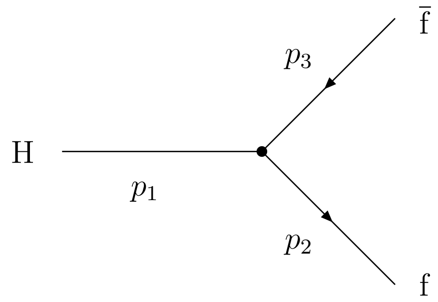
NST Part III Experimental and Theoretical Physics — Dr J R Batley

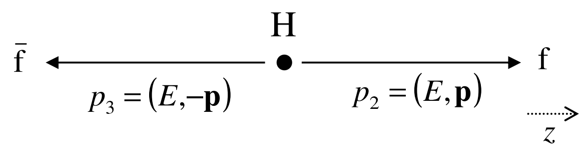
Particle Physics Major Option Exam, January 2002

SOLUTIONS

1. **Higgs boson decay, $\mathbf{H \to f\bar{f}}$:**

The leading-order Feynman diagram for the decay $\mathrm{H} \to \mathrm{f\bar{f}}$, and the configuration in the Higgs rest frame, are





The 4-momentum of the fermion f is $p_2 = (E, 0, 0, p)$ with $p > 0$ and $E^2 = p^2 + m_\mathrm{f}^2$. The basis spinors are:

$$u_1(p_2) = \sqrt{E + m_\mathrm{f}} \begin{pmatrix} 1 \\ 0 \\ p/(E + m_\mathrm{f}) \\ 0 \end{pmatrix}, \qquad u_2(p_2) = \sqrt{E + m_\mathrm{f}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -p/(E + m_\mathrm{f}) \end{pmatrix}$$

where $u_1(p_2)$ is positive helicity (right-handed) and $u_2(p_2)$ is negative helicity (left-handed). The corresponding adjoint spinors are:

$$\bar{u}_1(p_2) = \sqrt{E + m_\mathrm{f}}\ (1, 0, -p/(E + m_\mathrm{f}), 0)\,, \quad \bar{u}_2(p_2) = \sqrt{E + m_\mathrm{f}}\ (0, 1, 0, p/(E + m_\mathrm{f}))\ . \tag{1}$$

The antifermion 4-momentum is $p_3 = (E, 0, 0, -p)$ and the basis spinors are

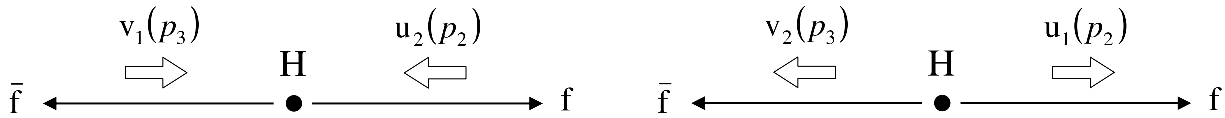
$$v_1(p_3) = \sqrt{E + m_\mathrm{f}} \begin{pmatrix} 0 \\ p/(E + m_\mathrm{f}) \\ 0 \\ 1 \end{pmatrix}, \qquad v_2(p_3) = \sqrt{E + m_\mathrm{f}} \begin{pmatrix} -p/(E + m_\mathrm{f}) \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{2}$$

where $v_1(p_3)$ corresponds to negative helicity and $v_2(p_3)$ to positive helicity.

Combining equations (1) and (2), the four possible combinations $\overline{u}(p_2)v(p_3)$ are

$$\bar{u}_1(p_2)v_1(p_3) = \quad \bar{u}_2(p_2)v_2(p_3) = 0 \tag{3}$$
$$\bar{u}_2(p_2)v_1(p_3) = -\bar{u}_1(p_2)v_2(p_3) = 2p\ . \tag{4}$$

Hence only two combinations, $\bar{u}_2(p_2)v_1(p_3)$ and $\bar{u}_1(p_2)v_2(p_3)$, give a non-zero matrix element. These correspond to the following spin configurations:



For $\bar{u}_2(p_2)v_1(p_3)$, the fermion and antifermion both have negative helicity, while for $\bar{u}_1(p_2)v_2(p_3)$ both particles have positive helicity. The total spin in the final state in both these cases is therefore zero, consistent with the fact that the Higgs boson has spin zero. The two other spinor combinations, $\bar{u}_1(p_2)v_1(p_3)$ and $\bar{u}_2(p_2)v_2(p_3)$ have total spin 1 in the final state, which is forbidden by angular momentum conservation.

The matrix element for the two non-zero spinor combinations of equation (4) is

$$M_{\mathrm{fi}} = \frac{g_{\mathrm{W}} m_{\mathrm{f}}}{2m_{\mathrm{W}}} \overline{u}(p_2)v(p_3) = \pm \frac{g_{\mathrm{W}} m_{\mathrm{f}}}{2m_{\mathrm{W}}} \cdot 2p$$

Summing over these two possibilities gives a decay rate

$$\Gamma = \frac{p}{8\pi m_{\mathrm{H}}^2} \langle |M_{\mathrm{fi}}|^2 \rangle = \frac{p}{8\pi m_{\mathrm{H}}^2} \cdot 2 \left(\frac{g_{\mathrm{W}} m_{\mathrm{f}}}{2m_{\mathrm{W}}} \cdot 2p \right)^2 = \frac{1}{4\pi m_{\mathrm{H}}^2} \left(\frac{g_{\mathrm{W}} m_{\mathrm{f}}}{m_{\mathrm{W}}} \right)^2 p^3$$

Since $E = m_H/2$ and $E^2 - p^2 = m_{\mathrm{f}}^2$, the centre of mass momentum p is given by

$$p = \sqrt{\tfrac{1}{4} m_H^2 - m_{\mathrm{f}}^2} \ .$$

Using $G_{\mathrm{F}}/\sqrt{2} = g_{\mathrm{W}}^2/8m_{\mathrm{W}}^2$ then gives

$$\boxed{\Gamma = N_{\mathrm{c}} \frac{G_{\mathrm{F}}}{\sqrt{2}} \frac{m_{\mathrm{f}}^2 m_{\mathrm{H}}}{4\pi} \left(1 - \frac{4m_{\mathrm{f}}^2}{m_{\mathrm{H}}^2} \right)^{3/2}} \ .$$

For $m_{\mathrm{H}} = 100\,\mathrm{GeV}$, the decay $\mathrm{H} \to \mathrm{t}\bar{\mathrm{t}}$ is forbidden because the top quark is too heavy, but decays to all other quark flavours and all lepton types are allowed:

$$\mathrm{H} \to \mathrm{b}\overline{\mathrm{b}}, \mathrm{c}\overline{\mathrm{c}}, \mathrm{s}\overline{\mathrm{s}}, \mathrm{u}\overline{\mathrm{u}}, \mathrm{d}\overline{\mathrm{d}}, \tau^+\tau^-, \mu^+\mu^-, \mathrm{e}^+\mathrm{e}^-, \nu_{\mathrm{e}}\overline{\nu}_{\mathrm{e}}, \nu_\mu\overline{\nu}_\mu, \nu_\tau\overline{\nu}_\tau \ .$$

For all these final states we have $m_{\mathrm{f}}^2 \ll m_H^2$ (the heaviest final state particle is the b quark with $m_b \sim 5\,\mathrm{GeV}$) and hence $(1 - 4m_{\mathrm{f}}^2/m_H^2)^{3/2} \approx 1$. We can therefore take

$$\Gamma \approx N_{\mathrm{c}} \frac{G_{\mathrm{F}}}{\sqrt{2}} \frac{m_{\mathrm{f}}^2 m_{\mathrm{H}}}{4\pi} \propto N_{\mathrm{c}} m_{\mathrm{f}}^2 \ .$$

Only b, c, τ are heavy enough to contribute significantly, with decay rates in the ratio [1]

$$3m_b^2 : 3m_c^2 : m_\tau^2 = 3 \times (5\,\mathrm{GeV})^2 : 3 \times (1.5\,\mathrm{GeV})^2 : (1.7\,\mathrm{GeV})^2 = 75 : 6.7 : 2.9$$

where the factors of 3 are for colour. For the b quark we have

$$\Gamma(\mathrm{H} \to \mathrm{b}\overline{\mathrm{b}}) = 3 \times \frac{1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}}{\sqrt{2}} \times \frac{(5\,\mathrm{GeV})^2 \times (100\,\mathrm{GeV})}{4\pi} = 4.9\,\mathrm{MeV}$$

[1] Accurate values of the quark and lepton masses are not expected here, just rough estimates.

giving an estimate of the total width Γ of

$$\Gamma \approx \frac{(75 + 6.7 + 2.9)}{75} \times 4.9 = 5.5\,\mathrm{MeV}\ .$$

The Higgs lifetime is then given by

$$\tau = \frac{\hbar}{\Gamma} = \frac{6.582 \times 10^{-25}\,\mathrm{GeV}.s}{5.5\,\mathrm{MeV}} = 1.2 \times 10^{-22}\,\mathrm{s}\ .$$

3. Neutral Kaons

The K^0 and $\overline{\mathrm{K}}^0$ belong to a $J^{PC} = 0^{-+}$ multiplet so

$$CP\left|\mathrm{K}^0\right\rangle = -\left|\overline{\mathrm{K}}^0\right\rangle, \qquad CP\left|\overline{\mathrm{K}}^0\right\rangle = -\left|\mathrm{K}^0\right\rangle \ .$$

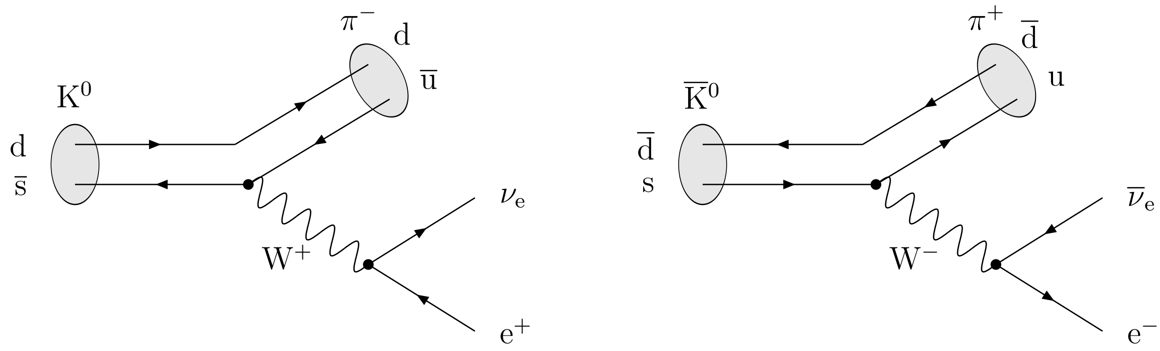
The CP eigenstates K_1 and K_2 can then be constructed as

$$\begin{aligned} |\mathrm{K}_1\rangle &= \frac{1}{\sqrt{2}}\left(\left|\mathrm{K}^0\right\rangle - \left|\overline{\mathrm{K}}^0\right\rangle\right) \qquad CP\,|\mathrm{K}_1\rangle = +\,|\mathrm{K}_1\rangle \\ |\mathrm{K}_2\rangle &= \frac{1}{\sqrt{2}}\left(\left|\mathrm{K}^0\right\rangle + \left|\overline{\mathrm{K}}^0\right\rangle\right) \qquad CP\,|\mathrm{K}_2\rangle = -\,|\mathrm{K}_2\rangle \end{aligned}$$

If CP violation is neglected, the states $\mathrm{K_S}$ and $\mathrm{K_L}$ decay only via $\mathrm{K_S} \to \pi\pi$ and $\mathrm{K_L} \to \pi\pi\pi$. The $\pi\pi$ system has $CP = +1$ and the $\pi\pi\pi$ system has $CP = -1$, and we can therefore identify

$$\begin{aligned} |\mathrm{K_S}\rangle = |\mathrm{K}_1\rangle &= \frac{1}{\sqrt{2}}\left(\left|\mathrm{K}^0\right\rangle - \left|\overline{\mathrm{K}}^0\right\rangle\right) \\ |\mathrm{K_L}\rangle = |\mathrm{K}_2\rangle &= \frac{1}{\sqrt{2}}\left(\left|\mathrm{K}^0\right\rangle + \left|\overline{\mathrm{K}}^0\right\rangle\right) \end{aligned}$$

Feynman diagrams for $\mathrm{K}^0 \to \pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\overline{\mathrm{K}}^0 \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$:



Thus the decays $\mathrm{K}^0 \to \pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\overline{\mathrm{K}}^0 \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$ are allowed, while the decays $\overline{\mathrm{K}}^0 \to \pi^-\mathrm{e}^+\nu_\mathrm{e}$ and $\mathrm{K}^0 \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$ are forbidden, *i.e.* the final state $\pi^-\mathrm{e}^+\nu_\mathrm{e}$ determines the K^0 component in the beam while $\pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}$ determines the $\overline{\mathrm{K}}^0$ component.

For a pure $|\mathrm{K}^0\rangle$ beam at $t = 0$, the initial wavefunction is

$$|\psi(0)\rangle = \left|\mathrm{K}^0\right\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{K_L}\rangle + |\mathrm{K_S}\rangle\right)$$

The wavefunction ψ evolves with time as

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}\left(|\mathrm{K_L}(t)\rangle + |\mathrm{K_S}(t)\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(|\mathrm{K_L}\rangle\, e^{-im_Lt-\Gamma_Lt/2} + |\mathrm{K_S}\rangle\, e^{-im_St-\Gamma_St/2}\right) \ . \end{aligned}$$

The decay rate into $\pi^-\mathrm{e}^+\nu_\mathrm{e}$ is determined by the K^0 component of the beam:

$$\begin{aligned}
\Gamma(\mathrm{K}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) &= \left|\left\langle \mathrm{K}^0 \middle| \psi(t) \right\rangle\right|^2 \\
&= \left| \left\langle \frac{1}{\sqrt{2}} (\mathrm{K_L} + \mathrm{K_S}) \middle| \frac{1}{\sqrt{2}} \left(\mathrm{K_L} e^{-im_L t - \Gamma_L t/2} + \mathrm{K_S} e^{-im_S t - \Gamma_S t/2} \right) \right\rangle \right|^2 \\
&= \frac{1}{4} \left| e^{-im_L t - \Gamma_L t/2} + e^{-im_S t - \Gamma_S t/2} \right|^2 \\
&= \tfrac{1}{4} \left(e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S + \Gamma_L)t/2} \cos \Delta m t \right)
\end{aligned}$$

where $\Delta m \equiv m_L - m_S$. Similarly,

$$\Gamma(\mathrm{K}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}) = \left|\left\langle \overline{\mathrm{K}}^0 \middle| \psi(t) \right\rangle\right|^2 = \tfrac{1}{4} \left(e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S + \Gamma_L)t/2} \cos \Delta m t \right) \ .$$

The two decay rates become equal when $\cos \Delta m t = 0$, *i.e.* when $\Delta m t = \pi/2$. Since $L = v t_{lab}$, $t_{lab} = \gamma t$, $\gamma = E/m$ and $v = p/E$, we have

$$\begin{aligned}
\Delta m &= \frac{\pi}{2}\frac{1}{t} = \frac{\pi}{2}\frac{\gamma}{t_{lab}} = \frac{\pi}{2}\frac{E/m}{L/v} = \frac{\pi}{2L}\frac{p}{m} \\
&= \frac{\pi}{2 \times (17.8\,\mathrm{m})} \times \frac{100\,\mathrm{GeV}}{0.498\,\mathrm{GeV}} \times (0.197\,\mathrm{GeV.fm}) = 3.5 \times 10^{-15}\,\mathrm{GeV} \ .
\end{aligned}$$

The $\mathrm{K_L}$ lifetime is about 500 times greater than the $\mathrm{K_S}$ lifetime, so at large times, only the $e^{-\Gamma_L t}$ term survives. The two decay rates are then approximately equal:

$$\Gamma(\mathrm{K}^0_{t=0} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) \approx \Gamma(\mathrm{K}^0_{t=0} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}) \approx \tfrac{1}{4} e^{-\Gamma_L t} \ .$$

Since the beam is almost pure $\mathrm{K_L}$ at large times, this gives (in the absence of CP violation)

$$\Gamma(\mathrm{K_L} \to \pi^-\mathrm{e}^+\nu_\mathrm{e}) = \Gamma(\mathrm{K_L} \to \pi^+\mathrm{e}^-\overline{\nu}_\mathrm{e}) \ .$$

CP violation was first discovered through the observation of a small fraction of $\mathrm{K_L} \to \pi^+\pi^-$ decays. The state $\mathrm{K_L}$ was therefore seen to decay both into a $CP = +1$ eigenstate ($\pi\pi$) and into a $CP = -1$ eigenstate ($\pi\pi\pi$), which is only possible if CP is violated. A comparison of $\mathrm{K_L} \to \pi^+\pi^-$ with $\mathrm{K_L} \to \pi^0\pi^0$ showed that CP violation is due dominantly to mixing:

$$|\mathrm{K_L}\rangle = \frac{1}{\sqrt{1 + |\epsilon|^2}} \left(|\mathrm{K_2}\rangle + \epsilon\, |\mathrm{K_1}\rangle \right)$$

where $|\epsilon| \sim 2 \times 10^{-3}$. The $\pi\pi$ decays can then be explained as coming from the $CP = +1$ $\mathrm{K_1}$ component of the $\mathrm{K_L}$ wavefunction.

With CP violation:

$$\begin{aligned}
|\mathrm{K_L}\rangle &= \frac{1}{\sqrt{1 + |\epsilon|^2}} \left(|\mathrm{K_2}\rangle + \epsilon\, |\mathrm{K_1}\rangle \right) \\
&= \frac{1}{\sqrt{1 + |\epsilon|^2}} \left[\frac{1}{\sqrt{2}} \left(\left|\mathrm{K}^0\right\rangle + \left|\overline{\mathrm{K}}^0\right\rangle \right) + \frac{\epsilon}{\sqrt{2}} \left(\left|\mathrm{K}^0\right\rangle - \left|\overline{\mathrm{K}}^0\right\rangle \right) \right] \\
&= \frac{1}{\sqrt{1 + |\epsilon|^2}} \cdot \frac{1}{\sqrt{2}} \left[(1 + \epsilon) \left|\mathrm{K}^0\right\rangle + (1 - \epsilon) \left|\overline{\mathrm{K}}^0\right\rangle \right] \ .
\end{aligned}$$

Hence the decay rates to $\pi^- e^+ \nu_e$ and $\pi^+ e^- \overline{\nu}_e$ are

$$
\begin{aligned}
I(\pi^- e^+ \nu_e) &\propto \left|\left\langle K^0 \middle| K_L \right\rangle\right|^2 \propto |1+\epsilon|^2 \\
I(\pi^+ e^- \overline{\nu}_e) &\propto \left|\left\langle \overline{K}^0 \middle| K_L \right\rangle\right|^2 \propto |1-\epsilon|^2
\end{aligned}
$$

The decay rate asymmetry is

$$
\begin{aligned}
\delta &\equiv \frac{\Gamma(K_L \to \pi^- e^+ \nu_e) - \Gamma(K_L \to \pi^+ e^- \overline{\nu}_e)}{\Gamma(K_L \to \pi^- e^+ \nu_e) + \Gamma(K_L \to \pi^+ e^- \overline{\nu}_e)} \\
&= \frac{|1+\epsilon|^2 - |1-\epsilon|^2}{|1+\epsilon|^2 + |1-\epsilon|^2} \\
&= \frac{(1+\epsilon)(1+\epsilon^*) - (1-\epsilon)(1-\epsilon^*)}{(1+\epsilon)(1+\epsilon^*) + (1-\epsilon)(1-\epsilon^*)} \\
&= \frac{\epsilon + \epsilon^*}{1+|\epsilon|^2} \\
&\approx 2\mathrm{Re}(\epsilon)
\end{aligned}
$$