UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics
MASTER OF ADVANCED STUDY IN PHYSICS

Monday 16th January 2023 10:00 to 12:00

MAJOR TOPICS
Paper 1/PP (Particle Physics)

*Answer **two** questions only. The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate. The paper has content on six sides, including this one, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

*You should use a **separate Answer Book** for each question.*

STATIONERY REQUIREMENTS
2x20-page answer books
Rough workpad

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

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The information in this box may be used in any question.

The Pauli-matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The gamma matrix representation of the Part III Particles lecture course was:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

which has the following properties:

$$(\gamma^0)^* = \gamma^0,\ \ (\gamma^1)^* = \gamma^1,\ \ (\gamma^2)^* = -\gamma^2,\ \ (\gamma^3)^* = \gamma^3 \ \ \textit{and}\ \ \gamma^2(\gamma^\mu)^* = -\gamma^\mu\gamma^2.$$

Using the above convention, the Part III Particles lecture course defined the following particle and anti-particle spinors:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\boldsymbol{p}|}{E+m}c \\ \frac{|\boldsymbol{p}|}{E+m}e^{i\phi}s \end{pmatrix}, \qquad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\boldsymbol{p}|}{E+m}s \\ -\frac{|\boldsymbol{p}|}{E+m}e^{i\phi}c \end{pmatrix},$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\boldsymbol{p}|}{E+m}s \\ -\frac{|\boldsymbol{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}, \qquad v_\downarrow = N\begin{pmatrix} \frac{|\boldsymbol{p}|}{E+m}c \\ \frac{|\boldsymbol{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

for objects whose three-momentum $\boldsymbol{p}$ *is given by* $|\boldsymbol{p}|(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ *where* $c = \cos\frac{\theta}{2}$ *and* $s = \sin\frac{\theta}{2}$. *The normalising constant is* $N = \sqrt{E+m}$.

$\hbar \approx 1.05 \times 10^{-34}$ kg m^2/s, $\qquad c \approx 3.00 \times 10^8$ m/s, $\qquad e \approx 1.60 \times 10^{-19}$ C.

$m_e = 5.11 \times 10^{-4}$ GeV. $\qquad m_p = m_n = 1.67 \times 10^{-27}$ kg.

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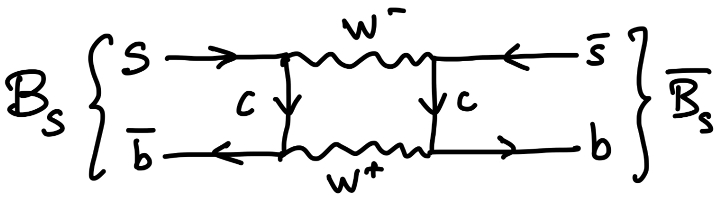
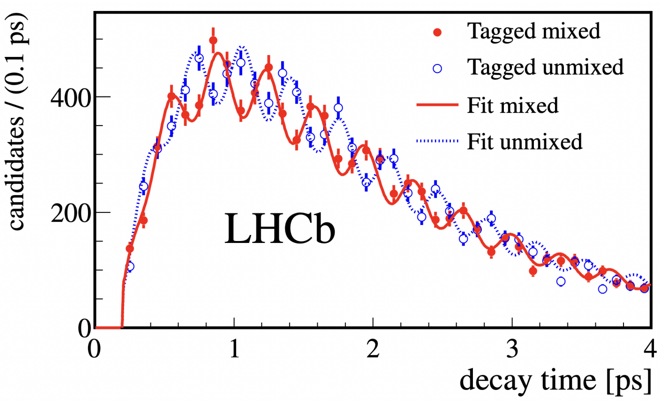
1 A B_s^0 meson contains an anti bottom quark together with a strange quark and so both it and its antiparticle are referred to as strange b-mesons. In 2013 the LHCb Collaboration published a paper[1] describing evidence for $B_s^0 - \overline{B}_s^0$ oscillations for strange b-mesons produced in proton-proton collisions at the Large Hadron Collider.

(a) Draw a Feynman diagram indicating the process(es) most likely to allow $B_s^0 - \overline{B}_s^0$ mixing. [5]

Bookwork (in the sense that knowledge of SM vertices is bookwork) but also marginal extension (in the sense that the meson oscillations seen in the course were predominantly kaon oscillations).



is one typical diagram, but one should also draw the variants that have the W bosons vertically rather than horizontally. Note that the question asks for the diagrams which are MOST LIKELY to lead to oscillation, so we should favour c here because: (1) there is plenty of mass for it, (2) t would be mass-suppressed, and (3) u would be Cabbibo suppressed relative to c. Nonetheless, someone not considering Cabbibo suppression would lose at most one mark by considering only the u. Considering only t would be bizarre, though, so someone thinking the most likely diagram here was top-dominated might lose two marks.



A figure from the LHCb paper is reproduced above. The filled-circle data points labelled 'Tagged mixed' count events in which the experimenters believe that a strange b-meson was produced with one flavour but decayed with another (i.e. $B_s^0 \to \overline{B}_s^0$ or $\overline{B}_s^0 \to B_s^0$), while the open circles labelled 'Tagged unmixed' count events in which the

[1] Precision measurement of the $B_s^0 - \bar{B}_s^0$ oscillation frequency with the decay $B_s^0 \to D_s^- \pi^+$, [arXiv:1304.4741]

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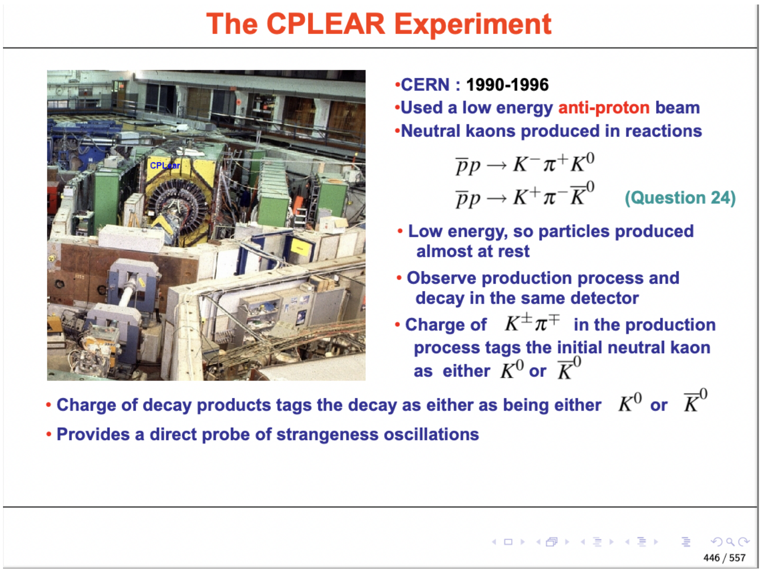
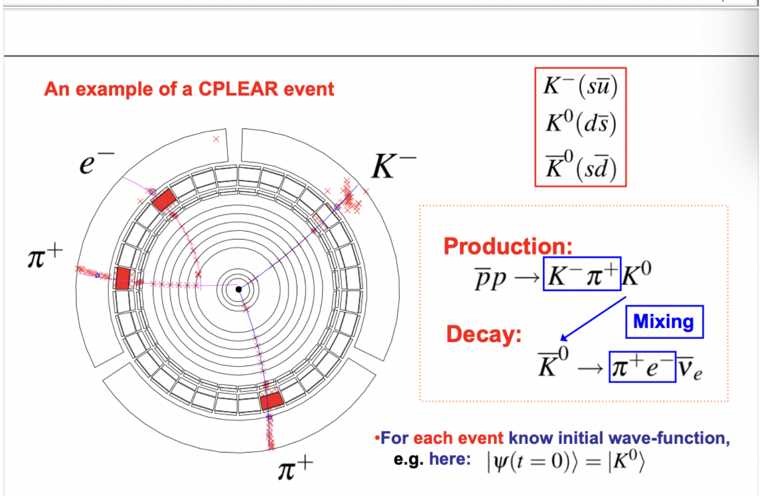
experimenters believe that a strange b-meson was produced and decayed without changing flavour (i.e. $B_s^0 \to B_s^0$ or $\overline{B}_s^0 \to \overline{B}_s^0$). In both cases, the 'candidate' counts are the number of events which were observed to have 'decay times' (i.e. times between birth and death) which lie in the relevant bin of the histogram.

(b) How might the members of the LHCb collaboration have been able to determine the flavour of the strange b-mesons at production? [8]

Half way between bookwork and unseen

The students on this course have been shown flavour mixing in the context of K0 K0bar mixing in the CPLEAR experiment. The CPLEAR explanation in lectures included examples of flavour tagging the produced K0 or K0bar by looking for the charge of the pion produced in association with the initial Kaon in slides like these:





This question therefore expects them to try and put that knowledge to use for strange b-mesons instead of Kaons.

This question is only looking for reasonable or plausible POSSIBILITIES which

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could CONCEIVABLY have been available for the LHCb collaboration to use for tagging the flavour of the initial state strange b-mesons.[2] After all, the course does not expect the candidates sitting the exam to have any specific prior knowledge of LHCb's Bs Bsbar oscillation measurements. It just wants to see if they understood the CPLEAR material well enough to translate it to a different flavour oscillation.

It is therefore intended to accept for full marks any answer which demonstrates that the candidate is aware that the job of the initial state tagger is to determine whether a $b\bar{s}$ or $\bar{b}s$ was made, and that this will require the use of a conserved flavour-based quantity. More specifically, good answers will therefore mention some/all of the following:

- b-production is expected with $\bar{b}$-production at a proton-proton collider,
- s-production is expected with $\bar{s}$-production,
- and that therefore one needs to look for one or more mesons in the same event with b or s flavours which are the opposite of those in the strange b-meson being tested for decay.
- marks for saying that since b-tagging is usually lifetime related, and since we really don't want long lived particles for the production tag (lest they oscillate too) the production tag is most likely to require tagging the sign of the s generated with the $\bar{s}$ (or vice versa), ...
- and therefore looking for a charged strange meson Kaon ($K^- = s\bar{u}$, $K^+ = \bar{s}u$) or charmed meson ($D^+ = c\bar{d}$, $D^- = \bar{c}d$) may be good places to start.

For what it's worth, although the students don't need to know what arXiv:1304.4741 ACTUALLY does, we note here that it says the following about it's own methods:

> To determine the flavour of the B_s^0 meson at production, both opposite-side (OST) and same-side (SST) tagging algorithms are used. The OST exploits the fact that b quarks at the LHC are predominantly produced in quark–antiquark pairs. By partially reconstructing the second b hadron in the event, conclusions on the flavour at production of the signal B_s^0 candidate can be drawn. The OST have been optimized on large samples of $B^+ \to J/\psi K^+$, $B \to \mu^+ D^{*-} X$, and $B^0 \to D^- \pi^+$ decays [23].
>
> The SST takes advantage of the fact that the net strangeness of the pp collision is zero. Therefore, the s quark needed for the hadronization of the B_s^0 meson must have been produced in association with an $\bar{s}$ quark, which in about 50% of the cases hadronizes to form a charged kaon. By identifying this kaon, the flavour at production of the signal B_s^0 candidate is determined. The optimization of the SST was performed on a data sample of

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(c) How might the experimenters have been able to determine the flavour of the strange b-mesons at decay? [6]

(Again half way between bookwork and unseen as it's just an extension of the last question)

Here candidates can respond with answers like "The tagging of the final state of the strange b-meson can proceed in largely the same way as the tagging of its initial state. The only difference is that we need to look for (a) the flavour or charge correlated tagging particles in its own decay products (rather than in things produced at the same time as its own creation), and (b) that these will all have to come from a displaced vertex so that we

[2]This is clear from the choice of wording 'How *might* [they] have been able to determine ...' which makes clear that the question does not demand the ACTUAL tagging method used by LHCb.

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know they are associated with a particle that has had a chance to travel some distance. This last requirement is because we can only look for oscillation in particles which have lived long enough (and have thus travelled far enough) to oscillate.

(d) Explain why, for positive decay times t, and in the absence of any oscillations or detector considerations, one might expect the probability density $p(t)$ for B_s^0 decay times to be proportional to

$$\Gamma_s e^{-\Gamma_s t} \cosh\left(\frac{\Delta\Gamma_s}{2} t\right)$$

where Γ_s is the B_s^0 decay width and $\Delta\Gamma_s$ the decay width difference between the light and heavy mass eigenstates. [6]

Unseen but not a million miles away from similar things seen in lectures

Technically it's the mass eigenstates (which we might call states '1' and '2') which have masses and widths Γ_1 and Γ_2. However, those masses and widths are going to be so similar that we can talk about 'the width of the B_s^0' or the width of the $\bar{B}_s^0$ as if they were a single quantity Γ_s which we could technically define to be $\Gamma_s = (\Gamma_1 + \Gamma_2)/2$. Anything that actually cares about the width differences we will write as a function of $\Delta\Gamma_s = \Gamma_2 - \Gamma_1$ with $\Gamma_2 \geq \Gamma_1 \geq 0$. We will assume that $\Gamma_s \gg \Delta\Gamma_s$.

Then, making the PDF the sum of two independent PDFs each representing exponential decays with time constants $\tau_1 < \tau_2$ we have:

$$\begin{aligned}
p(t) &\propto \frac{1}{\tau_1} e^{-t/\tau_1} + \frac{1}{\tau_2} e^{-t/\tau_2} \\
&= \Gamma_1 e^{-t\Gamma_1} + \Gamma_2 e^{-t\Gamma_2} \\
&= e^{-t(\Gamma_1+\Gamma_2)/2} \left(\Gamma_1 e^{-t(\Gamma_1-\Gamma_2)/2} + \Gamma_2 e^{-t(\Gamma_2-\Gamma_1)/2}\right) \\
&= e^{-t\Gamma_s} \left(\frac{\Gamma_s - \frac{1}{2}\Delta\Gamma_s}{2} e^{t\Delta\Gamma_s/2} + \frac{\Gamma_s + \frac{1}{2}\Delta\Gamma_s}{2} e^{-t\Delta\Gamma_s/2}\right) \\
&= e^{-t\Gamma_s} \left(\Gamma_s \cosh\left(\frac{\Delta\Gamma_s}{2} t\right) - \frac{\Delta\Gamma_s}{2} \sinh\left(\frac{\Delta\Gamma_s}{2} t\right)\right) \\
&= \Gamma_s e^{-t\Gamma_s} \cosh\left(\frac{\Delta\Gamma_s}{2} t\right) + O((\Delta\Gamma_s)^2)
\end{aligned}$$

which is equal to the result we are trying to find, given that $\Gamma_s \gg \Delta\Gamma_s$.

(e) The experimentally observed data shown in the LHCb plot do not look much like the function we have just derived. What are the likely causes of the differences you can see, and what do they tell us? [5]

Unseen, discursive.

FIRSTLY:

The experimental results do not see any decay times below 0.2 ps, and they see much fewer below 1 ps. In contrast, our naive formula predicts that the biggest decay rates are at

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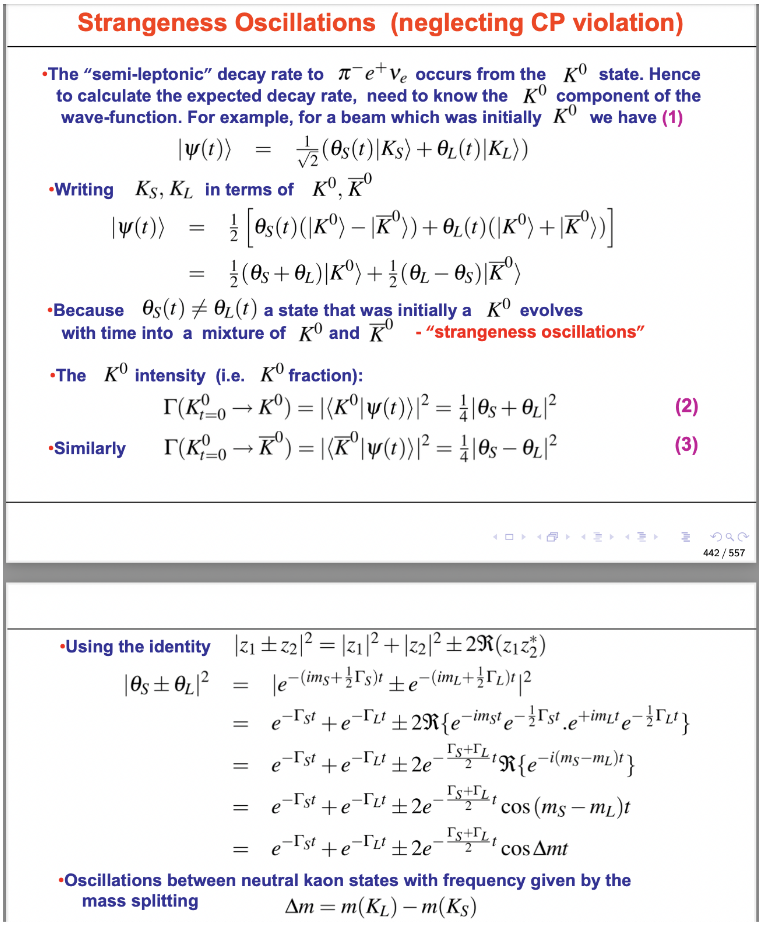
$t - 0$. This is presumably an acceptance effect. I.e. to see a decay at very short time would correspond to seeing a decay at short distance from the main interaction point. However an decay at the main interaction point would likely be unattributable to mixed-vs-unmixed: the opposite and same-side tagging clues would overlap and conflict if the strange b-meson (which we are trying to monitor) has not had a chance to move out of the primary vertex. Presumably, therefore, the LHCb result blinds itself (intentionally or unintentionally) to early decays – and focus instead on those with longer decay times. Note that 0.1ps corresponds to 0.03mmm which is far too small for the problem to be caused by any mechanical boundary (such as beam pipe diameter, or inner tracking layer).

SECONDLY:

We can see oscillations in the red which are out of phase with oscillations in the blue. These clearly show that our naive model was wrong to ignore interference effects.

If we put them in, we expect to get oscillations with a structure similar to that determined in lectures for the kaons, namely something like this:



so the period T of oscillation (in natural units) satisfies

$$\Delta M T = 2\pi$$

and so

$$\Delta M = 2\pi/T.$$

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We see 9 oscillations between $t = 0.6$ and $t = 3.8$ ps, so the LHCb data suggests $T = (3.8 - 0.6)/9 = 0.356$ ps, and so this data suggests $\Delta M = 2\pi/(0.356\ \mathrm{ps}) = 17.7/\mathrm{ps} = 17.7/\mathrm{ps}$ or, multiplying a $\hbar/c^2$ in to get actual mass units, we have $\Delta M = 17.7/\mathrm{ps} * \hbar/\mathrm{c}^2 = 2.1 \times 10^{-38}$ kg $= 0.012$ eV/c^2.

FYI: The paper reports $\Delta m_s = 17\ \mathrm{ps}^{-1}$.

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2 In the deep inelastic scattering of an electron with four-momentum p_1 by a proton with four-momentum p_2, the following Lorentz invariant variables can be defined:

$$Q^2 \equiv -q^2; \qquad x \equiv \frac{Q^2}{2p_2 \cdot q}; \qquad y \equiv \frac{p_2 \cdot q}{p_2 \cdot p_1}$$

where $q = p_1 - p_3$ and p_3 is the four-momentum of the scattered electron. Neglecting the mass of the electron:

(a) show that $Q^2 > 0$; [3]

bookwork

Neglecting the electron mass ($p_1^2 = p_3^2 = 0$) we have

$$\begin{aligned} q^2 &= (p_1 - p_3)^2 \\ -Q^2 &= p_1^2 + p_2^2 - 2p_1.p_3 \\ -Q^2 &= -2(E_1, 0, 0, E_1).(E_3, E_1 \sin\theta, 0, E_3 \cos\theta) \\ Q^2 &= 2E_1E_3(1 - \cos\theta) \\ Q^2 &> 0 \end{aligned}$$

(b) by considering the invariant mass of the final state hadronic system, or otherwise, determine the range of values which x can take. Comment on the physical significance of x, and comment on the interpretation which can be attached to events with $x = 1$; [6]

bookwork

The final state hadronic system must contain at least one baryon, hence $M_X^2 > m_p^2$:

$$\begin{aligned} (p_2 + q)^2 &\geq m_p^2 \\ p_2^2 + 2p_2.q + q^2 &\geq m_p^2 \\ m_p^2 + 2p_2.q - Q^2 &\geq m_p^2 \\ -Q^2 &\geq -2p_2.q \\ Q^2 &\leq 2p_2.q \\ \therefore \quad x = \frac{Q^2}{2p_2.q} &\leq 1 \end{aligned}$$

Furthermore, we know from the parton model that x has two additional (related) properties: when equal to 1 it indicates that the collision was elastic (the hadron was neither broken up nor placed in an excited state). When not equal to one it indicates the fraction of the hadron's momentum which was present in the struck parton, as measured in the infinite momentum frame.

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(c) deduce the nature of the limiting case or condition in which y relates to the centre-of-mass-frame scattering angle $\theta^\star$ of the electron via the formula:

$$y = \frac{1}{2}\left(1 - \cos\theta^\star\right). \quad [4]$$

(very close to bookwork but not commonly voiced this way)

Working in the centre-of-mass frame and treating all particles as (effectively) massless, we have $p_1 = (E, 0, 0, E)$, $p_2 = (E, 0, 0, -E)$ and $p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$ and so:

$$\begin{aligned} y &= \frac{p_2.q}{p_2.p_1} \\ &= \frac{p_2.(p_1 - p_3)}{2E^2} \\ &= \frac{p_2.(0, 0, 0, E(1 - \cos\theta^*))}{2E^2} \\ &= \frac{1}{2}(1 - \cos\theta^*) \end{aligned}$$

The question does not ask for an explicit proof, so all four marks could be obtained by a clear statement of which particles need to be massless to make the relationship work. However, it is assumed that most people will set out the proof as shown above, if only to satisfy themselves of the truth of the result. The fact that they are not told to make the massless approximation, but are rather asked to determine which approximation makes the formula right, is to test whether the candidates understood why we were able to use the formula so frequently in this form in the lecture course.

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The so-called 'Drell-Yan' process is the production of lepton pairs ($\ell^+\ell^-$) in hadron-hadron collisions through the annihilation of a quark and an anti-quark into a photon.

(d) Draw the Feynman diagram for this process and explain why the cross section is non-zero for proton-proton collisions. [2]

Feynman diagram is just the QED process $q\overline{q} \to \mu^+\mu^-$.

Students should state there are no valence antiquarks in protons to allow for $q\overline{q}$ annihilation. However, the presence of sea quarks in protons, which are virtual and low energy, allows for a small, non-zero cross-section.

The cross section for $q\bar{q}$ annihilation is

$$\sigma = \frac{4\pi}{3}\frac{\alpha^2}{\hat{s}}e_q^2$$

where e_q is the quark charge (i.e. $e_u = +\frac{2}{3}$ and $e_d = -\frac{1}{3}$ and $\hat{s}$ is the quark-anti-quark energy in their centre-of-mass frame.

(e) Neglecting any strange quark contributions, show that the parton model prediction for the $pp \to \mu^+\mu^- X$ differential cross section can be written in the form:

$$\frac{d^2\sigma}{dx_1 dx_2} = \left\{A\left[u(x_1)\bar{u}(x_2) + u(x_2)\bar{u}(x_1)\right] + B\left[d(x_1)\bar{d}(x_2) + d(x_2)\bar{d}(x_1)\right]\right\}$$

in which s is the centre of mass energy of the proton-proton system, and in which A and B are two real constants which you should determine. Your answer should make clear what the functions $u(x)$, $d(x)$, $\bar{u}(x)$ and $\bar{d}(x)$ represent, and to what x_1 and x_2 refer. [10]

(unseen)

Of the 10 marks here, five are for the wordy parts (i.e. describing, as requested, what $u(x)$ etc actually are, (i.e. describing a PDF qualitatively and quantitatively in terms of its count of particles with Bjorken x in between x and $x + dx$ etc ...) and specifically which partons and parent particles x_1 and x_2 are related.).

The rest of the marks are for the calculation, and are loosely as follows (though variance in the way people answer means that, in practice the mark awarded was determined subjectively out of five for the whole calculation based on how many features of the correct answer the candidate successfully obtained (notably $x_1 x_2$ dependence, ratio of $A/B = 4$, correct constant prefactors of A and B, colour suppression factor of $1/3$) and how reasonable the general argument was.

Note that (with hindsight) the question should have been worded better: it should have said "... where A and B are two real functions which you should determine." The change there is 'constants' to 'functions' since A and B contain x_1 and x_2 dependence.

Let the colliding protons have energy E in the C.o.M. frame. The four-momenta of the colliding quark and anti-quark are $p_1 = (x_1 E, 0, 0, x_1 E)$ and $p_2 = (x_2 E, 0, 0, -x_2 E)$. The centre-of-mass energy of the $q\bar{q}$ collision is $\hat{s}$:

$$\begin{aligned}\hat{s} &= (p_1 + p_2)^2 \\ &= (x_1 + x_2)^2 E^2 - (x_1 - x_2)^2 E^2 \\ &= 4 x_1 x_2 E^2 \\ &= x_1 x_2 s\end{aligned}$$

[1]

The number quarks in proton 1 with momentum fraction between x_1 and x_1+dx_1 is $f_q(x_1)\mathrm{d}x_q$. Hence for a particular quark flavour in proton 1 colliding with the appropriate flavour anti-quark in proton 2

$$d^2\sigma = \frac{1}{3}\sigma(\hat{s}) f_q(x_1) f_{\bar{q}}(x_2)\mathrm{d}x_1\mathrm{d}x_2$$ [1]

Note the factor of 1/3 arises because a quark can only annihilate with an anti-quark of the same colour (2 of the above 3 marks are given for this).

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Can also have the same interaction between an anti-quark in proton 1 and a quark in proton 2

$$d^2\sigma = \frac{1}{3}\sigma(\hat{s})(f_q(x_1)f_{\overline{q}}(x_2) + f_q(x_2)f_{\overline{q}}(x_1)\mathrm{d}x_1\mathrm{d}x_2$$ [1]

Summing over quarks and using the above expression for $\hat{s}$ gives:

$$\frac{d^2\sigma}{\mathrm{d}x_1\mathrm{d}x_2} = \frac{4\pi\alpha^2}{9x_1x_2s}\sum_q e_q^2[f_q(x_1)f_{\overline{q}}(x_2) + f_q(x_2)f_{\overline{q}}(x_1)]$$ [1]

Finally, considering only u and d quarks

$$\frac{d^2\sigma}{\mathrm{d}x_1\mathrm{d}x_2} = \frac{4\pi\alpha^2}{81x_1x_2s}\{4[u(x_1)\overline{u}(x_2) + u(x_2)\overline{u}(x_1)] + d(x_1)\overline{d}(x_2) + d(x_2)\overline{d}(x_1)\}$$ [1]

In this expression x_1 and x_2 are the fractional momenta carried by the partons involved in the collistion, s is the centre of mass energy of the proton-proton collision, and $u(x)$, $d(x)$, $\bar{u}(x)$ and $\bar{d}(x)$ are the up and down quark/anti-quark parton distribution functions. These count the number of quarks (or antiquarks) of a given flavour which carry a given fraction x of the hadron's momentum in the infinite momentum frame. Specifically they are normalised such that $u(x)dx$ is the unumber of up quarks with momentum fraction x lying between x and $x + dx$. [5]

Drell-Yan production of muon paris has been studied in pion collisions with carbon targets. Carbon (C) contains an equal number of protons and neutrons.

(f) If the invariant mass of the observed $\mu^+\mu^-$ system is Q^2, explain why the ratio ρ defined by

$$\rho = \frac{\sigma(\pi^+\mathrm{C} \to \mu^+\mu^- X)}{\sigma(\pi^-\mathrm{C} \to \mu^+\mu^- X)}$$

might approach unity for small Q^2. Furthermore: what might you expect it to approach to as Q^2 approaches s? [5]

(unseen in this form, but the course did mention the analogous results for a different ratio obtained from a different scattering process)

At low Q^2 mainly see annihilation from sea quarks (dominate at low x). Would expect the same sea quark distributions in pi^+/pi^- and hence the ratio is one. [2] At high Q^2 mainly see annihilation of pion valence quarks. Hence cross-sections in ratio of squares of anti-quark charges giving a ratio of about one quarter. [3]

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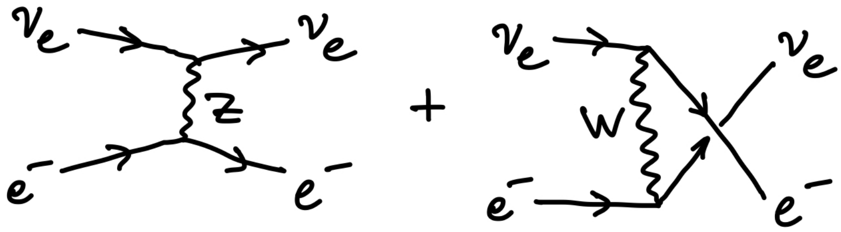
3 The Super-Kamiokande water Čerenkov detector observes solar neutrinos through the elastic scattering of electron neutrinos from atomic electrons.

(a) Draw the two lowest order Feynman diagrams for the process of $\nu_e e^- \to \nu_e e^-$ scattering. [2]



The Lorentz invariant matrix element for the charged current contribution to $\nu_e e^- \to \nu_e e^-$ scattering can be written

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\rho} \left[\bar{u}(p'_e)\gamma^\mu \frac{1}{2}(1-\gamma^5)u(p_{\nu_e}) \right] \left[\bar{u}(p'_{\nu_e})\gamma^\rho \frac{1}{2}(1-\gamma^5)u(p_e) \right].$$

(b) Show that, in the limit where the electron and neutrino masses can be neglected, the only scattering processes to have non-zero matrix elements are between left-handed helicity particles, and that for those the matrix element can be written as follows:

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\rho} \left[\bar{u}_\downarrow(p'_e)\gamma^\mu u_\downarrow(p_{\nu_e}) \right] \left[\bar{u}_\downarrow(p'_{\nu_e})\gamma^\rho u_\downarrow(p_e) \right]$$

where $u_\downarrow$ is a left-handed helicity eigenstate. [5]

(part bookwork (in terms of definitions) and part mathematical manipulation)
What we need to prove is:

1. that the RHS of the first given ME goes to zero if we replace any of the spinors it contains with $u_\uparrow(a^\mu)$ and then let $a^\mu a_\mu \to 0$, and
2. that the RHS of the first given expression goes to the second supplied RHS if we replace all of the spinors in the first one with $u_\downarrow$ (for the appropriate momentum) and then let all the masses tend to zero.

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We know (or can easily prove from either the supplied γ-matrices, or $\{\gamma^\alpha,\gamma^\beta\} = 2g^{\alpha\beta}\mathbb{1}_{4\times4}$) that for $P_{\substack{R\\L}} = \frac{1}{2}(\mathbb{1} \pm \gamma^5)$ the following hold:

① $P_L = P_L^2$

② $P_L\gamma^\mu = \gamma^\mu P_R$

③ $P_R\gamma^\mu = \gamma^\mu P_L$

④ $(P_{L/R})^\dagger = P_{L/R}$

These facts are enough for us to show that

$$\bar{u}(a)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)u(b) = \bar{u}(a)\gamma^\mu P_L u(b)$$
$$= \overline{(P_L u(a))}\,\gamma^\mu\,(P_L u(b))$$

and so it suffices for us to show that

⑤ $$L_{⑤} \overset{\text{def}}{\equiv} \lim_{a^2\to 0}\left(u_\downarrow(a) - P_L u_\downarrow(a)\right) = 0$$

⑥ $$L_{⑥} \overset{\text{def}}{\equiv} \lim_{a^2\to 0}\left(\qquad P_L u_\uparrow(a)\right) = 0 .$$

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With $\alpha = \frac{|p_a|}{E_a+m_a}$, know $a^2 \to 0$ is same as $\alpha \to 1$

Since $u_\downarrow(a) = N\begin{pmatrix}-s\\ e^{i\phi}c\\ \alpha s\\ -\alpha e^{i\phi}c\end{pmatrix}$ and $u_\uparrow(a) = N\begin{pmatrix}c\\ e^{i\phi}s\\ \alpha c\\ \alpha e^{i\phi}s\end{pmatrix}$

we have $\lim\limits_{a^2\to 0} u_\downarrow(a) = N\begin{pmatrix}-s\\ e^{i\phi}c\\ s\\ -e^{i\phi}c\end{pmatrix}$ and $\lim\limits_{a^2\to 0} u_\uparrow(a) = N\begin{pmatrix}c\\ e^{i\phi}s\\ c\\ e^{i\phi}s\end{pmatrix}$

and so
$$\left\{\begin{array}{l}\lim\limits_{a^2\to 0} P_L u_\downarrow(a) = \frac{1}{2}\begin{pmatrix}\mathbb{1} & -\mathbb{1}\\ -\mathbb{1} & \mathbb{1}\end{pmatrix}\lim\limits_{a^2\to 0} u_\downarrow(a) = \lim\limits_{a^2\to 0} u_\downarrow(a)\\ \lim\limits_{a^2\to 0} P_L u_\uparrow(a) = \frac{1}{2}\begin{pmatrix}\mathbb{1} & -\mathbb{1}\\ -\mathbb{1} & \mathbb{1}\end{pmatrix}\lim\limits_{a^2\to 0} u_\uparrow(a) = 0\end{array}\right\}$$

and so $L_{5} = 0$ & $L_{6} = 0$ as desired.

(c) Considering only the charged current contribution to $\nu_e e^- \to \nu_e e^-$ scattering, and working in the centre-of-mass frame, express M_{fi} in terms of the centre-of-mass energy, $\sqrt{s}$, and show that the total cross section is given by

$$\sigma(\nu_e e^- \to \nu_e e^-) = \frac{G_F^2 s}{\pi},$$

where $G_F/\sqrt{2} = g_W^2/8m_W^2$. [10]

(technically unseen, but similar content has been explained in the course many times)

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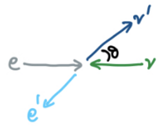
We are given the hint that $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|p_f^*|}{|p_i^*|}\langle|M_{fi}|\rangle^2$

For our situation $|p_f^*| = |p_i|^*$ and there is only one matrix element, but there are two initial spin possibilities (the electron can be $\uparrow$ or $\downarrow$) so $\langle|M_{fi}|^2\rangle = \underbrace{\tfrac{1}{2}}_{\text{average over 2 spins}}|M_{fi}|^2$.

$$\therefore \quad \frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{1}{2}|M_{fi}|^2 \quad \text{with} \quad M_{fi} = \frac{g_W^2}{2m_W^2}\, j^\mu k_\mu$$



where $j^\mu = \bar{u}_\downarrow(p_e')\gamma^\mu u_\downarrow(p_\nu)\Big|_{\text{massless}}$

and $k^\mu = \bar{u}_\downarrow(p_\nu')\gamma^\mu u_\downarrow(p_e)\Big|_{\text{massless}}$

$$\left\{\begin{matrix} p_e' = \begin{pmatrix} p \\ -p\sin\theta \\ 0 \\ -p\cos\theta \end{pmatrix} & p_\nu = \begin{pmatrix} p \\ 0 \\ 0 \\ -p \end{pmatrix} \\ p_\nu' = \begin{pmatrix} p \\ p\sin\theta \\ 0 \\ p\cos\theta \end{pmatrix} & p_e = \begin{pmatrix} p \\ 0 \\ 0 \\ p \end{pmatrix} \end{matrix}\right\} \text{ and so } \left\{\begin{matrix} \theta = \pi - \theta \to c\to s,\ s \to c & \theta = \pi \to c = 0,\ s = 1 \\ \phi = \pi \quad e = -1 & \phi = 0 \quad e = 1 \\ \theta = \theta \to c \to c,\ s \to s & \theta = 0 \to c = 1,\ s = 0 \\ \phi = 0 \quad e = 1 & \phi = 0 \quad e = 1 \end{matrix}\right\}$$

$$u_\downarrow = N\begin{pmatrix} -s \\ ec \\ s \\ -ec \end{pmatrix} \Rightarrow \left\{\begin{matrix} u_\downarrow = N\begin{pmatrix} -c \\ -s \\ c \\ s \end{pmatrix} & u_\downarrow = N\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ u_\downarrow = N\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix} & u_\downarrow = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \end{matrix}\right\}$$

where in all cases $s = \sin\theta/2$, $s = \cos\theta/2$

Using the paper's long-hint expression for $\bar{\psi}\gamma^\mu\phi$:

$$\begin{aligned} \bar{\psi}\gamma^0\phi &= \overset{-c}{\psi_1^*}\overset{-1}{\phi_1} + \cancel{\psi_2^*\phi_2} + \overset{c}{\psi_3^*}\phi_3 + \cancel{\psi_4^*\phi_4}, \\ \bar{\psi}\gamma^1\phi &= \cancel{\psi_1^*\phi_4} + \overset{-s}{\psi_2^*}\phi_3 + \cancel{\psi_3^*\phi_2} + \overset{s}{\psi_4^*}\overset{-1}{\phi_1}, \\ \bar{\psi}\gamma^2\phi &= -i(\cancel{\psi_1^*\phi_4} - \overset{-s}{\psi_2^*}\phi_3 + \cancel{\psi_3^*\phi_2} - \overset{s}{\psi_4^*}\overset{-1}{\phi_1}), \\ \bar{\psi}\gamma^3\phi &= \overset{-c}{\psi_1^*}\phi_3 - \cancel{\psi_2^*\phi_4} + \overset{c}{\psi_3^*}\overset{-1}{\phi_1} - \cancel{\psi_4^*\phi_2}, \end{aligned}$$

we see $j^\mu = N^2\begin{pmatrix} c+c \\ -s-s \\ -i(s+s) \\ -c-c \end{pmatrix} = 2N^2\begin{pmatrix} c \\ -s \\ -is \\ -c \end{pmatrix}$

and

$$\begin{aligned} \bar{\psi}\gamma^0\phi &= \cancel{\psi_1^*\phi_1} + \overset{c}{\psi_2^*}\phi_2 + \cancel{\psi_3^*\phi_3} + \overset{-c}{\psi_4^*}\overset{-1}{\phi_4}, \\ \bar{\psi}\gamma^1\phi &= \overset{-s}{\psi_1^*}\overset{-1}{\phi_4} + \cancel{\psi_2^*\phi_3} + \overset{s}{\psi_3^*}\phi_2 + \cancel{\psi_4^*\phi_1}, \\ \bar{\psi}\gamma^2\phi &= -i(\overset{-s}{\psi_1^*}\overset{-1}{\phi_4} - \cancel{\psi_2^*\phi_3} + \overset{s}{\psi_3^*}\phi_2 - \cancel{\psi_4^*\phi_1}), \\ \bar{\psi}\gamma^3\phi &= \cancel{\psi_1^*\phi_3} - \overset{c}{\psi_2^*}\overset{-1}{\phi_4} + \cancel{\psi_3^*\phi_1} - \overset{-c}{\psi_4^*}\phi_2, \end{aligned}$$

we see $k^\mu = N^2\begin{pmatrix} c+c \\ s+s \\ -i(s+s) \\ c+c \end{pmatrix} = 2N^2\begin{pmatrix} c \\ s \\ -is \\ c \end{pmatrix}$

$$\begin{aligned} \therefore \quad j^\mu k_\mu &= (2N^2)(2N^2)\left[c^2 - (-s^2 - s^2 - c^2)\right] \\ &= (2N^2)(2N^2)\,2 \\ &= 2s \end{aligned}$$

since $N = \sqrt{E+m} \overset{\text{since } m=0}{=} \sqrt{p} = \sqrt{\tfrac{1}{2}\sqrt{s}}$

$\therefore\ 2N^2 = \sqrt{s}$

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and hence

$$\begin{aligned}
\frac{d\sigma}{d\Omega^\star} &= \frac{1}{64\pi^2 s}\cdot\frac{1}{2}\left|\frac{g_W^2}{2m_W^2} j^\mu k_\mu\right|^2 \\
&= \frac{1}{64\pi^2 s}\cdot\frac{1}{2}\left|\frac{g_W^2}{2m_W^2} 2s\right|^2 \qquad \text{(using the hand written results)} \\
&= \frac{s}{\pi^2}\cdot\frac{1}{2}\left|\frac{g_W^2}{8m_W^2}\right|^2 \\
&= \frac{s}{\pi^2}\cdot\frac{1}{2}\left|\frac{G_F}{\sqrt{2}}\right|^2 \\
&= \frac{G_F^2 s}{4\pi^2}
\end{aligned}$$

and therefore after integrating over all 4π solid angle we conclude that:

$$\sigma = \frac{G_F^2 s}{\pi}.$$

Solar neutrinos detected in Super-Kamiokande are produced primarily from the $^8\mathrm{B} \to {}^7\mathrm{Be} + e^+ + \nu_e$ process and have a mean energy of approximately 10 MeV.

(d) Obtain the value of $\sigma(\nu_e e^- \to \nu_e e^-)$ at this energy, expressing your answer in S.I. units. [3]

(Unseen - not much unit manipulation in the course. Provides place to test if candidates have feel for order of magnitude of expected result(s).)

The when massless neutrinos with energy E_{ν_e} meet stationary electrons with mass m_e, then the initial state has four-momentum

$$(E_{\nu_e} + m_e, 0, 0, E_{\nu_e})$$

and so

$$\begin{aligned}
s &= (E_{\nu_e} + m_e, 0, 0, E_{\nu_e}) \\
&= (E_{\nu_e} + m_e)^2 - E_{\nu_e}^2 \\
&= 2E_{\nu_e} m_e + m_e^2 \\
&= 2 \times (10\ \mathrm{MeV}) \times (5.11 \times 10^{-1}\ \mathrm{MeV}) + (5.11 \times 10^{-1}\ \mathrm{MeV})^2 \\
&= (10.22 + 0.511^2) \times \mathrm{MeV}^2 \\
&= 10.48 \times \mathrm{MeV}^2
\end{aligned}$$

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and so

$$\begin{aligned}
\sigma &= \frac{G_F^2 s}{\pi} \\
&= (1.166 \times 10^{-5} \times \mathrm{GeV}^{-2})^2 \times (10.48 \times \mathrm{MeV}^2)/\pi \\
&= (1.166 \times 10^{-5} \times (10^3\ \mathrm{MeV})^{-2})^2 \times (10.48 \times \mathrm{MeV}^2)/\pi \\
&= \frac{1.166^2 \times 10.48}{\pi} \times 10^{-22} \times \mathrm{MeV}^{-2} \\
&= \frac{1.166^2 \times 10.48}{\pi} \times 10^{-22} \times \mathrm{MeV}^{-2} \times (197 \times \mathrm{MeV} \times \mathrm{fm})^2
\end{aligned}$$

since $197 \times \mathrm{MeV} \times \mathrm{fm} = \hbar c = 1$ in natural units[3], and so

$$\begin{aligned}
\sigma &= \frac{1.166^2 \times 10.48}{\pi} \times 10^{-22} \times (197 \times \mathrm{fm})^2 \\
&= \frac{1.166^2 \times 10.48 \times 197^2}{\pi} \times 10^{-22} \times (10^{-15} \times \mathrm{m})^2 \\
&= \frac{1.166^2 \times 10.48 \times 197^2}{\pi} \times 10^{-52} \times \mathrm{m}^2 \\
&\approx 1.76 \times 10^{-47} \times \mathrm{m}^2
\end{aligned}$$

The flux of ^{8}B solar electron neutrinos at the Earth is expected to be $2.3 \times 10^{10}\ \mathrm{m}^{-2}\mathrm{s}^{-1}$.

(e) Neglecting all interactions other than the charged current interaction, estimate the number of ^{8}B solar electron neutrino interactions per day in the Super-Kamiokande detector, of mass 5×10^7 kg. [4]

(Unseen - not much unit manipulation in the course. Provides place to test if candidates have feel for order of magnitude of expected result(s).)

The rate of interaction Γ is related to the flux ϕ, the cross section σ and the number of scatterers N by the formula

$$\Gamma = N\phi\sigma.$$

Furthermore, the number of scatterers is the the number of electrons in the detector, which is also the number of protons N_p. I.e.

$$N = N_p.$$

Since the detector is mostly water, and since Oxygen has 8 protons and 8 neutrons, the detector has protons and neutrons in the ratio $N_n/N_p = 8/(8+1+1) = 8/10$. The number of nucleons $N_N = N_p + N_n$ can be calculated from the mass M of the detector: taking the mass to be dominated by nuclei, and taking $m_p = m_n$, we will have

$$M = N_N m_p$$

[3]this fact was emphasised in the course, however if the students cannot remember it, they can derive it from the value of the electron charge given in the hint

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and so

$$\begin{aligned} M &= (N_p + N_n)m_p \\ &= N_p\left(1 + \frac{N_n}{N_p}\right)m_p \\ &= N_p\left(1 + \frac{8}{10}\right)m_p \\ &= \frac{18N_p m_p}{10} \end{aligned}$$

and so

$$N = N_p = \frac{10M}{18m_p}$$

and so

$$\begin{aligned} \Gamma &= N\phi\sigma \\ &= \frac{10M\phi\sigma}{18m_p} \\ &= \frac{10 \times (5 \times 10^7 \times \mathrm{kg}) \times (2.3 \times 10^{10} \times \mathrm{m}^{-2} \times \mathrm{s}^{-1}) \times (1.76 \times 10^{-47} \times \mathrm{m}^2)}{18 \times (1.67 \times 10^{-27} \times \mathrm{kg})} \\ &= \frac{10 \times (5 \times 10^7) \times (2.3 \times 10^{10}) \times (1.76 \times 10^{-47})}{18 \times (1.67 \times 10^{-27})} \times \mathrm{s}^{-1} \\ &= \frac{11.5 \times 1.76}{18 \times 1.67} \times 10^{1+7+10-47+27} \times \mathrm{s}^{-1} \\ &= \frac{11.5 \times 1.76}{18 \times 1.67} \times 10^{-2} \times \mathrm{s}^{-1} \\ &= 6.73 \times 10^{-3} \times \mathrm{s}^{-1} \\ &= 6.73 \times 10^{-3} \times \mathrm{day} \times \mathrm{s}^{-1} \times \mathrm{day}^{-1} \\ &= 6.73 \times 10^{-3} \times (86400 \times \mathrm{s}) \times \mathrm{s}^{-1} \times \mathrm{day}^{-1} \\ &= 582 \times \mathrm{day}^{-1} \\ &\approx 600 \times \mathrm{day}^{-1} \end{aligned}$$

(f) Briefly explain how solar neutrinos are detected in the Super-Kamiokande experiment and how they are distinguished experimentally from the background due to radioactive decays. [6]

(Unseen - discursive)

This answer should describe Cerenkov radiation emitted from the electrons or muons which are kicked out by the nuetrinos at faster than the speed of light in the water. The Cerenkov light is picked up by photomultiplier tubes on the walls floor and ceiling of the derctor where the light arrives in rings. Timing information enables the detector to work out from which direction the particle came. The fuzziness of the ring distinguishes

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electrons from muons as the electrons scatter more and make fuzzier rings. The solar neutrinos are distinguished from others by using the directional information: those coming from the sun should point back to the sun. The background from other sources is isotropic.

You may make use of the following pieces of information which make use of the gamma-matrix conventions adopted by the lecture course:

i) For spinors ψ and ϕ:

$$\begin{aligned}
\bar{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4, \\
\bar{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1, \\
\bar{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1), \\
\bar{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2,
\end{aligned}$$

ii) With starred quantities being defined in the centre-of-mass frame, the differential cross section for a two-to-two scattering process satisfies:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}_f^*|}{|\boldsymbol{p}_i^*|}\left\langle\left|M_{fi}\right|^2\right\rangle.$$

iii) $G_F = 1.166\times10^{-5}$ GeV^{-2}, $m_e = 5.11\times10^{-4}$ GeV, $m_p = 0.94$ GeV $= 1.67\times10^{-27}$ kg *and* $e = 1.6\times10^{-19}$ C.

END OF PAPER

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