NST Part III Experimental and Theoretical Physics Dr J R Batley

Particle Physics Major Option Exam, January 2000

SOLUTIONS

1. **$\pi^- \to \mu^- \overline{\nu}_\mu$ decay**:

The μ^- 4-momentum is $p_2 = (E, 0, 0, p)$ with $E^2 = p^2 + m_\mu^2$. The possible μ^- spinors are:

$$u_1(p_2) = \sqrt{E + m_\mu} \begin{pmatrix} 1 \\ 0 \\ p/(E+m_\mu) \\ 0 \end{pmatrix}, \qquad u_2(p_2) = \sqrt{E + m_\mu} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -p/(E+m_\mu) \end{pmatrix}$$

with corresponding adjoint spinors

$$\bar{u}_1(p_2) = \sqrt{E + m_\mu}\ \ (1, 0, -p/(E+m_\mu), 0)\,, \qquad \bar{u}_2(p_2) = \sqrt{E + m_\mu}\ \ (0, 1, 0, p/(E+m_\mu))\ \ .$$

The spinor $u_1(p_2)$ corresponds to a positive helicity μ^-, $u_2(p_2)$ to a negative helicity μ^-.

The $\overline{\nu}_\mu$ 4-momentum is $p_3 = (p, 0, 0, -p)$, and the $\overline{\nu}_\mu$ spinors are therefore

$$v_1(p_3) = \sqrt{p} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad v_2(p_3) = \sqrt{p} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

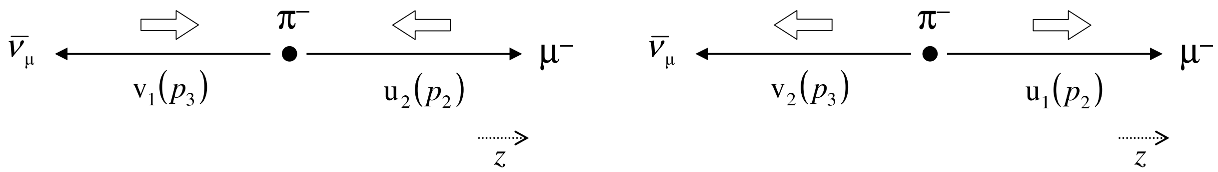
where $v_1(p_3)$ corresponds to negative helicity and $v_2(p_3)$ to positive helicity. The four possible combinations $\overline{u}(p_2)v(p_3)$ are

$$\bar{u}_1(p_2)v_1(p_3) = \bar{u}_2(p_2)v_2(p_3) = 0$$

$$\bar{u}_2(p_2)v_1(p_3) = -\bar{u}_1(p_2)v_2(p_3) = \sqrt{\frac{p}{E+m_\mu}}\,(E + m_\mu + p) = \sqrt{\frac{p}{E+m_\mu}}\,(m_\pi + m_\mu) \qquad (1)$$

where energy conservation, $m_\pi = E + p$, has been used in the last step.

Hence only two combinations, $\bar{u}_2(p_2)v_1(p_3)$ and $\bar{u}_1(p_2)v_2(p_3)$, are non-zero:



For $\bar{u}_2(p_2)v_1(p_3)$, the μ^- and $\overline{\nu}_\mu$ both have negative helicity, while for $\bar{u}_1(p_2)v_2(p_3)$ both particles have positive helicity. The total spin in the final state in both these cases is therefore zero, consistent with the fact that the π^- has spin zero. The two other spinor combinations,

$\bar{u}_1(p_2)v_1(p_3)$ and $\bar{u}_2(p_2)v_2(p_3)$ have total spin 1 in the final state, which is forbidden by angular momentum conservation.

[Note that this is different from the standard V−A charged current weak interaction, where only the right-hand diagram, with a right-handed antineutrino, is allowed.]

To find p (the centre of mass momentum), start from energy conservation, $m_\pi = E + p$, and square:

$$m_\pi^2 = (E+p)^2 = E^2 + p^2 + 2Ep = 2p^2 + m_\mu^2 + 2p\sqrt{p^2 + m_\mu^2}$$
$$\Rightarrow \qquad 4p^2(p^2 + m_\mu^2) = \left(m_\pi^2 - m_\mu^2 - 2p^2\right)^2$$
$$\Rightarrow \qquad 4p^2 m_\mu^2 = \left(m_\pi^2 - m_\mu^2\right)^2 - 4p^2\left(m_\pi^2 - m_\mu^2\right)$$
$$\Rightarrow \qquad p = (m_\pi^2 - m_\mu^2)/2m_\pi$$

Hence

$$E + m_\mu = m_\pi + m_\mu - p = (m_\pi + m_\mu)^2/2m_\pi$$

giving

$$\sqrt{\frac{p}{E+m_\mu}} \cdot (m_\pi + m_\mu) = \sqrt{\frac{m_\pi^2 - m_\mu^2}{(m_\pi + m_\mu)^2}} \cdot (m_\pi + m_\mu) = \sqrt{m_\pi^2 - m_\mu^2}$$

The two non-zero spinor combinations of equation (1) are therefore

$$\bar{u}_2(p_2)v_1(p_3) = -\bar{u}_1(p_2)v_2(p_3) = \sqrt{m_\pi^2 - m_\mu^2}$$

The matrix element for these is

$$M_{\mathrm{fi}} = i\frac{G_\mathrm{F}}{\sqrt{2}} f_\pi \overline{u}(p_2)v(p_3) = i\frac{G_\mathrm{F}}{\sqrt{2}} f_\pi \sqrt{m_\pi^2 - m_\mu^2}$$

and summing over the two non-zero final state spin configurations gives

$$\langle |M_{\mathrm{fi}}|^2 \rangle = \tfrac{1}{2} G_\mathrm{F}^2 f_\pi^2 \cdot 2(m_\pi^2 - m_\mu^2)$$

Since $p^* = p = (m_\pi^2 - m_\mu^2)/2m_\pi$, the decay rate Γ is given by

$$\Gamma = \frac{p^*}{8\pi m_\pi^2}\langle |M_{\mathrm{fi}}|^2 \rangle = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}\frac{1}{8\pi m_\pi^2} G_\mathrm{F}^2 f_\pi^2 \cdot (m_\pi^2 - m_\mu^2) = \frac{1}{16\pi m_\pi^3} G_\mathrm{F}^2 f_\pi^2 \cdot (m_\pi^2 - m_\mu^2)^2$$

Hence, for a scalar weak charged current, we predict

$$R = \frac{\Gamma(\pi^- \to \mathrm{e}^- \overline{\nu}_\mathrm{e})}{\Gamma(\pi^- \to \mu^- \overline{\nu}_\mu)} = \frac{(m_\pi^2 - m_e^2)^2}{(m_\pi^2 - m_\mu^2)^2} = \left(\frac{139.6^2 - 0.51^2}{139.6^2 - 105.7^2}\right)^2 = 5.49$$

The correct structure of the weak charged current is V−A, and the matrix element becomes

$$M_{\mathrm{fi}} = i\frac{G_\mathrm{F}}{\sqrt{2}} f_\pi \bar{u}(p_2)\tfrac{1}{2}(1-\gamma^5)\gamma^\mu v(p_3)\ .$$

Given that this results in an extra factor in the decay rate of $2m_\mu^2$ for $\pi^- \to \mu^-\overline{\nu}_\mu$ and $2m_e^2$ for $\pi^- \to \mathrm{e}^-\overline{\nu}_\mathrm{e}$, the V−A prediction for R is

$$R = \left(\frac{m_e}{m_\mu}\right)^2 \times 5.49 = \left(\frac{0.51}{105.7}\right)^2 \times 5.49 = 1.28 \times 10^{-4}$$

The predicted decay rates for $\pi^- \to \mu^-\overline{\nu}_\mu$ and $\mathrm{K}^- \to \mu^-\overline{\nu}_\mu$ are:

$$\begin{aligned}
\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu) &= \frac{m_\mu^2}{8\pi m_\pi^3} G_\mathrm{F}^2 f_\pi^2 \cdot (m_\pi^2 - m_\mu^2)^2 \\
&= \frac{(0.1057)^2}{8\pi(0.1396)^3}(1.166 \times 10^{-5})^2 \times (0.132)^2 \times (0.1396^2 - 0.1057^2)^2 \\
&= 2.67 \times 10^{-17}\,\mathrm{GeV}
\end{aligned}$$

$$\begin{aligned}
\Gamma(\mathrm{K}^- \to \mu^-\overline{\nu}_\mu) &= \frac{m_\mu^2}{8\pi m_K^3} G_\mathrm{F}^2 f_K^2 \cdot (m_K^2 - m_\mu^2)^2 \\
&= \frac{(0.1057)^2}{8\pi(0.4937)^3}(1.166 \times 10^{-5})^2 \times (0.160)^2 \times (0.4937^2 - 0.1057^2)^2 \\
&= 6.95 \times 10^{-16}\,\mathrm{GeV}
\end{aligned}$$

In comparison, the *measured* decay rates are:

$$\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu) = \frac{6.582 \times 10^{-25}\,\mathrm{GeV.s}}{2.60 \times 10^{-8}\,\mathrm{s}} = 2.53 \times 10^{-17}\,\mathrm{GeV}$$

$$\Gamma(\mathrm{K}^- \to \mu^-\overline{\nu}_\mu) = \frac{6.582 \times 10^{-25}\,\mathrm{GeV.s} \times 0.635}{1.24 \times 10^{-8}\,\mathrm{s}} = 3.37 \times 10^{-17}\,\mathrm{GeV}$$

where we make use of the fact that $\pi^- \to \mu^-\overline{\nu}_\mu$ has a branching ratio of almost 100% while $\mathrm{K}^- \to \mu^-\overline{\nu}_\mu$ has a branching ratio of 63.5%.

The $\pi^- \to \mu^-\overline{\nu}_\mu$ measured rate is about 5% below the V−A prediction, while $\mathrm{K}^- \to \mu^-\overline{\nu}_\mu$ is about a factor of 20 too low. Agreement is restored by including extra factors $\cos^2\theta_C \approx 0.95$ and $\sin^2\theta_C \approx 0.05$ respectively, where $\theta_C \approx 13.1^\circ$ is the Cabibbo angle.

2. **Colour forces**

For the colour singlet state

$$\psi = \frac{1}{\sqrt{3}}(r\overline{r} + g\overline{g} + b\overline{b})$$

the overall colour factor C contains three diagonal terms of the form $\frac{1}{3}C(r\overline{r} \to r\overline{r})$ and six cross-terms of the form $\frac{1}{3}C(r\overline{r} \to g\overline{g})$. In Handout 7, these colour factors were shown to have the values $C(r\overline{r} \to r\overline{r}) = \frac{1}{3}$ and $C(r\overline{r} \to g\overline{g}) = \frac{1}{2}$. Therefore the overall colour factor is

$$C = 3 \cdot \frac{1}{3} \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{4}{3}\ .$$

For a quark-antiquark state, the short-range potential is

$$V(r) = -C\frac{\alpha_{\mathrm{s}}}{r}$$

which is negative, corresponding to an *attractive* force between the quark and antiquark.

For the colour octet state

$$\psi = \frac{1}{\sqrt{6}}(r\overline{r} + g\overline{g} - 2b\overline{b})$$

the overall colour factor is

$$C = \frac{1}{6}\left[6 \times C(r\overline{r} \to r\overline{r}) - 6 \times C(r\overline{r} \to g\overline{g})\right] = \frac{1}{6}\left[6 \times \frac{1}{3} - 6 \times \frac{1}{2}\right] = -\frac{1}{6}$$

Similarly, for the colour octet state

$$\psi = \frac{1}{\sqrt{2}}(r\overline{r} - g\overline{g})$$

we have

$$C = \frac{1}{2}\left[2 \times C(r\overline{r} \to r\overline{r}) - 2 \times C(r\overline{r} \to g\overline{g})\right] = \frac{1}{2}\left[2 \times \frac{1}{3} - 2 \times \frac{1}{2}\right] = -\frac{1}{6}$$

Finally, for the colour octet states $r\overline{g}, r\overline{b}, g\overline{r}, g\overline{b}, b\overline{r}, b\overline{g}$, it was shown in Handout 7 that

$$C(r\overline{g} \to r\overline{g}) = C(r\overline{b} \to r\overline{b}) = C(g\overline{r} \to g\overline{r}) = C(g\overline{b} \to g\overline{b}) = C(b\overline{r} \to b\overline{r}) = C(b\overline{g} \to b\overline{g}) = -\frac{1}{6}$$

In summary, the colour factor for each octet state is negative, corresponding to a *repulsive* force between the quark and antiquark, and has the same value ($C = -\frac{1}{6}$), reflecting invariance of the strong interactions under SU(3) colour transformations. Only if the quark-antiquark pair is in a colour singlet state do we get a positive overall colour factor, and hence a *binding* potential at short range.

Neutral kaons

For mesons with the quark-antiquark pair in an orbital angular momentum state L and total spin S, the parity is $P = P_1 P_2 (-1)^L = (-1)^{L+1}$, while C is equivalent to parity followed by spin exchange giving $C = (-1)^{L+1}(-1)^{S+1} = (-1)^{L+S}$. Only neutral mesons are charge conjugation eigenstates. We get meson nonets with $J^{PC} = 0^{-+}, 1^{--}, 1^{+-}, 0^{++}, 1^{++}, 2^{++}$ etc.

Since $P\left|\mathrm{K}^0\right\rangle = -\left|\mathrm{K}^0\right\rangle$, $P\left|\overline{\mathrm{K}}^0\right\rangle = -\left|\overline{\mathrm{K}}^0\right\rangle$, $C\left|\mathrm{K}^0\right\rangle = +\left|\overline{\mathrm{K}}^0\right\rangle$, $C\left|\overline{\mathrm{K}}^0\right\rangle = +\left|\mathrm{K}^0\right\rangle$, we have

$$\widehat{C}\widehat{P}\left|\mathrm{K}^0\right\rangle = -\left|\overline{\mathrm{K}}^0\right\rangle, \qquad \widehat{C}\widehat{P}\left|\overline{\mathrm{K}}^0\right\rangle = -\left|\mathrm{K}^0\right\rangle$$

so K^0 and $\overline{\mathrm{K}}^0$ are *not* CP eigenstates. But for

$$\left|\mathrm{K}_1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\mathrm{K}^0\right\rangle - \left|\overline{\mathrm{K}}^0\right\rangle\right), \qquad \left|\mathrm{K}_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\mathrm{K}^0\right\rangle + \left|\overline{\mathrm{K}}^0\right\rangle\right)$$

we have $CP\left|\mathrm{K}_1\right\rangle = +\left|\mathrm{K}_1\right\rangle$, $CP\left|\mathrm{K}_2\right\rangle = -\left|\mathrm{K}_2\right\rangle$, so these *are* CP eigenstates.

For $\mathrm{K}^0 \to \pi\pi$, we must have $L = 0$ in the final state. Hence $P = +1$. For $\pi^0\pi^0$, the two pions are charge conjugation eigenstates so $C = +1$ directly. For $\pi^+\pi^-$, C and P are identical giving $C = +1$ also. Hence $CP = +1$ for $\pi\pi$.

For $\mathrm{K}^0 \to \pi\pi\pi$, angular momentum conservation gives $L_1 = L_2$ for the two final state angular momentum quantum numbers, so $P = -1.-1.-1.(-1)^{L_1}.(-1)^{L_2} = -1$. For $\mathrm{K}^0 \to \pi^0\pi^0\pi^0$, we have $C = +1$ directly. For $\mathrm{K}^0 \to \pi^+\pi^-\pi^0$, we have $L_1 = 0$ experimentally, giving $C = +1.C(\pi^+\pi^-) = +1.(-1)^{L_1} = +1$. Hence $CP = -1$ for $\pi\pi\pi$.

So, if CP is conserved, only the decays $\mathrm{K}_1 \to \pi\pi$ and $\mathrm{K}_2 \to \pi\pi\pi$ are allowed.