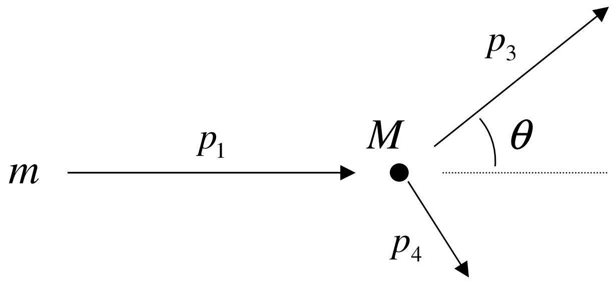
NST Part III Experimental and Theoretical Physics Dr J R Batley

Particle Physics Major Option Exam, January 2004

SOLUTIONS

1. **Elastic ep and deep-inelastic scattering**

Consider elastic scattering in the lab frame, neglecting the mass of the incoming particle:



The four-momenta of the beam particle before and after the scattering are $p_1 = (E_1, 0, 0, E_1)$ and $p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta)$, with $p_1^2 = p_3^2 = 0$. The 4-momentum transfer is $q = p_1 - p_3$, and squaring this equation gives

$$q^2 = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1 - \cos\theta) \ . \qquad (1)$$

Conservation of 4-momentum, $p_1 + p_2 = p_3 + p_4$, gives $p_2 + q = p_4$. Squaring this equation and using $p_2^2 = p_4^2 = M^2$ gives

$$M^2 + 2p_2.q + q^2 = M^2 \ .$$

Since $p_2 = (M, 0, 0, 0)$, we have $p_2.q = M(E_1 - E_3)$ and hence

$$E_1 - E_3 = \frac{-q^2}{2M} \ . \qquad (2)$$

With $q^2 = -2.5\,\mathrm{GeV}^2$ and $\theta = 60^\circ$, Equation (1) gives

$$E_1E_3 = \frac{-q^2}{2(1 - \cos\theta)} = \frac{2.5\,\mathrm{GeV}^2}{2(1 - \cos 60^\circ)} = 2.5\,\mathrm{GeV}^2$$

while Equation (2) gives

$$E_1 - E_3 = \frac{2.5\,\mathrm{GeV}^2}{2 \times 0.938\,\mathrm{GeV}} = 1.33\,\mathrm{GeV} \ .$$

Eliminating E_3 gives a quadratic equation for E_1:

$$E_1(E_1 - 1.33) = 2.5$$

which can be solved to give $E_1 = 2.38\,\text{GeV}$.

In deep-inelastic scattering, the Lorentz invariant variables x and y are defined as

$$x \equiv \frac{-q^2}{2M\nu}, \qquad y \equiv \frac{p_2 . q}{p_2 . p_1}$$

where

$$\nu = E_1 - E_3 \ .$$

The underlying process is elastic scattering of the incoming lepton from a quark of mass m. Applying Equation (2) to the quark scattering gives

$$\nu = E_1 - E_3 = \frac{-q^2}{2m}$$

and a comparison with the definition of x immediately gives

$$x = \frac{m}{M} \ .$$

Now consider the infinite momentum frame and suppose that the quark carries a fraction x of the target proton's momentum in this frame. Then the quark has 4-momentum (xE, xp) and therefore

$$m^2 = (xE)^2 - (xp)^2 = x^2(E^2 - p^2) = x^2 M^2 \ .$$

Hence we again obtain $x = m/M$, validating the supposition that x can be interpreted as the fractional momentum.

In the lepton-quark centre of mass frame in the relativistic limit, the 4-momenta are $p_1 = (E, 0, 0, E)$, $p_2 = (E, 0, 0, -E)$, and $p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$. Therefore the 4-momentum transfer is

$$q = p_1 - p_3 = (0, -E\sin\theta^*, 0, E(1-\cos\theta^*))$$

and the variable y is

$$y = \frac{p_2 . q}{p_2 . p_1} = \frac{E^2(1-\cos\theta^*)}{2E^2} = \frac{1}{2}(1-\cos\theta^*) \ .$$

A differential cross section $\mathrm{d}\sigma/\mathrm{d}y \propto (1-y)^2$ corresponds to

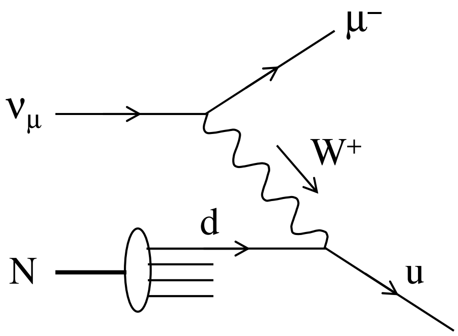
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta^*} = \frac{\mathrm{d}y}{\mathrm{d}\cos\theta^*}\frac{\mathrm{d}\sigma}{\mathrm{d}y} \propto \frac{1}{2}(1-y)^2 = \frac{1}{8}(1+\cos\theta^*)^2 \ .$$

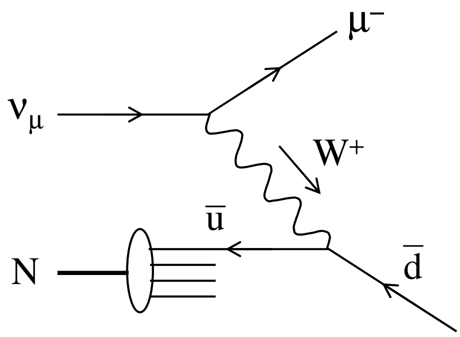
Similarly, a differential cross section $\mathrm{d}\sigma/\mathrm{d}y = \text{constant}$ corresponds to $\mathrm{d}\sigma/\mathrm{d}\cos\theta^* = \text{constant}$, namely to isotropic scattering in the centre of mass.

The differential cross section for neutrino scattering, $\nu_\mu \mathrm{p} \to \mu^- + X$ for example, is

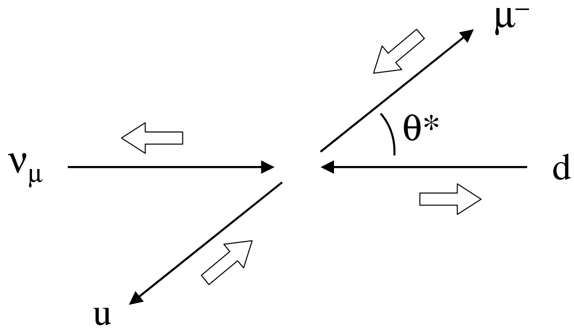
$$\frac{\mathrm{d}^2\sigma^{\nu\mathrm{p}}}{\mathrm{d}x\mathrm{d}y} = \frac{G_\mathrm{F}^2 s x}{\pi}\left[d(x) + (1-y)^2\overline{u}(x)\right] \ .$$

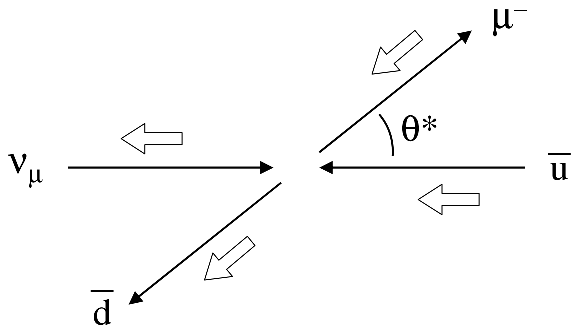
The first term on the right-hand side is the contribution from $\nu_\mu \mathrm{d} \to \mu^- \mathrm{u}$ scattering, while the second term arises from $\nu_\mu \overline{\mathrm{u}} \to \mu^- \overline{\mathrm{d}}$ scattering. The leading-order Feynman diagrams are:





The scattering involves the $W^{\pm}$ boson and is therefore $V-A$ in structure. In the relativistic limit, all the particles involved must be left-handed, while antiparticles must be right-handed. In the lepton-(anti)quark centre of mass frame we therefore have:



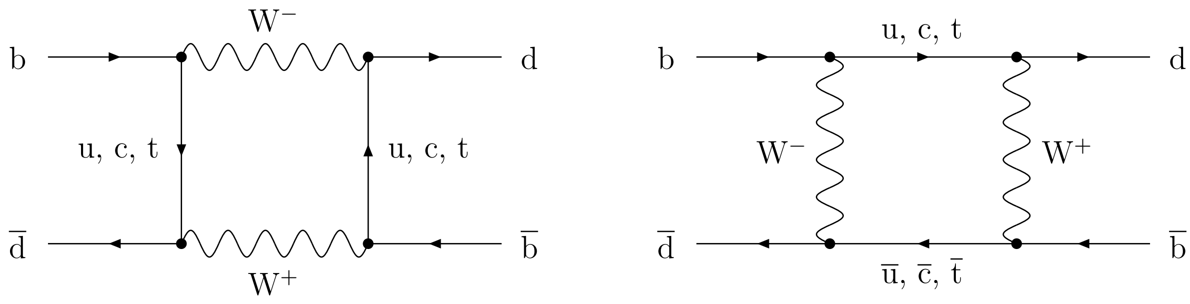


In the first case, the total spin is zero, and there is no preferred spatial direction. The scattering is therefore isotropic in the neutrino-quark centre of mass frame, and hence does not depend on y.

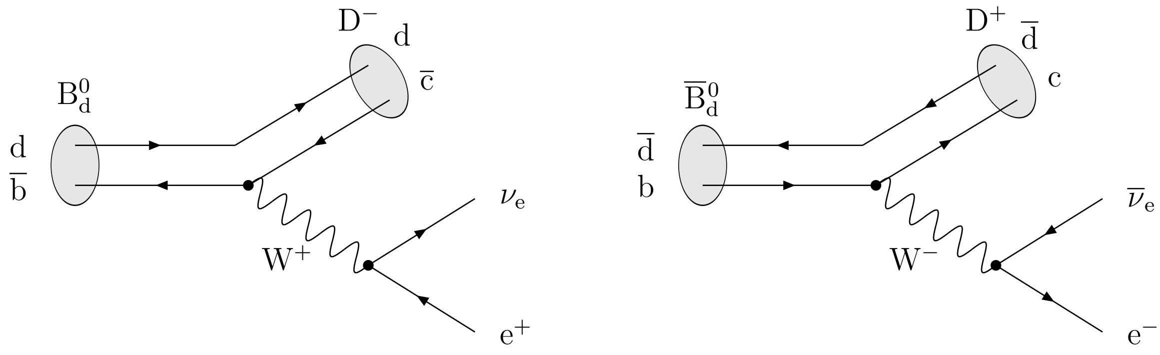
In the second case, the total spin is one in the initial and final states, giving a cross section proportional to $(1+\cos\theta^*)^2$. This gives $\mathrm{d}\sigma/\mathrm{d}y \propto (1-y)^2$.

3. **B meson and neutrino oscillations:**

The leading order Feynman diagrams for $B_d^0 - \overline{B}_d^0$ mixing are:



Examples of Feynman diagrams for semileptonic B_d^0 and $\overline{B}_d^0$ decay are:



(The name of the final state meson, $D^{\pm}$ in this example, is not expected to be known). The semileptonic decay of B_d^0 always produces a *positive* charged lepton ℓ^+ while the semileptonic decay of $\overline{B}_d^0$ always produces a *negative* charged lepton ℓ^-.

Neglecting CP violation, the mass eigenstates are

$$B_H = \frac{1}{\sqrt{2}}(B_d^0 + \overline{B}_d^0), \qquad B_L = \frac{1}{\sqrt{2}}(B_d^0 - \overline{B}_d^0)\ .$$

These equations can be inverted to give

$$B_d^0 = \frac{1}{\sqrt{2}}(B_H + B_L), \qquad \overline{B}_d^0 = \frac{1}{\sqrt{2}}(B_H - B_L)\ .$$

For an initial B_d^0, the wavefunction evolves with time as

$$\begin{aligned} B_d^0(t) &= \frac{1}{\sqrt{2}}\left[B_H e^{-im_H t - \Gamma t/2} + B_L e^{-im_L t - \Gamma t/2}\right] \\ &= \frac{1}{2}e^{-\Gamma t/2}\left[(B_d^0 + \overline{B}_d^0)e^{-im_H t} + (B_d^0 - \overline{B}_d^0)e^{-im_L t}\right] \\ &= \frac{1}{2}e^{-\Gamma t/2}\left[B_d^0(e^{-im_H t} + e^{-im_L t}) + \overline{B}_d^0(e^{-im_H t} - e^{-im_L t})\right] \end{aligned}$$

Hence the transition rates are

$$\begin{aligned} \Gamma(B_d^0 \to \overline{B}_d^0) &\propto e^{-\Gamma t}|e^{-im_H t} - e^{-im_L t}|^2 = 2e^{-\Gamma t}(1 - \cos\Delta m t) \\ \Gamma(B_d^0 \to B_d^0) &\propto e^{-\Gamma t}|e^{-im_H t} + e^{-im_L t}|^2 = 2e^{-\Gamma t}(1 + \cos\Delta m t) \end{aligned}$$

where $\Delta m \equiv m_{\mathrm{H}} - m_{\mathrm{L}}$. Using

$$\int_0^\infty e^{-\alpha t} \cos \beta t \mathrm{d}t = \frac{\alpha}{\alpha^2 + \beta^2}$$

the integrated rate for the transition $\mathrm{B_d^0} \to \overline{\mathrm{B}}{}_\mathrm{d}^0$ is

$$\Gamma(\mathrm{B_d^0} \to \overline{\mathrm{B}}{}_\mathrm{d}^0) \propto \int_0^\infty e^{-\Gamma t}(1 - \cos \Delta m t)\mathrm{d}t = \frac{1}{\Gamma}\frac{x^2}{1+x^2}$$

where $x \equiv \Delta m / \Gamma$. The integrated rate for no oscillation is

$$\Gamma(\mathrm{B_d^0} \to \mathrm{B_d^0}) \propto \int_0^\infty e^{-\Gamma t}(1 + \cos \Delta m t)\mathrm{d}t = \frac{1}{\Gamma}\frac{2+x^2}{1+x^2} \ .$$

Normalising these transition rates so that they sum to unity gives the transition probabilities

$$P(\mathrm{B_d^0} \to \overline{\mathrm{B}}{}_\mathrm{d}^0) = \frac{1}{2}\frac{x^2}{1+x^2}, \qquad P(\mathrm{B_d^0} \to \mathrm{B_d^0}) = \frac{1}{2}\frac{2+x^2}{1+x^2} \ .$$

The experiment produces $\mathrm{B_d^0}\overline{\mathrm{B}}{}_\mathrm{d}^0$ pairs (for example in a process such as $\mathrm{p\overline{p}} \to \mathrm{B_d^0}\overline{\mathrm{B}}{}_\mathrm{d}^0 + \mathrm{X}$), where the $\mathrm{B_d^0}$ and $\overline{\mathrm{B}}{}_\mathrm{d}^0$ are uncorrelated, *i.e.* each B meson evolves independently. Since the semileptonic decay of the $\mathrm{B_d^0}$ produces a positive lepton, ℓ^+, while the semileptonic decay of the $\overline{\mathrm{B}}{}_\mathrm{d}^0$ produces a negative lepton, ℓ^-, the only way to get a pair of leptons with the same charge, $\ell^\pm\ell^\pm$, is for one of the B mesons to undergo mixing while the other B meson does not:

$$\text{either} \qquad (\mathrm{B_d^0} \to \overline{\mathrm{B}}{}_\mathrm{d}^0)(\overline{\mathrm{B}}{}_\mathrm{d}^0 \to \overline{\mathrm{B}}{}_\mathrm{d}^0) \qquad \text{or} \qquad (\mathrm{B_d^0} \to \mathrm{B_d^0})(\overline{\mathrm{B}}{}_\mathrm{d}^0 \to \mathrm{B_d^0}) \ .$$

Defining

$$p \equiv P(\mathrm{B_d^0} \to \overline{\mathrm{B}}{}_\mathrm{d}^0) = P(\overline{\mathrm{B}}{}_\mathrm{d}^0 \to \mathrm{B_d^0}) = \frac{1}{2}\frac{x^2}{1+x^2} \ ,$$

the fraction of like-sign dileptons is

$$\begin{aligned} f(\ell^\pm\ell^\pm) &= P(\mathrm{B_d^0} \to \overline{\mathrm{B}}{}_\mathrm{d}^0).P(\overline{\mathrm{B}}{}_\mathrm{d}^0 \to \overline{\mathrm{B}}{}_\mathrm{d}^0) + P(\mathrm{B_d^0} \to \mathrm{B_d^0}).P(\overline{\mathrm{B}}{}_\mathrm{d}^0 \to \mathrm{B_d^0}) \\ &= p(1-p) + (1-p)p = 2p(1-p) = 0.297 \ . \end{aligned}$$

Solving the quadratic equation gives $p = 0.1814$. Hence

$$\frac{1}{2}\frac{x^2}{1+x^2} = 0.1814$$

which gives $x = 0.755$. The mass difference Δm is therefore

$$\Delta m = x\Gamma = \frac{x}{\tau} = \frac{0.755}{1.54 \times 10^{-12}\,\mathrm{s}} \times (6.582 \times 10^{-25}\,\mathrm{GeV.\,s}) = 3.23 \times 10^{-13}\,\mathrm{GeV} \ .$$

(b) Solar neutrino experiments reveal $\nu_\mathrm{e} \to \nu_\mu$ and/or $\nu_\mathrm{e} \to \nu_\tau$ oscillations with a mass-squared splitting

$$\Delta m_{12}^2 \approx 7 \times 10^{-5}\,\mathrm{eV}^2 \ .$$

Atmospheric neutrino experiments reveal $\nu_\mu \to \nu_\tau$ oscillations with

$$\Delta m_{13}^2 \approx \Delta m_{23}^2 \approx 2 \times 10^{-3}\,\mathrm{eV}^2 \ .$$

(Precise numerical values for Δm^2 are not expected). For the K2K experiment, the oscillation wavelengths corresponding to these two mass-squared splittings are

$$\lambda_{12} \approx \frac{4\pi E}{\Delta m_{12}^2} = \frac{4\pi \times 1.3\,\mathrm{GeV}}{7 \times 10^{-5}\,\mathrm{eV}^2} \times (0.197\,\mathrm{GeV\,fm}) = 4.6 \times 10^4\,\mathrm{km}$$
$$\lambda_{23} \approx \frac{4\pi E}{\Delta m_{23}^2} = \frac{4\pi \times 1.3\,\mathrm{GeV}}{2 \times 10^{-3}\,\mathrm{eV}^2} \times (0.197\,\mathrm{GeV\,fm}) = 1600\,\mathrm{km}$$

Hence, for $L = 250\,\mathrm{km}$, only atmospheric oscillations play a significant role. The most likely explanation is therefore that the K2K is observing $\nu_\mu \to \nu_\tau$ oscillations. The oscillation probability can be estimated as

$$P(\nu_\mu \to \nu_\tau) \approx \sin^2 2\theta \sin^2\left(\frac{\Delta m_{23}^2 L}{4E}\right) = \sin^2\left(\frac{2 \times 10^{-3}\,\mathrm{eV}^2 \times 250\,\mathrm{km}}{4 \times 1.3\,\mathrm{GeV}} \times \frac{1}{0.197\,\mathrm{GeV\,fm}}\right)$$
$$= \sin^2 0.488 = 0.22$$

which is consistent with the observation of $(80 - 56)/80 = 0.30 \pm 0.06$.