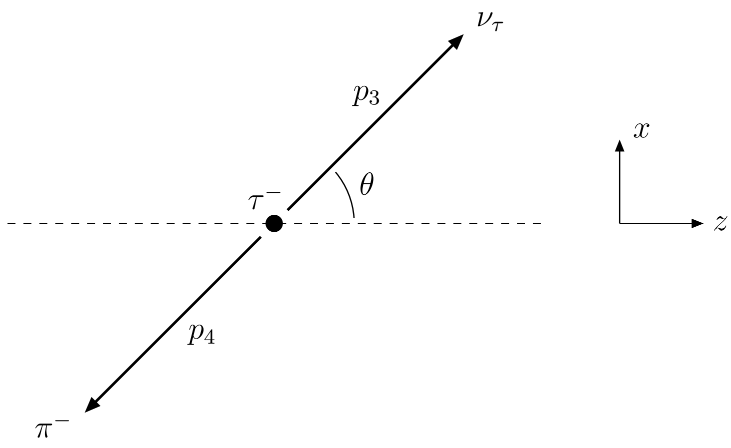
where the lepton current is

$$j^\mu = \overline{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1) \ .$$

In the τ^- rest frame, and assuming a massless neutrino, the four-momenta can be taken to be

$$p_1 = (m_\tau, 0, 0, 0)\,, \qquad p_3 = (p^*, p^* \sin\theta, 0, p^*\cos\theta)\,, \qquad p_4 = (E_\pi, -p^*\sin\theta, 0, -p^*\cos\theta) \ ,$$

where $E_\pi^2 = (p^*)^2 + m_\pi^2$ and $p^* = |\boldsymbol{p}_3| = |\boldsymbol{p}_4|$.



For a particle of 4-momentum $p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$, the spinors are

$$u_\uparrow(p) = \sqrt{E+m}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ \frac{p}{E+m}\cos\theta/2 \\ \frac{p}{E+m}e^{i\phi}\sin\theta/2 \end{pmatrix}, \qquad u_\downarrow(p) = \sqrt{E+m}\begin{pmatrix} -\sin\theta/2 \\ e^{i\phi}\cos\theta/2 \\ \frac{p}{E+m}\sin\theta/2 \\ -\frac{p}{E+m}e^{i\phi}\cos\theta/2 \end{pmatrix} \qquad (1)$$

For a massless neutrino, only the left-handed helicity eigenstate $u_\downarrow(p_3)$ can contribute. Since the neutrino four-momentum is $p_3 = (p^*, p^* \sin\theta, 0, p^* \cos\theta)$, the spinor $u_\downarrow(p_3)$ can be obtained from Equation (1) by setting $E = p = p^*$, $m = 0$, $\phi = 0$:

$$u_\downarrow(p_3) = \sqrt{p^*} \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \\ \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix} .$$

The corresponding adjoint spinor is

$$\overline{u}_\downarrow(p_3) = u_\downarrow^\dagger(p_3)\gamma^0 = \sqrt{p^*} \begin{pmatrix} -\sin\theta/2 & \cos\theta/2 & -\sin\theta/2 & \cos\theta/2 \end{pmatrix} .$$

The τ^- is in a spin eigenstate with the spin pointing in the $+z$ direction. The spinor $u(p_1)$ describing this spin state can be obtained from $u_\uparrow(p)$ of Equation (1) by setting $\theta = 0$ and taking the zero-momentum limit $E = m_\tau$, $p = 0$ (the value of ϕ is irrelevant): [1]

$$u(p_1) = \sqrt{2m_\tau} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{2}$$

The lepton current $j^\mu = \overline{u}_\downarrow(p_3)\gamma^\mu \frac{1}{2}(1-\gamma^5)u(p_1)$ can now be evaluated using standard matrix multiplication:

$$\tfrac{1}{2}(1-\gamma^5)u(p_1) = \tfrac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \sqrt{2m_\tau} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \tfrac{1}{2}\sqrt{2m_\tau} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} .$$

$$\begin{aligned}
\overline{u}_\downarrow(p_3)\gamma^0 &= \sqrt{p^*} \begin{pmatrix} -\sin\theta/2 & \cos\theta/2 & \sin\theta/2 & -\cos\theta/2 \end{pmatrix} \\
\overline{u}_\downarrow(p_3)\gamma^1 &= \sqrt{p^*} \begin{pmatrix} -\cos\theta/2 & \sin\theta/2 & \cos\theta/2 & -\sin\theta/2 \end{pmatrix} \\
\overline{u}_\downarrow(p_3)\gamma^2 &= \sqrt{p^*} \begin{pmatrix} -i\cos\theta/2 & -i\sin\theta/2 & i\cos\theta/2 & i\sin\theta/2 \end{pmatrix} \\
\overline{u}_\downarrow(p_3)\gamma^3 &= \sqrt{p^*} \begin{pmatrix} \sin\theta/2 & \cos\theta/2 & -\sin\theta/2 & -\cos\theta/2 \end{pmatrix}
\end{aligned}$$

which combine to give

$$\boxed{j^\mu = \overline{u}_\downarrow(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_1) = \sqrt{2m_\tau p^*}\,(-\sin\theta/2, -\cos\theta/2, -i\cos\theta/2, \sin\theta/2)} .$$

The 4-vector scalar product of the lepton current with p_4 is then

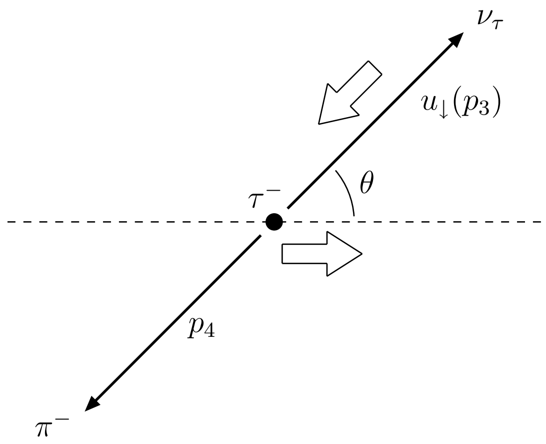
$$\begin{aligned}
j^\mu.p_4 &= \sqrt{2m_\tau p^*}(-E_\pi \sin\theta/2 - p^* \sin\theta\cos\theta/2 + p^*\cos\theta\sin\theta/2) \\
&= -\sqrt{2m_\tau p^*}(E_\pi + p^*)\sin\theta/2 \\
&= -\sqrt{2m_\tau p^*}m_\tau \sin\theta/2
\end{aligned}$$

[1]Equivalently, we could use the spinor $u_\downarrow(p)$ of Equation (1) and set $\theta = \pi$, since a negative helicity particle travelling in the $-z$ direction ($\theta = \pi$) has its spin pointing in the $+z$ direction, as required. This gives the same form for the spinor $u(p_1)$ as in Equation (2), up to an overall minus sign.

where energy conservation, $m_\tau = E_\pi + p^*$, has been used in the last step. Hence

$$\boxed{M_{\rm fi} = \sqrt{2} f_\pi V_{\rm ud} G_{\rm F} (j^\mu . p_4) = -2 f_\pi V_{\rm ud} G_{\rm F} m_\tau \sqrt{m_\tau p^*} \sin\theta/2} \ .$$

An angular distribution $\sin\theta/2$ is to be expected for the overlap of two spin-half wavefunctions. In particular, the matrix element vanishes for $\theta = 0$, where the τ^- and ν_τ spins are oppositely directed, and reaches a maximum for $\theta = \pi$, where the τ^- and ν_τ spins both point in the $+z$ direction.



If the τ^- spin points in the $-z$ direction, the matrix element must vanish at $\theta = \pi$ and reach a maximum at $\theta = 0$. The matrix element can be written down on symmetry grounds as

$$\boxed{M_{\rm fi} = -2 f_\pi V_{\rm ud} G_{\rm F} \sqrt{m_\tau p^*} m_\tau \cos\theta/2} \ .$$

Squaring the energy conservation equation $m_\tau = E_\pi + p^*$ gives

$$(m_\tau - p^*)^2 = E_\pi^2 = (p^*)^2 + m_\pi^2 \ .$$

This gives the centre of mass momentum as

$$p^* = \frac{m_\tau^2 - m_\pi^2}{2m_\tau} \ .$$

When the τ^- spin points in the $+z$ direction, the differential decay rate is then

$$\begin{aligned} \frac{d\Gamma}{d\Omega} &= \frac{p^*}{32\pi^2 m_\tau^2} |M_{\rm fi}|^2 = \frac{1}{32\pi^2 m_\tau^2} \cdot \frac{m_\tau^2 - m_\pi^2}{2m_\tau} \cdot 2 f_\pi^2 |V_{\rm ud}|^2 G_{\rm F}^2 (m_\tau^2 - m_\pi^2) m_\tau^2 \sin^2\theta/2 \\ &= \frac{f_\pi^2 G_{\rm F}^2}{32\pi^2} |V_{\rm ud}|^2 m_\tau^3 \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2 \sin^2\theta/2 \end{aligned}$$

where $d\Omega = d\cos\theta \, d\phi$. When the τ^- spin points in the $-z$ direction, the factor $\sin^2\theta/2$ is replaced by $\cos^2\theta/2$.

An unpolarised τ^- sample is effectively an equal mix of $S_z = +\frac{1}{2}$ and $S_z = -\frac{1}{2}$ τ^- leptons, and the differential decay rate can be obtained by averaging over the two possible τ^- spin states:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{f_\pi^2 G_\mathrm{F}^2}{64\pi^2}|V_\mathrm{ud}|^2 m_\tau^3 \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2 .$$

This decay rate is independent of θ (since $\sin^2\theta/2 + \cos^2\theta/2 = 1$), so that the decay is isotropic, as expected for an initially unpolarised sample.

The partial width for unpolarised $\tau^- \to \pi^-\nu_\tau$ decay is obtained by integrating over $\mathrm{d}\Omega$, giving a factor of 4π :

$$\boxed{\Gamma(\tau^- \to \pi^-\nu_\tau) = \frac{G_\mathrm{F}^2 f_\pi^2}{16\pi}|V_\mathrm{ud}|^2 m_\tau^3 \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2} .$$

The branching ratio is

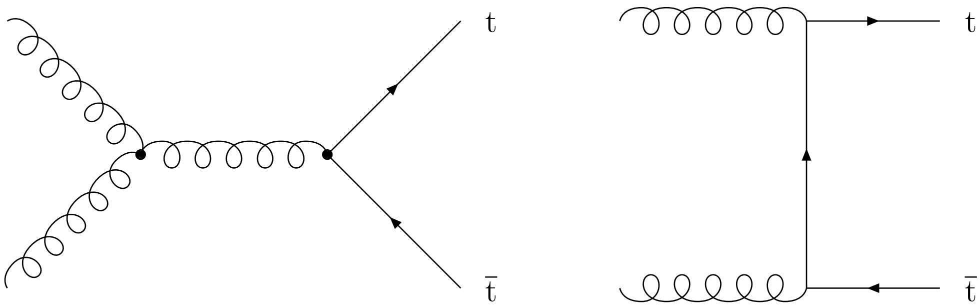
$$\mathrm{BR}(\tau^- \to \pi^-\nu_\tau) = \frac{\Gamma(\tau^- \to \pi^-\nu_\tau)}{\Gamma} = \tau_\tau\Gamma(\tau^- \to \pi^-\nu_\tau) = \tau_\tau\frac{G_\mathrm{F}^2 f_\pi^2}{16\pi}|V_\mathrm{ud}|^2 m_\tau^3 \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2 .$$

Estimating $|V_\mathrm{ud}| \approx 1$ (or $|V_\mathrm{ud}|^2 \approx \cos^2\theta_\mathrm{C} \approx 0.95$ to be more precise), taking $f_\pi = m_\pi$, and using $\tau_\tau = 2.91 \times 10^{-13}\,$s gives

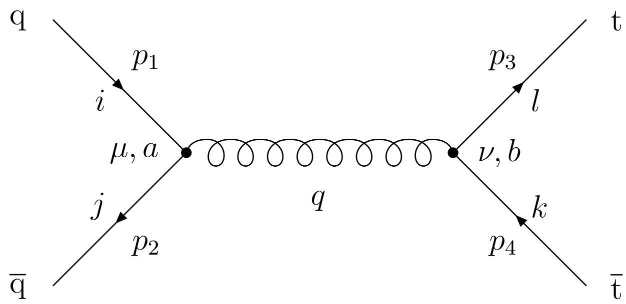
$$\begin{aligned}\mathrm{BR}(\tau^- \to \pi^-\nu_\tau) &= \frac{(2.91 \times 10^{-13})}{(6.582 \times 10^{-25})}\frac{(1.166 \times 10^{-5})^2(0.1396)^2}{16\pi}(1.777)^3 \left(1 - \frac{(0.1396)^2}{(1.777)^2}\right)^2 \\ &= \boxed{12.9\%} .\end{aligned}$$

3. **Colour and partons:**

Leading-order Feynman diagrams for $\mathrm{gg} \to \mathrm{t\bar{t}}$:



Leading order Feynman diagram for $\mathrm{q\bar{q}} \to \mathrm{t\bar{t}}$:



Colour factors for $\mathrm{q\bar{q}} \to \mathrm{t\bar{t}}$: the $\mathrm{q\bar{q}}$ and $\mathrm{t\bar{t}}$ vertices contribute $\frac{1}{2}\lambda^a_{ij}$ and $\frac{1}{2}\lambda^a_{kl}$ respectively:

$$C(i\bar{j} \to l\bar{k}) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ij}\lambda^a_{kl} \ .$$

For $r\bar{r} \to r\bar{r}$, we have $i = j = k = l = 1$:

$$C(r\bar{r} \to r\bar{r}) = \frac{1}{4}\sum_{a=1}^{8}(\lambda^a_{11})^2 = \frac{1}{4}\left[(\lambda^3_{11})^2 + (\lambda^8_{11})^2\right] = \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3} \ .$$

Similarly:

$$C(g\bar{g} \to g\bar{g}) = \frac{1}{4}\left[(\lambda^3_{22})^2 + (\lambda^8_{22})^2\right] = \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3} \ .$$

$$C(b\bar{b} \to b\bar{b}) = \frac{1}{4}(\lambda^8_{33})^2 = \frac{1}{4}\left(\frac{-2}{\sqrt{3}}\right)^2 = \frac{1}{3} \ .$$

For $r\bar{r} \to g\bar{g}$, we have $i = j = 1$ and $k = l = 2$, so

$$C(r\bar{r} \to g\bar{g}) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{22} = \frac{1}{4}\left(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22}\right) = \frac{1}{4}\left(1 \cdot -1 + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right) = -\frac{1}{6} \ .$$

For $r\overline{g} \to r\overline{g}$, we have $i = 1, j = 2$ and $k = 2, l = 1$, so

$$C(r\overline{g} \to r\overline{g}) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{12}\lambda^a_{21} = \frac{1}{4}\left(\lambda^1_{12}\lambda^1_{21} + \lambda^2_{12}\lambda^2_{21}\right) = \frac{1}{4}\left(1\cdot 1 + i\cdot -i\right) = \frac{1}{2} \; .$$

In summary, the allowed colour factors contributing to the matrix element $M_{\rm fi}$ are

$$C(r\overline{r} \to r\overline{r}) = C(g\overline{g} \to g\overline{g}) = C(b\overline{b} \to b\overline{b}) = \frac{1}{3}$$

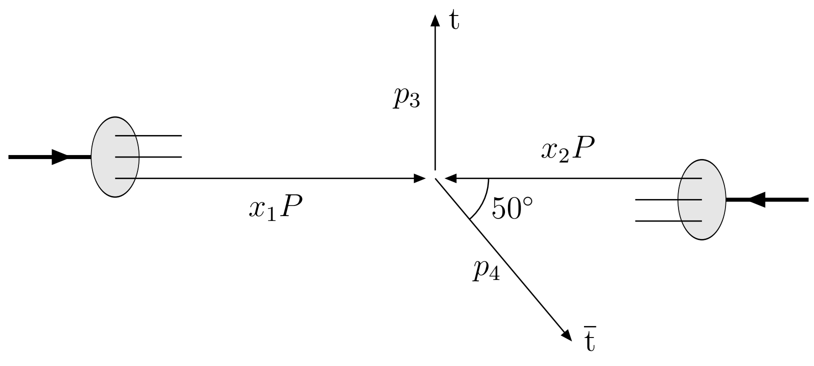
$$C(r\overline{g} \to r\overline{g}) = C(r\overline{b} \to r\overline{b}) = C(g\overline{r} \to g\overline{r}) = C(g\overline{b} \to g\overline{b}) = C(b\overline{r} \to b\overline{r}) = C(b\overline{g} \to b\overline{g}) = \frac{1}{2}$$

$$C(r\overline{r} \to g\overline{g}) = C(r\overline{r} \to b\overline{b}) = C(g\overline{g} \to r\overline{r}) = C(g\overline{g} \to b\overline{b}) = C(b\overline{b} \to r\overline{r}) = C(b\overline{b} \to g\overline{g}) = -\frac{1}{6}$$

In $\mathrm{q\overline{q}} \to \mathrm{t\overline{t}}$ scattering in high energy hadron-hadron collisions, the initial state q and $\overline{\mathrm{q}}$ are not in a well-defined colour state, but rather each is effectively an equal mix (unpolarised mixture) of red, green and blue. The colour factor appearing in the $\mathrm{q\overline{q}} \to \mathrm{t\overline{t}}$ cross section (which contains $|M_{\rm fi}|^2$) is obtained by summing over all allowed colour configurations for the scattering, and averaging over the possible colours of the initial q and $\overline{\mathrm{q}}$ (factor of 1/3 for each):

$$\left\langle |C(\mathrm{q\overline{q}} \to \mathrm{t\overline{t}})|^2 \right\rangle = \frac{1}{3}\cdot\frac{1}{3}\left[3\times\left(\frac{1}{3}\right)^2 + 6\times\left(-\frac{1}{6}\right)^2 + 6\times\left(\frac{1}{2}\right)^2\right] = \frac{2}{9} \; .$$

Consider the production of a $\mathrm{t\overline{t}}$ pair in a hadron-hadron collision, due to the interaction of two partons with momentum fractions x_1 and x_2:



The t quark has four-momentum (with $p = 40\,\mathrm{GeV}$)

$$p_3 = (\sqrt{p^2 + m_t^2}, p, 0, 0) = (\sqrt{40^2 + 175^2}, 40, 0, 0) = (179.513, 40, 0, 0) \; .$$

Since the transverse momentum of the $\overline{\mathrm{t}}$ is the same as that of the t, namely $40\,\mathrm{GeV}$, the $\overline{\mathrm{t}}$ momentum is $40/\sin 50^\circ = 52.216\,\mathrm{GeV}$. Hence the $\overline{\mathrm{t}}$ four-momentum is

$$p_4 = (\sqrt{(p/\sin 50^\circ)^2 + m_t^2}, -p, 0, p\cot 50^\circ) = (182.624, -40, 0, 33.564) \; ,$$

and the four-momentum of the $t\bar{t}$ system is

$$p_3 + p_4 = (362.137, 0, 0, 33.564)\ .$$

The incoming partons have 4-momenta $(x_1 P, 0, 0, x_1 P)$ and $(x_2 P, 0, 0, -x_2 P)$. Conservation of energy and momentum then gives

$$\begin{aligned}(x_1 + x_2)P &= 362.137\,\text{GeV}\\ (x_1 - x_2)P &= \ \ 33.564\,\text{GeV}\end{aligned}$$

At the Tevatron, with beam momenta $P = 980\,\text{GeV}$, these equations give

$$\begin{aligned}x_1 &= (362.14 + 33.56)/(2 \times 980) = 0.202\\ x_2 &= (362.14 - 33.56)/(2 \times 980) = 0.168\end{aligned}$$

At the LHC, with beam momenta $P = 7000\,\text{GeV}$, we have

$$\begin{aligned}x_1 &= (362.14 + 33.56)/(2 \times 7000) = 0.028\\ x_2 &= (362.14 - 33.56)/(2 \times 7000) = 0.023\end{aligned}$$

Measurements of parton distribution functions $q(x)$ and $g(x)$ show that quarks dominate for momentum fractions $x > 0.15$-0.2, and gluons dominate below this. Hence, at the Tevatron, $q\bar{q} \rightarrow t\bar{t}$ dominates, while at the LHC, the most likely production mechanism is $gg \rightarrow t\bar{t}$.