UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics
MASTER OF ADVANCED STUDY IN PHYSICS

Monday 17th January 2022 10:00 to 12:00

MAJOR TOPICS
Paper 1/PP (Particle Physics)

Answer ***two*** *questions only. The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate. The paper has content on 31 sides, including this one, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

You should use a ***separate Answer Book*** *for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2x20-page answer books	Mathematical Formulae Handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

The information in this box may be used in any question.

The Pauli-matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The gamma matrix representation of the Part III Particles lecture course was:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

which has the following properties:

$$(\gamma^0)^* = \gamma^0,\ \ (\gamma^1)^* = \gamma^1,\ \ (\gamma^2)^* = -\gamma^2,\ \ (\gamma^3)^* = \gamma^3 \ \ \textit{and}\ \ \gamma^2(\gamma^\mu)^* = -\gamma^\mu\gamma^2.$$

Using the above convention, the Part III Particles lecture course defined the following particle and anti-particle spinors:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\boldsymbol{p}|}{E+m}c \\ \frac{|\boldsymbol{p}|}{E+m}e^{i\phi}s \end{pmatrix}, \qquad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\boldsymbol{p}|}{E+m}s \\ -\frac{|\boldsymbol{p}|}{E+m}e^{i\phi}c \end{pmatrix},$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\boldsymbol{p}|}{E+m}s \\ -\frac{|\boldsymbol{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}, \qquad v_\downarrow = N\begin{pmatrix} \frac{|\boldsymbol{p}|}{E+m}c \\ \frac{|\boldsymbol{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

for objects whose three-momentum $\boldsymbol{p}$ *is given by* $|\boldsymbol{p}|(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ *where* $c = \cos\frac{\theta}{2}$ *and* $s = \sin\frac{\theta}{2}$*. The normalising constant is* $N = \sqrt{E+m}$*.*

$\hbar \approx 1.05 \times 10^{-34}$ kg m^2/s, $\qquad c \approx 3.00 \times 10^8$ m/s, $\qquad e \approx 1.60 \times 10^{-19}$ C.

$m_e = 5.11 \times 10^{-4}$ GeV. $\qquad m_p = m_n = 1.67 \times 10^{-27}$ kg.

1 Neutrinos produced inside the sun by electron capture in the following process $^7\text{Be} + e^- \to {}^7\text{Li} + \nu_e$ are almost mono-energetic with an energy of $E_\nu = 862$ keV and are produced at a rate of $4.5 \times \pi \times 10^{36}$ per second. The Earth's mean distance from the sun is 150 million km.

The Borexino experiment in Italy is optimised to look for such neutrinos. It observes neutrinos through their elastic scattering $\nu_e e^- \to \nu_e e^-$ with electrons in molecules of an organic solvent called 1,2,4-trimethylbenzene, $C_6H_3(CH_3)_3$ within which a small quantity of a scintillating additive is dissolved. The additive generates scintillation photons in proportion to the lab-frame *kinetic* energy ($\text{KE}_e = E_e - m_e$) of each scattered electron, and it is by this means that scattering events are observed. Many tonnes of solvent are used in the experiment (1 tonne=1000 kg). [*Each of the carbon atoms of the solvent contains six protons and six neutrons.*]

If the above scattering process were mediated exclusively by the W-boson (and if electron and neutrino masses could be neglected) then the total cross section in natural units would be:

$$\sigma(\nu_e e^- \to \nu_e e^-) = \frac{G_F^2 s}{\pi} \qquad (\star)$$

where $\sqrt{s}$ is the centre-of-mass energy and $G_F = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$.

(a) Write down in S.I. units a numerical value for the cross section predicted by ($\star$) for ^{7}Be solar neutrinos. [4]

Bookwork

We are told that $G_F = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$. The quantity s is the square of a centre-of-mass energy, and so the units of

$$\frac{G_F^2 s}{\pi}$$

will be 'per energy squared'. However, we know that cross sections should come out as 'per unit area' i.e. 'per length squared'. We recognise, therefore, that we should use the fact that $\hbar c = 197$ MeV fm (emphasised in the course) is 1 in natural units to get back lost unit-inducing factors. We will need two powers of $\hbar c$ to convert back to actual area. Hence, with that correction we actually evaluate:

$$\sigma_{\text{S.I.}} = \frac{G_F^2 s}{\pi} (\hbar c)^2 .$$

(TURN OVER

Furthermore,

$$
\begin{aligned}
s &= (p_e^\mu + p_\nu^\mu)^2 \\
&= \left(\begin{pmatrix} E_\nu \\ 0 \\ 0 \\ E_\nu \end{pmatrix} + \begin{pmatrix} m_e \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)^2 = \begin{pmatrix} E_\nu + m_e \\ 0 \\ 0 \\ E_\nu \end{pmatrix}^2 \\
&= (E_\nu + m_e)^2 - E_\nu^2 \\
&= 2m_e E_\nu + m_e^2
\end{aligned}
$$

(assuming all quantities are internally in Natural Units). Paper rubric says $m_e = 511 keV$ and question description says $E_\nu = 862$ keV so cannot neglect electron mass.

In summary:

$$
\begin{aligned}
\sigma_{\text{S.I.}} &= \left[\frac{G_F^2}{\pi} (2m_e E_\nu + m_e^2) \right]_{\text{Natural Units}} (\hbar c)^2 \\
&= \frac{1}{\pi} \left(1.166 \times 10^{-5}\ \text{GeV}^{-2} \right)^2 \times ((511 \times 862 + 511^2) \times \text{keV}^2) \times \\
&\qquad (197\ \text{MeV fm})^2 \\
&\approx 1.92 \times \frac{\text{keV} \times \text{keV} \times \text{MeV} \times \text{MeV}}{\text{GeV} \times \text{GeV} \times \text{GeV} \times \text{GeV}} \times \text{fm}^2 \\
&= 1.92 \times \frac{10^{18}}{10^{36}} \times \left(10^{-15} \times \text{m} \right)^2 \\
&= 1.92 \times 10^{-48}\ \text{m}^2
\end{aligned}
$$

[Aside: You get $\sigma_{\text{S.I.}} = 1.48 \times 10^{-48}\ \text{m}^2$ if you assume $m_e \ll E_\nu$. Though this is not really the case, no significant penalties are applied to those who assume that.]

Note that a student who has not learned $\hbar c$=197 MeV fm will still be able to complete the question by computing this value from constants given in the text between rubric and question. However this will cost them time and effort, and the intention is that those who did learn $\hbar c$=197 MeV fm will benefit from having learned this. If they did want to work this value out for themselves they would have to do this:

```
In[1]:= hbar = 1.05*10^-34 kg m^2/s
```

$$\text{Out[1]}= \frac{1.05\times10^{-34}\ \text{kg m}^2}{\text{s}}$$

```
In[2]:= c = 3.00*10^8 m/s
```

$$\text{Out[2]}= \frac{3.\times10^{8}\ \text{m}}{\text{s}}$$

```
In[3]:= MeV = 10^6 × 1.60*10^-19 kg m^2/s^2
```

$$\text{Out[3]}= \frac{1.6\times10^{-13}\ \text{kg m}^2}{\text{s}^2}$$

```
In[4]:= fm = 10^-15 m
```

$$\text{Out[4]}= \frac{\text{m}}{1\,000\,000\,000\,000\,000}$$

```
In[5]:= hbar c / (MeV fm)
Out[5]= 196.875
```

Note added after marking the 2022 exam scripts:

Though it was reassuring to see that the majority of candidates who sat the exam (approximately 90% of them) could (and did) instantly write down $\hbar c = 197$ MeV fm, it was nonetheless surprising that very few candidates could either (a) answer this question correctly, or (b) that many who got huge answers did record any concerns about the size based on 'feeling' that neutrino cross sections should be 'small'. In fact, the list of answers received in 2022 (sorted into random order) was as follows:[1]

1.91×10^{-39} m^2, $0.00\times10^{+00}$ m^2, 1.48×10^{-48} m^2, $0.00\times10^{+00}$ m^2, 5.20×10^{-47} m^2, 4.97×10^{-56} m^2, 1.48×10^{-48} m^2, 1.93×10^{-48} m^2, 7.90×10^{-53} m^2, 4.37×10^{-49} m^2, 1.91×10^{-48} m^2, 4.43×10^{-38} m^2, 1.15×10^{-48} m^2, 1.48×10^{-48} m^2, 6.04×10^{-31} m^2, 3.79×10^{-73} m^2, $0.00\times10^{+00}$ m^2, 1.25×10^{-48} m^2, 1.48×10^{-45} m^2, 1.26×10^{-42} m^2, 2.00×10^{-47} m^2, 1.92×10^{-46} m^2, 1.65×10^{-42} m^2, 3.17×10^{-33} m^2, 1.90×10^{-48} m^2, 1.30×10^{-48} m^2, 1.25×10^{-48} m^2, 1.40×10^{-30} m^2, 1.27×10^{-43} m^2, 1.47×10^{-26} m^2, 1.58×10^{-46} m^2, 9.30×10^{-09} m^2, 1.38×10^{-42} m^2, 3.13×10^{-49} m^2, 1.45×10^{-45} m^2, 7.25×10^{-35} m^2, 1.91×10^{-48} m^2, 3.27×10^{-12} m^2, 7.70×10^{-12} m^2, 1.00×10^{-54} m^2, $1.20\times10^{+15}$ m^2, 1.10×10^{-41} m^2, 1.49×10^{-48} m^2, 1.92×10^{-48} m^2, 5.00×10^{-48} m^2, 3.17×10^{-48} m^2, 7.29×10^{-41} m^2, $1.07\times10^{+09}$ m^2, 1.92×10^{-48} m^2, 1.92×10^{-48} m^2.

Note that the range of values reported extends over nearly 100 orders of magnitude from 10^{-73} m^2 to 10^{+15} m^2. The upper extreme is more than double the surface area of The Earth! One would have imagined that **all** candidates with answers close to or above a

[1]Some people gave answers in fm^2 rather than m^2 for no penalty. The answers given in this list are what those answers would have become after conversion to m^2. A very small number of people gave incommensurate answers. E.g. at least one person reported a value for the cross section as a volume (m^3), another as a length, another as a frequency, and another as dimensionless. To protect the identities of those persons I have rendered those small number of answers in the list here as m^2 and/or changed them in some other way even though they were not reported as such on the scripts. The number of such changes is so small, however, as to not change the meaning of the list as a whole.

square fermi or a barn (i.e. above $\sim 10^{-30}$ m^2 or $\sim 10^{-28}$ m^2, the size nucleii) would have had *huge* reason to express concern that their answers answers being correct given the expectation that neutrinos should be the most slippery and hard-to-detect particles of the Standard Model. Alas few with large cross sections did report on or notice excess sizes, though those who were able to spot their answers were too big were usually given some credit for their wisdom even if they lost credit for their inability to divide or convert between units.

(b) Estimate the number of electrons in one tonne of 1,2,4-trimethylbenzene. Make clear your assumptions and estimate the accuracy of your answer. [4]

Straightforward

1,2,4-trimethylbenzene has 9 carbon atoms and 12 hydrogen atoms. That implies that in one molecule $N_p = 9 \times 6 + 12 = 66$ while $N_n = 9 \times 6 = 54$. Ignoring the full SEMF, binding energies, electron masses, *etc*, the approximate number of molecules in one tonne, $N_{\text{molecules-per-tonne}}$, could be given by:

$$
\begin{aligned}
N_{\text{molecules-per-tonne}} &= \frac{1 \text{ tonne}}{66 m_p + 54 m_n} \\
&= \frac{1 \text{ tonne}}{120 \times m_p} \qquad \text{(since we may assume } m_p = m_n\text{)} \\
&= \frac{10^3 \times \text{kg}}{120 \times 1.67 \times 10^{-27} \times \text{kg}} \\
&= 4.99 \times 10^{27}.
\end{aligned}
$$

Using this, the number of electrons per tonne, $N_{\text{electrons-per-tonne}}$, is given by

$$
\begin{aligned}
N_{\text{electrons-per-tonne}} &= N_{\text{molecules-per-tonne}} \times N_p \\
&= 4.99 \times 10^{27} \times 66 \\
&= 3.29 \times 10^{29}.
\end{aligned}
$$

This answer is certainly not better than 1% as m_p and m_n are only given to that precision. Neglection of the SEMF and/or binding energies and could easily be an $\sim O(5\%)$ effect. Ignoring electron masses will be an O(0.1%) effect. Probably our final estimate is therefore accurate to somewhere between 1% and 10%. A candidate may produce an estimate by a different method and still get full credit so long as the method chosen is not likely to be out by more than 10%.

(c) Estimate the maximum number of ^{7}Be solar neutrino interactions which occur per day per tonne of the Borexino detector, in the absence of neutrino oscillations, assuming that the cross section for scattering is described by ($\star$). [4]

Straightforward

The $4.5 \times \pi \times 10^{36}\ \mathrm{s}^{-1}$ solar neutrinos at the source are spread out over a sphere of area $4\pi \times (150 \times 10^9\ m)^2$ by the time they reach Earth, meaning that the flux of neutrinos at the Earth, ϕ, will be

$$\begin{aligned}\phi &= \frac{4.5 \times \cancel{\pi} \times 10^{36}\ \mathrm{s}^{-1}}{4 \times \cancel{\pi} \times (150 \times 10^9\ \mathrm{m})^2} \\ &= 5.0 \times 10^{13}\ \mathrm{m}^{-2}\mathrm{s}^{-1} \\ &= 5.0 \times 10^{9}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}.\end{aligned}$$

If the cross section for neutrino scattering from part (a) is

$$\sigma = 1.48 \times 10^{-48}\ \mathrm{m}^2$$

and the number of scatterers per tonne from part (b) is

$$N = 3.29 \times 10^{29}\ \mathrm{tonne}^{-1}$$

then the event rate Γ (in the absence of neutrino oscillations) would be

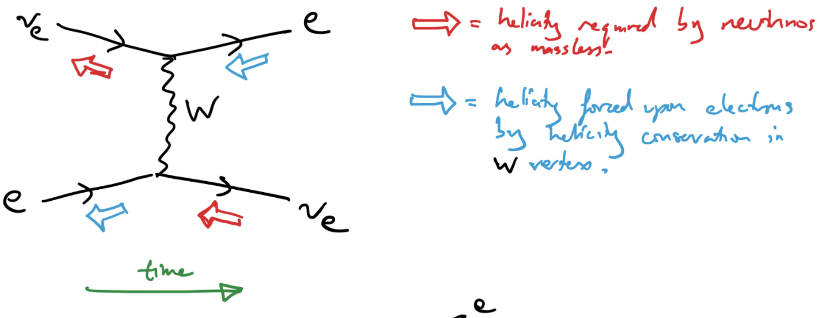
$$\begin{aligned}\Gamma &= \phi\sigma N \\ &= (5.0 \times 10^{13}\ \mathrm{m}^{-2}\mathrm{s}^{-1}) \times (1.92 \times 10^{-48}\ \mathrm{m}^2) \times (3.29 \times 10^{29}\ \mathrm{tonne}^{-1}) \\ &= 3.16 \times 10^{-5}\ \mathrm{tonne}^{-1}\mathrm{s}^{-1} \\ &= 3.16 \times 10^{-5}\ \mathrm{tonne}^{-1}\mathrm{s}^{-1} \times (86400\ \mathrm{s\ day}^{-1}) \\ &= 2.73\ \mathrm{day}^{-1}\mathrm{tonne}^{-1}.\end{aligned}$$

[Aside: This is 273 events per day per hundred tons, which is relevant to part (h) of this question.]

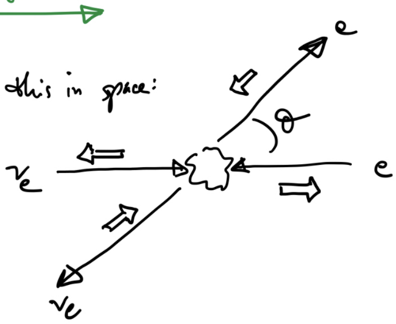
(d) Describe the angular and energy distributions in the centre-of-mass frame for electrons scattered by neutrinos as a result of charged-current interactions. Include within your answer an algebraic expression for $d\sigma/d\Omega^*$ showing its dependence on the cosine of the angle θ^* between the outgoing electron and the incoming neutrino. [4]

Applied knowledge

The incoming and outgoing neutrinos are necessarily left-chiral. The charged current of the weak interaction preserves chirality, so the incoming and outgoing electron chiralities may be deduced from those of the neutrinos and area as shown in the next diagram (which incorrectly labels chiralities as helicities!):



Looks like this in space:



Hence $S=0$ initial state (and final state)
Hence NO $\cos\theta$ preference.

If the particles are relativistic,[2] states of definite chirality are also states of definite helicity, and so the initial state and final state are $S = 0$. Consequently the radiation is spherically symmetric in the c.o.m. frame, i.e. uniform in $\cos\theta^*$ and uniform in ϕ. The former ranges over $[-1, 1]$ and the latter over $[0, 2\pi]$ which are size 2 and 2π respectively, and so the differential cross section is simply a factor 4π less than the total:

$$\frac{d\sigma}{d\Omega^*} = \frac{G_F^2 s}{4\pi^2}.$$

In this frame the energies for the electrons (both incoming and outgoing) are fixed by momentum conservation and are equal to each other, and the same can be said for the neutrino energies. The energies could be found by solving the energy conservation equation in

$$\begin{pmatrix} \sqrt{s} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p^* \\ 0 \\ 0 \\ p^* \end{pmatrix} + \begin{pmatrix} \sqrt{m_e^2 + (p^*)^2} \\ 0 \\ 0 \\ -p^* \end{pmatrix}$$

yielding

$$p^* = \frac{s - m_e^2}{2\sqrt{s}} \qquad \text{and} \qquad E_e^* = \sqrt{m_e^2 + (p_e^*)^2} = \frac{s + m_e^2}{2\sqrt{s}}.$$

[2] In actual fact the particles are not relativistic but in a grey area between relativistic and non-relativistic. From past experience it is not expected that candidates will worry about this distinction. It is expected that all/most will just assume the particles are relativistic and will answer accordingly. To guide them in this direction the opening question paragraph mentions that the supplied cross section assumes that the electron mass has been neglected. Of course, a candidate who takes a more nuanced approach and does not neglect electron masses will not be penalised (and could even be rewarded). But dealing with $m_e \neq 0$ is certainly not expected, and the fact that there are only 4 marks here should make very clear to the candidates that they are being asked for a simple answer, not a full-on first-principles spinorial calculation. For 4 marks they should realised that a simple argument is being requested. FWIW: the possibility to express concerns about relativistic/non-relativistic an be explicitly rewarded later in part (h).

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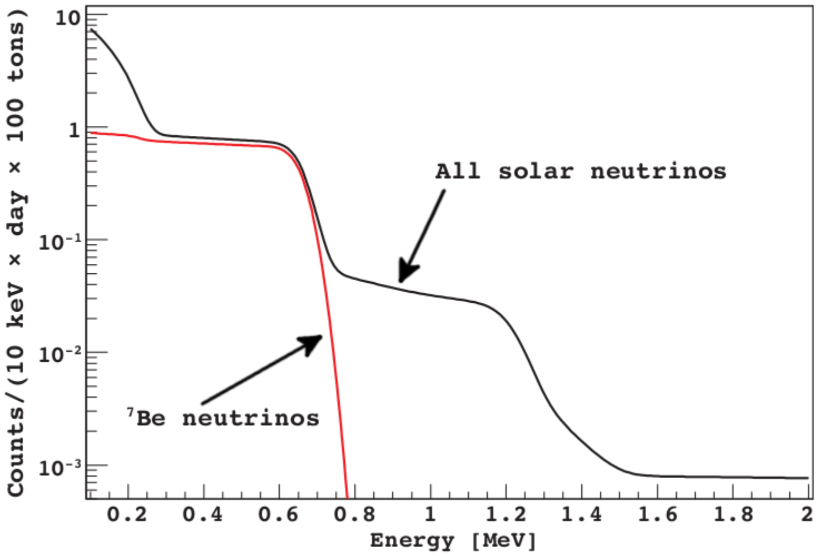
The first Borexino paper to describe observations of ^{7}Be neutrinos (arXiv:0708.2251) noted that:

> "... the recoil electron [kinetic energy distribution has] a rectangular shape with a sharp cut-off edge at 665 keV in the case of ^{7}Be neutrinos (see Fig. 1). The background from the 156 keV β-decay of ^{14}C, intrinsic to the scintillator, limits neutrino observation to [scattered electrons with kinetic] energies above 200 keV."

The figure referred to in the text above shows the following plot:



(e) Check that that $\mathrm{KE}_e^{\max} = 665$ keV is indeed the maximum kinetic energy of an electron scattered by a ^{7}Be solar neutrino by: (i) deriving an algebraic expression for $\mathrm{KE}_e^{\max}$ in terms of m_e and E_ν, and then (ii) confirming that it evaluates to 665 keV. [*You will not be able to neglect the electron mass.*] [4]

Straightforward

$$\begin{pmatrix} E_\nu \\ 0 \\ 0 \\ E_\nu \end{pmatrix} + \begin{pmatrix} m_e \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ 0 \\ -k \end{pmatrix} + \begin{pmatrix} \sqrt{m_e^2 + p_e^2} \\ 0 \\ 0 \\ p_e \end{pmatrix} \tag{1}$$

is solved as follows. First, equate total energy before and after:

$$E_\nu + m_e = k + \sqrt{m_e^2 + p_e^2} \qquad (1)$$

and equate total momentum before and after:

$$E_\nu = -k + p_e \qquad (2)$$

Then (1)+(2) gives

$$2E_\nu + m_e = p_e + \sqrt{m_e^2 + p_e^2}$$

so

$$(2E_\nu + m_e - p_e)^2 = m_e^2 + p_e^2$$

or

$$4E_\nu^2 + 2E_\nu m_e - 2E_\nu p_e + 2E_\nu m_e + m_e^2 - m_e p_e - 2E_\nu p_e + m_e p_e + p_e^2 = m_e^2 + p_e^2$$

so

$$2E_\nu^2 + 2E_\nu m_e - 2E_\nu p_e - m_e p_e = 0,$$

giving

$$p_e = \frac{2E_\nu^2 + 2E_\nu m_e}{2E_\nu + m_e} = 1059\,\text{keV}.$$

Using that value of p_e with the supplied electron mass in the supplied KE formula:

$$KE^{max} = \sqrt{p_e^2 + m_e^2} - m_e$$

gives $E^{max} = 664.9\,\text{keV}$.

ASIDE 1: One student who took the exam took a different (but equivalent) approach with his/her algebra and arrived at an answer which is the same as the above but which is more explicit. He/she (correctly) found that one can write:

$$KE^{max} = \frac{2E_\nu^2}{m_e + 2E_\nu} = 664.9\,\text{keV}.$$

ASIDE 2: In the exam quite a few candidates attempted a bit of (what one might call) 'speculative numerology' rather than physics to answer this 'show that' question. E.g. at least two candidates appeared observe that $\sqrt{E_\nu m_e} \approx 664$ keV, and so created what appeared to be pseudo-justifications for this being the right formula for KE max. Perhaps they hoped that 664 keV was close enough to 665 keV to be telling them something? Alas, any argument leading to a KE max value pf $\sqrt{E_\nu m_e}$ must be wrong: the similarity between 664 and 665 keV is purely a co-incidence. The plot following this para (produced during marking – it would not have been needed by people taking the exam) shows that as E_ν is varied the alternative numerology answer $\sqrt{E_\nu m_e}$ (in orange) does not track the correct answer (in blue) even though they happen to pass very nearby each other for the value of $E\nu$ supplied in the question (dotted vertical line). A similar coincidence was found by another candidate who reported $\text{KE}_{\text{max}} = \sqrt{E_\nu^2 - m_e^2}$ which again is very close to the desired answer (see red line in plot below) but only at the given value of E_ν. Numerology is (sadly) not a suitable way of solving this question!

(TURN OVER

```
EνGiven = 862; (* kEV *)

pe = (2 Eν^2 + 2 Eν me) / (2 Eν + me)
```

$$\frac{2\,E\nu^2 + 2\,E\nu\,me}{2\,E\nu + me}$$

```
KEMAX = Sqrt[pe^2 + me^2] - me;

Wrong1_ALTKEMAX = Sqrt[me Eν];

OK_ALTKEMAX = 2 Eν^2 / (me + 2 Eν);

Wrong2_ALTKEMAX = me ((Eν + me) / (Sqrt[me^2 + 2 me Eν]) - 1);

Wrong3_ALTKEMAX = Sqrt[Eν^2 - me^2];

Series[KEMAX, {me, 0, 4}]
```

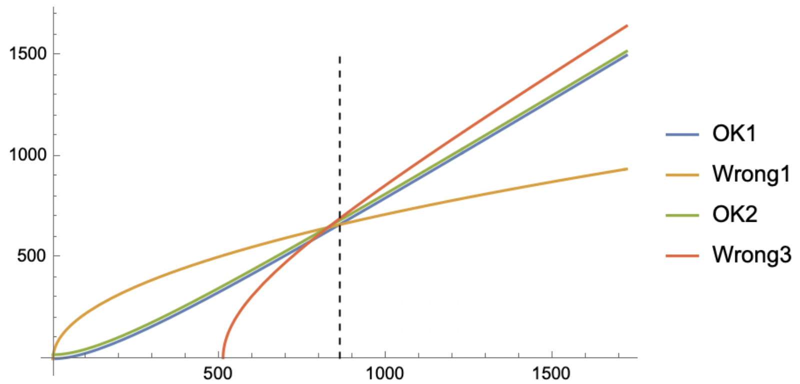
$$\sqrt{E\nu^2} + \left(-1 + \frac{\sqrt{E\nu^2}}{2\,E\nu}\right) me + \frac{me^2}{4\sqrt{E\nu^2}} - \frac{\sqrt{E\nu^2}\;me^3}{8\,E\nu^3} + \frac{\sqrt{E\nu^2}\;me^4}{16\,E\nu^4} + O[me]^5$$

```
subs = {me → 511 (*keV*)};

{KEMAX,  OK_ALTKEMAX, Wrong1_ALTKEMAX, Wrong2_ALTKEMAX, Wrong3_ALTKEMAX} /. subs /.
  Eν → EνGiven // N

{664.916, 664.916, 663.688, 145.511, 694.207}

Plot[{KEMAX /. subs, Wrong1_ALTKEMAX /. subs, 20 + OK_ALTKEMAX /. subs,
  Wrong3_ALTKEMAX /. subs}, {Eν, 0, 2 EνGiven},
 Epilog → {Directive[{Dashed}], Line[{{EνGiven, 0}, {EνGiven, 1500}}]},
 PlotLegends → {OK1, Wrong1, OK2, Wrong3}]
```



(f) Explain how you could find out whether the Borexino paper's description of the shape of the kinetic energy spectrum of scattered electrons is consistent with your answer to part (d) [*You are not* ***required*** *to compute an analytic form for the lab-frame kinetic energy distribution of the electrons, but your answer must provide a clear description of the nature of the computations you would perform.*] [4]

Unseen

[Note: This question is not asking a candidate to PERFORM a calculation, it is only asking then to outline how they would go about performing a calculation. There are thus as many ways of answering it as there are valid approaches. The example given below is just one way a candidate might consider answering it. It is important to stress that answers very different to the one given below could easily gain full marks – all that matters is that the process described make sense overall and achieve the desired aim.]

The uniform radiation in the COM frame for the radiated electron will, after a boost to the lab frame, lead to a preference for forward peaked electrons. Whether or not this leads to a flat (or flattish) distribution for the kinetic energy would require one to transform the mono-energetic distribution from the c.o.m. frame to the required distribution in the lab frame. The student could report that such a transformation is, in effect, what we did in the lecture course when we worked our way from the c.o.m.-frame expression

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{|\bar{p}^*_f|}{|\bar{p}^*_i|}|M_{fi}|^2$$

to the lab-frame expression

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left(\frac{1}{M + E - E\cos\theta}\right)^2 |M_{fi}|^2 \qquad \text{(A)}$$

via Jacobians computed by expressing final-state dependent nvariants (like Mandlestam t) in terms of first (i) com quantities and then (ii) lab frame quantities. Eq (A) shown above is not itself directly applicable to us as its $\cos\theta$ is for the outgoing massless particle not the outgoing massive particle (the electron) – but the nature of the computation done would be similar.

If we **had** been interested in the distribution of the $\cos\theta$ of the neutrino (rather than the electron) then the expression (A) just given shows us that (since the matrix element has no preference for any special final state directions) the probability density $\cos\theta$ can be simply read off as proportional to

$$\left(\frac{1}{M + E - E\cos\theta}\right)^2.$$

Since we are actually interested in the distribution of KE_e our goal would be to get $\frac{d\sigma}{d\mathrm{KE}_e}$ rather than $\frac{d\sigma}{d\cos\theta} \propto \frac{d\sigma}{d\Omega}$ so that we could read off whatever pre-factor **then** appeared. We would therefore expect to need to calculate the Jacobian $\frac{d\mathrm{KE}_e}{d\cos\theta}$ so that the desired differential cross could be computed from

$$\frac{d\sigma}{d\mathrm{KE}_e} = \frac{d\sigma}{d\cos\theta}\frac{d\cos\theta}{d\mathrm{KE}_e}.$$

If, having done so, we found an expression that looked like:

$$\frac{d\sigma}{d(\mathrm{KE}_e)} = f(s, \mathrm{KE}_e)|M_{fi}|^2$$

(TURN OVER

then consistency would be established with the Borexino description if the function $f(s, \mathrm{KE}_e)$ were shown to be approximately independent of KE_e. Calculating $\frac{d\cos\theta}{d\mathrm{KE}_e}$ could be ugly and might be best done using a computer algebra package.

(g) According to the Borexino paper, what fraction of ^{7}Be solar neutrino elastic scattering events are observable, i.e. would have kinetic energies above the β-decay background? [2]

Straightforward

As the distribution is flat, the observed fraction is approx

$$1 - 200/665 = 70\%.$$

1 mark for the general idea, and a second one for using 200 keV rather than 156 keV in the formula. [To first approximation, answers good to ±10 % were accepted.]

(h) The Borexino experiment reported 49 ± 4 counts/(day · 100 tonne) from neutrino interactions, while 75 ± 4 counts/(day · 100 tonne) were expected without neutrino oscillations. The rate found in part (c) should be significantly larger than either of these two rates. What might be the main cause or causes of the discrepancy? [4]

Unseen

[Aside: One cause of discrepancy could be that the student just made a mistake and got an astronomically high prediction through simple error. While it would be correct for a student to say 'my answer is bigger than Borexino's own prediction because I made a mistake and they did not' that sort of answer is clearly not what the examiners are looking for and will not be awarded marks! Marks will not be deducted for students who note something along these lines, though.]

[REASON 1 of 3] : The notes highlight the fact that electron neutrino detection via elastic scattering from electrons is complicated by the fact that the neutral current (Z-boson) interaction is also possible. It was reported in the notes that the Z-boson matrix element interferes destructively with the W-boson one, ($\star$), leading to the total cross section for elastic scattering being about 0.6 times lower than would have been expected from the ($\star$) alone. This might be a good place for candidates to draw the two lowest order Feynman diagrams for the process of $\nu_e e^- \to \nu_e e^-$ scattering: one with a Z and one with a W.

[REASON 2 of 3] : We saw in part (g) that only 70% of the of the scattering events are observable in Borexino.

[REASON 3 of 3] : There will inevitably be other sources of inefficiency though we are not able to quantify such sources with the information in the question. Photomultiplier down time might be one source. A change in angular distribution caused by the Z-boson interference changing the angular distribution to something different to that described in (d) may perhaps account for a small inefficiency. E.g. the plot from the paper shows a non-flat energy spectrum whose red line actually RISES slightly at energies beklow 0.25

MeV, and perhaps this is due to Z-interference. This rise would mean that the number of events above the 200 keV threshold would not be 70% but slightly less than 70%.

Taking into account the two multiplicative correction factors we can quantify above (the 60% from Z-boson interference, and the 70% from KE thresholds) results in our original count-rate prediction of

$$210\ \text{day}^{-1}(100\ \text{tonne})^{-1}$$

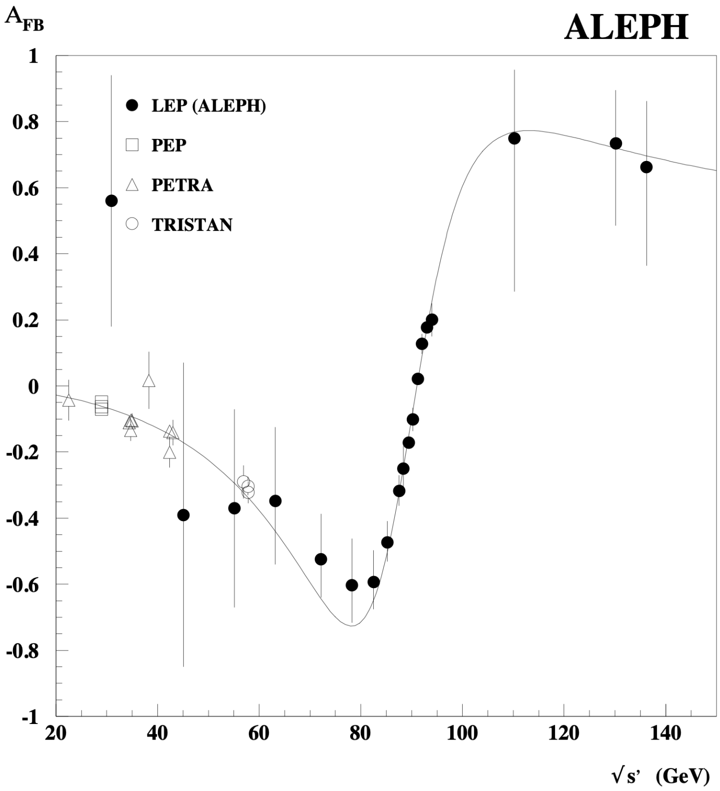
predicted in part (c) being reduced to

$$0.6 \times 0.7 \times 210\ \text{day}^{-1}(100\ \text{tonne})^{-1} = 88\ \text{day}^{-1}(100\ \text{tonne})^{-1}.$$

The rate reported by Borexino's paper ($75 \pm$ $\text{day}^{-1}(100\ \text{tonne})^{-1}$) is only 85% of the value we determined, so it is not too hard to imagine that this remaining difference could due to the other sources of unquantified inefficiency similar to those described in '**[REASON 3 of 3]**'.

2 (a) **Forward-Backward Asymmetry of the Z-boson:**

In 1997, one of the LEP experiments (ALEPH) published[3] the following plot of their measurements of the Z-boson's forward-backward asymmetry.



The plot above was captioned:

> 'Measured forward-backward asymmetries of muon-pair production compared with the fit results. ... For comparison the measurements at lower energies from [other experiments called] PEP, PETRA and TRISTAN are included.'

The fitted curve in the plot above passes through the point $(m_Z, 0.020)$ where m_Z is the mass of the Z-boson.

(i) From an experimental perspective: how is A_{FB} defined and how are measurements of A_{FB} like those shown in the above plot made? [4]

Bookwork

Key features of a good answer might include saying:

- How forward and backward directions are defined in $e^+e^- \to \mu^+\mu^-$ events (forward meaning outgoing μ^- has same sign of z-component of its momentum as was had by the incoming e^-).
- What σ_F and σ_B actually are.

[3] Phys. Lett. B **399** (1997) 329

•That $A_{\mathrm{FB}} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}$.

•That as a cross-section ratio the asymmetry insensitive to luminosity uncertainties and acceptances (so long as the same for positive and negative particles) which is a desirable feature.

(ii) From a theoretical perspective, why is A_{FB} worth measuring? What theoretical features of the standard model does it constrain, and how? [8]

Bookwork

The word ‘features’ in this question was deliberately chosen (when ‘symmetries’ or ‘parameters’ would have been more specific) to avoid making the question too ‘leading’. That is to say, part of the point of phrasing the question this way is to see if candidates answering it appreciate that there are *at least two* different reasons A_{FB}-measurements are interesting: (i) because they tell us something about a symmetry (Parity Violation in the standard model) via parameters like $c^{e/\mu}_{L/R}$ which control that parity violation, and (ii) because they allow us to make measurements of parameters, like $\sin\theta_W$, which are less directly related to parity breaking. [The above (i)/(ii) split is not a clear-cut dichotomy that should be enforced in marking. Someone could easily argue that $\sin\theta_W$ is very much parity related since it tells us how much the W_3-boson (which only couples to left-handed doublets) should mix with the even-handed B-boson. Nonetheless, it will be expected that somehow candidates bring out in some way the idea that there are both measurements of $\sin\theta_W$ and measurements of $c_L \neq c_R$ to be got out of A_{FB}-measurements.]

Key features of a good answer might therefore include saying:

•That

$$< |M_{fi}|^2 > \propto \left[(c_L^e)^2 + (c_R^e)^2\right]\left[(c_L^\mu)^2 + (c_R^\mu)^2\right](1+\cos^2\theta) +$$

$$+2\left[(c_L^e)^2 - (c_R^e)^2\right]\left[(c_L^\mu)^2 - (c_R^\mu)^2\right]\cos\theta$$

and that therefore

$$\frac{d\sigma}{d\Omega} \propto A(1+\cos^2\theta) + B\cos\theta$$

with $A = \left[(c_L^e)^2 + (c_R^e)^2\right]\left[(c_L^\mu)^2 + (c_R^\mu)^2\right]$ and
$B = 2\left[(c_L^e)^2 - (c_R^e)^2\right]\left[(c_L^\mu)^2 - (c_R^\mu)^2\right]$ and so

$$\sigma_F \propto \int_0^1 \frac{d\sigma}{d\Omega} d\cos\theta = \frac{4}{3}A + \frac{1}{2}B$$

and

$$\sigma_B \propto \int_{-1}^0 \frac{d\sigma}{d\Omega} d\cos\theta = \frac{4}{3}A - \frac{1}{2}B$$

and so

$$A_{\mathrm{FB}} = \frac{B}{\frac{8}{3}A} = \frac{3}{4}A_e A_f$$

(TURN OVER

if we define

$$A_f \equiv \frac{(c_L^f)^2 - (c_R^f)^2}{(c_L^f)^2 + (c_R^f)^2} \equiv \frac{2c_V^f c_A^f}{(c_V^f)^2 + (c_A^f)^2} \equiv \frac{2c_V^f/c_A^f}{(c_V^f/c_A^f)^2 + 1}.$$

- A candidate may wish to add some justification for the link between the Matrix Element given above and the helicity conservation at the Z-boson vertex?
- That within the Standard Model σ_F and σ_B are different from each other (or equivalently that $A_{\rm FB} \neq 0$) if and only if (i) $c_L^e \neq c_R^e$, and (ii) $c_L^\mu \neq c_R^\mu$ are different.
- This means that $A_{\rm FB}$ gives us a handle on a parity violation within the Standard Model, and a that a non-zero value of $A_{\rm FB}$ shows us that the SM violates parity.
- Another reason $A_{\rm FB}$ is good to measure is that it provides a clean way of measuring $\sin^2\theta_W$. A candidate could substantiate this claim by saying: (i) that $A_{\rm FB}$-measurements constrain A_e and A_μ, (ii) that any A_f constrains c_V^f/c_A^f, and (iii) that (by reproducing the argument given in lectures) charged leptons satisfy:

$$\frac{c_V^f}{c_A^f} = 1 - 4\sin^2\theta_W.$$

- Bonus marks for nothing that $A_{\rm FB} \neq 0$ does not **by itself** tell us that parity is violated in nature. It only tells us that parity is violated **if the SM describes nature**. This is because $\sigma_F - \sigma_B$ is that $A_{\rm FB}$ (as a quantity) is invariant (not odd) under a parity transform (a property it inherits from the invariance under parity of the angle θ between the outgoing muon and the incoming electron).

(iii) What general conclusions (if any) can be drawn from the ALEPH plot as a whole? What quantitative conclusions (if any) can be drawn from the value of $A_{\rm FB}$ at $\sqrt{s} = m_Z$? What quantitative conclusions about electroweak unification (if any) can we draw from the same data point if it is also assumed that electrons and muons have identical couplings to the Z-boson? [8]

Bookwork

Key features of a good answer might include:

- Broadly speaking, the fact that $A_{\rm FB} \neq 0$ at some values shows us that the SM's c_L and c_R values are unequal. [8]
- Broadly speaking the fitted curve's$A_{\rm FB}$ value is very close to 0 when $\sqrt{s} = m_Z$. This means that $\sin^2\theta_W$ is close to one quarter.
- Broadly speaking, the fact that the curve is not constant suggests that there is more than just one diagram involved – that there is some kind of interference between processes with different angular preferences.

To make specific use of $A_{\rm FB}(m_Z) = 0.02$ we would need to interpret it. To do this needs a theory. The only theory that the students on this course have access to is a leading-order calculation done assuming that only the Z-boson matters. An interpretation can be offered with this theory, but of course it should be recognised that a full calculation would have to be done (and might give have a slightly different result) taking into account things neglected here, if it were desired to get a result closer to ‘reality’.

Our leading order calculation said that:

$$A_{\rm FB}(m_Z) = \frac{3}{4}A_e A_\mu.$$

If we are unwilling to make any further assumptions, then the only thing we can conclude is that

$$\frac{3}{4}A_e A_\mu \approx 0.02.$$

However, if we are willing to assume lepton universality we have also that $A_e = A_\mu = A$ and so

$$\frac{3}{4}A^2 \approx 0.02$$

and so

$$A \approx \pm\sqrt{\frac{0.08}{3}}.$$

As already mentioned, we are expecting:

$$A = \frac{2\rho}{1+\rho^2}$$

in which we have defined $\rho = c_V^e/c_A^e = c_V^\mu/c_A^\mu$. This means that

$$A\rho^2 - 2\rho + A = 0$$

or

$$\rho = \frac{1 \pm \sqrt{1-A^2}}{A}.$$

Given the $\pm$ in the formula for ρ and the (independent) $\pm$ in A there are therefore four possible values of ρ:

$$\rho \in \left\{ +\frac{1+\sqrt{1-\frac{0.08}{3}}}{\sqrt{\frac{0.08}{3}}}, -\frac{1+\sqrt{1-\frac{0.08}{3}}}{\sqrt{\frac{0.08}{3}}}, +\frac{1-\sqrt{1-\frac{0.08}{3}}}{\sqrt{\frac{0.08}{3}}}, -\frac{1-\sqrt{1-\frac{0.08}{3}}}{\sqrt{\frac{0.08}{3}}} \right\}$$
$$\approx \{+12.2, -12.2, +0.0822, -0.0822\}.$$

We also already mentioned that

$$c_V^e/c_A^e = \rho = 1 - 4\sin^2\theta_W$$

and so

$$\sin^2\theta_W = \frac{1}{4}(1-\rho).$$

Using our four values of ρ we get

$$\sin^2\theta_W \in \{-2.79, 3.29, 0.229, 0.271\}.$$

(TURN OVER

We can discount the first two values as being outside ±1. But the last two are plausible. The data point at the centre of the plot is therefore consistent (at least if analysed using only the Z-diagram and only at leading order) with

$$\sin^2\theta_W \in \{0.229, 0.271\}\,.$$

One of these is not overly inconsistent with the current experimental value of $\sin^2\theta_W \approx 0.23$.

(b) **The Quark Model of the Hadrons:** The Σ^0, Σ^{0*} and Λ^0 baryons all have the same uds flavour, yet each has a different mass from the other. Why is this? Are there any physical (rather than, say, simply notational or conventional) reasons for them to have different masses? [10]

Unseen

Despite the fact that a good answer to this question could be wordy and need not involve calculations, per-se, it is not a 'brief notes'-style question. In particular, there is no section of the lecture notes that a student with a photographic memory can regurgitate and thereby obtain full marks. A student might be able to paste some parts together to achieve a reasonably high mark. But what I am looking for above all is to reward candidates who can find some pithy/simple way summarising in their own words the nuances of what multiplets are, which symmetries are exact, which are approximate, and what that even means. The 'worked answer' below is not 'the right answer; it is just an example of the sorts of issues I hope will be covered. From past experience as an examiner I am aware that many candidates will write much better answers than that provided here, or may find very different ways of conveying the same information that are also able to gain full credit. Above all, the intention when marking will be to reward candidates who are able to convince the marker that they understand the answer to the question at a level beyond photographic recall of notes.

Messages I would like to see conveyed could include:

- That flavour symmetry is apparently broken by both mass (e.g. $m_d \neq m_u < m_c < m_s$ etc.) and electric charge ($Q_u \neq Q_d$, etc).
- But that if an n-colour flavour symmetry were **not** broken, then the hadrons (collections of quarks) would group together into $SU(n)$-multiplets for which EVERY element of a given multiplet would be identical with respect to external properties (mass, spin, orbital angular momentum, parity, etc) even if those external properties could be different in other $SU(n)$-multiplets.
- To underline the above point, the difference between the elements of a SU(n)-flavour-multiplet would be no more than that signifying the difference between a spin-up or a spin-down electron (both members of an SU(2)-spin doublet). By this we mean that spin-up can be turned into spin-down by a mere rotation of co-ordinate axes, thus showing that (in a universe with rotational symmetry) there is no fundamental difference between spin-up and spin down. And in the same way any element of an $SU(n)$-flavour-multiplet (were it a true symmetry) could be turned into any other by a redefinition of the flavour basis.

- In that sense, the difference between the Σ^0, Σ^{0*} and Λ^0 baryons could be approached by our asking "Are they all in the same SU(3)-flavour-multiplet? Or are they in different ones?" This is a useful question to ask, because if are they in different multiplets, they can have very different masses or spins purely because they are put together with different spin and flavour wave-functions. In contrast, those which are in the same flavour multiplet as each other would be literally identical, were it not for other effects we will come to.
- It is not expected that candidates will be able to name and place every baryon by name onto the appropriate multiplet, so it is not expected that candidates will instantly recognise the spin and flavour wavefunctions of the Σ^0, Σ^{0*} and Λ^0. However, it can be expected that candidates for the exam can spot that uds is at the CENTRE of any SU(3)-flavour multiplet, and it can be expected that they will recall (or will be able to work out on a scrap of paper) that three-quark (i.e. baryon) SU(3)-flavour multiplets constructed from $3 \times 3 \times 3$ decompose into a symmetric decuplet, two mixed-symmetry octets and a singlet state ($3^3 = 10 + 8 + 8 + 1$) and that since the SU(2) spin wavefunctions from $2 \times 2 \times 2$ decompose into a symmetric quadruplet and two mixed symmetry doublets ($2^3 = 4 + 2 + 2$) so that the only combinations yielding the correct overall antisymmetry between quarks are an $S = 3/2$-decuplet and an $S = 1/2$-octet.
- Given the above, and knowing that octets have two states at their centre (more on that later) they should know or be able to deduce that, between them, two of the baryons in the set $B = \{\Sigma^0, \Sigma^{0*}, \Lambda^0\}$ are in the octet, whereas one is in the decuplet.
- As examiner I don't really mind which one they call which. Names are just names, after all, and little meaning is directly attached to them. However, the candidate who notices that super-script $*$ is often often attached to excited states might well assume that the two Σ baryons cannot be in the same multiplet, and this means that Λ^0 has to be in a multiplet with one of the other two, and so the Λ^0 has to be in an octet. Bonus marks to anyone who notices this perhaps?
- Similarly, 'hats off' to anyone who spotted (when revising the baryons part of the course) that baryons in a given row of the (S=3/2) decuplet are always heavier than the baryons in the corresonding row of the (S=1/2) octet. If they did spot this they could have reasonably intuited this as being consistent with the idea that it's harder to trap three magnets together with their north poles all pointing upwards and south poles all down (repelling each other) than to place one magnet anti-parallel to the other two, and so the higher internal energy of the $S = 3/2$ naturally makes them heaver and more excited than the others. Someone vaguely recalling that could (reasonably) put this fact together with that in the last bullet point to deduce (correctly) that the excited Σ^{0*} is the one on his own in the $S = 3/2$-decuplet while it is the Λ^0 and Σ^0 which live together in the octet.
- This means that the biggest difference between the states in B is that the Σ^* has $S = 3/2$ and therefore is probably heavier than both the other two as a result of its magnetic moments being parallel. Some people may just have remembered that photographically [the lectures notes said $m_{\Sigma^{0*}} = 1318$ GeV, while $m_{\Sigma^0} = 1193$ MeV and $m_{\Lambda^0} = 1116$ MeV.] but more credit can be given to those who can explain where all this is coming from.

(TURN OVER

- Arguably the hardest remaining part of the question is to then explain what the difference is between the Λ^0 and Σ^0 which live together in the sane octet. These two have, after all, the same spin and parity as each other, and (as we've already said) would be IDENTICAL to each other were it not for SU(3)-flavour being a broken symmetry. The 'trite' answer to this might be to say:

  > As one uses the T_- isospin ladder operator to traverse step-wise across the middle row of the baryon octet from the right-most state (uus or Σ^+) to the left-most sate (dds or Σ^-) one will inevitably see each member of an isospin triplet of states as the $uu \to \frac{1}{\sqrt{2}}(ud + du) \to dd$ or $(I_3 = 1) \to (I_3 = 0) \to (I_3 = -1)$, and so there should be a state in the middle which is the central member of an isospin-triplet (this is the state Σ^0. Therefore there must be a different state (which we might call the Λ^0) which is in an isospin-singlet state.

  The reason that this is a 'trite' or cop-out answer is that: (i) it does not tell us why we should care any more or any less about the $T_\pm$ operators than about the $V_\pm$ or $U_\pm$ ladder operators, and (ii) even if we did see a reason to view $T_\pm$ as special[4] we would still not really have provided a satisfactory answer to the question of: 'In what observable way is an iso-singlet state any different from an iso-triplet state?'. It is to flesh out an answer to the above that the candidates will have to put their answer together with information from other parts of the course.

- What I hope that some students will notice is that, in truth, the real difference between the Λ^0 and Σ^0 cannot be worked out, *ab initio* from arguments dealing only with ladder operators[5] Instead I hope they will connect this part of the course to what they saw with the K^0 and $\bar{K}^0$ in the CP-violation part of the course. There they saw that the K^0 and the $\bar{K}^0$ could change into one another, and that (a priori) the linear combinations of them that had well defined masses were related to what decays each was able to undergo. In that case the things with masses (K_L and K_S were very closely related to states with given CP (K_1 and K_2) since decays with some CP values were harder than others. In the same way, what *actually* separates Σ^0 and Λ^0 is their masses, and what drives their masses to (subtly) different values will depend on what sorts of decays things with a uds flavour content can decay to ...
- Although world describes Λ^0 as the iso-singlet and Σ^0 the iso-triplet ... really we should just give a name to the iso-singlet (call it λ^0) and a name to the iso-triplet (call it σ^0) and then we should just accept that the PHYSICAL states with well defined masses (Λ^0 and Σ^0) are just some as-yet undetermined orthogonal linear combinations of λ^0 and σ^0. Critically we can say that (at this point) we do know what the flavour wavefunctions for λ^0 and σ^0 are, even if we do not know what the flavour wavefunctions for Λ^0 and Σ^0 are.

[4] A prime candidate for making $T_\pm$ special is that the u and d quarks are by far the lightest, and so as far as mass is concerned could be viewed as more similar to each other than to any other quarks, making T the 'best' symmetry. But the Devlis Advocate could complain that, nonetheless, $Q_d = Q_s = -2Q_u$ which would give a special role to the ladder operators ($U_\pm$) interchanging d and s quarks.

[5] If simple ladder operator arguments were good enough, then the three flavour wave functions given in the lecture course for the three mesons at the centre of the pseudoscalar nonet would not looks so different from the flavour wave functions given for the three mesons at the centre of the vector nonet!

- We do not expect the candidates to know the different decay modes of the Σ^0 and Λ^0, but they should be able to see that uds decays to lighter final states would (angular momentum conservation and and parity conservation permitting, etc) either: (i) involve changing an s to a u by emission of W^- leading to either $uds \to udue^-\bar{\nu}_e \to p + e^-\bar{\nu}_e$ or $uds \to udu\bar{u}d \to ((p+\pi^-)$ or $(n+\pi^0))$, or (ii) would simply involve a neutral emission like $uds \to uds + \gamma$.
- We know that it CANNOT be the case that only decays of $uds \to uds + \gamma$ are possible (for both λ^0 and σ^0) since if that were the case then a uds in the ground state (i.e. after radiation of a sufficiently large number of photons) could not decay, contradicting our knowledge that the only stable baryons are neutrons and protons.
- Accordingly: at least one of the λ^0 and σ^0 must be able to decay to 'a proton or a neutron and something else', and since the proton and neutron have well defined isospin wavefunction, it will inevitably be the case that one of σ^0 or λ^0 (which have different isospin wavefunctions, by definition!) will find it easier to perform such decays than will the other. Accordingly this difference in decay preference will drive the difference in identity between Λ^0 and Σ^0 in a similar way to the way that the different decay modes of the K_1 and K_2 lead to different masses and mixings for K_L and K_S.
- Put another way, if it were the case that neither the Λ^0 nor the Σ^0 could decay[6], then presumably we would not really be able to distinguish the Λ^0 and the Σ^0 in any meaningful way other than simply 'by definition'.
- The shorter but less helpful answer which summarises all the above (and may be all that many students actually will write, and could therefore still be enough for them to get full marks) would be to say that:

  The lightest baryons (protons and neutrons) are states with definite isospin. The two states in the centre of a baryon octet will have different isospin wavefunctions from each other since they will be orthogonal linear combinations of iso-singlet and iso-triplet states. Accordingly, decays from one state to 'protons or neutrons and other stuff' will be easier for the other, and this will break the symmetry between the two. Experiment has identified the Λ^0 as an iso-singlet state and the Σ^0 as an iso-triuplet state. The concrete experimental manifestation of this difference, which is not taught in the course, and which is not expected to be known by the examinees (and so is merely documented here 'for fun') is that: (i) the Λ^0 has a greater than 99% branching ratio to proton-pion, while (ii) the Σ^0 has a 100% branching ratio to $\Lambda^0 + \gamma$.
- Since that was a very long answer, we repeat here that a good candidate does not need to give anywhere near the amount of verbiage provided above, but the more that their answer conveys the idea that they are thinking about what makes (or could make!) two states at the centre of a baryon octet different from each other as a result of more than just 'definition' then the more marks the student will get!

[6] E.g. if you removed the Weak interaction from the SM you could no longer change flavours, and if you removed QED too (leaving just QCD), you could no longer emit (non-virtual) neutral gauge bosons. Emission of virtual gluons going to $q\bar{q}$-mesons would also not work as the π^0 is too heavy. In that environment a uds-baryon in the octet could no longer decay!

(TURN OVER

This answer has not explored the possibility that some candidates might choose to write about something altogether different, like baryon magnetic moments.[7]

[7] Baryon magnetic moments are not an examinable part of the course, but it could be the case (not checked) that the a successful argument could be built on top of them. As ever, any valid answers will be accepted.

3 The 'Zarquon' is a hypothetical particle of mass $M > 0$ composed of indestructable 'Zarks'. There are two types of Zark: the 'Fat Zark' and the 'Ferret Zark', with respective masses of $\chi/2$ and $\chi/4$. Every Zarquon contains exactly one Fat Zark and two Ferret Zarks (and nothing else). It is known that if, at some moment in time, a Fat Zark has a three-momentum $\boldsymbol{k}$ in the rest frame of its Zarquon, then at the same moment in time each of the Ferret Zarks has a three-momentum $-\frac{1}{2}\boldsymbol{k}$ in the same frame. Zarquons cannot be polarised, so all directions for $\boldsymbol{k}$ are equally likely.

(a) Write down $|\boldsymbol{k}|$ in terms of M and χ, and then determine the range of values the parameter χ could take (given a value of M). [3]

Bookwork

To find $|\boldsymbol{k}|$ we need to solve the energy-conservation equation for the Zarquon (i.e. that its mass is the sum of the energies of its components in the rest frame) i.e.:

$$2\sqrt{\left(\frac{\chi}{4}\right)^2 + \left(\frac{k}{2}\right)^2} + \sqrt{\left(\frac{\chi}{2}\right)^2 + k^2} = M.$$

The above is the same as

$$\sqrt{\left(\frac{\chi}{2}\right)^2 + k^2} + \sqrt{\left(\frac{\chi}{2}\right)^2 + k^2} = M$$

or

$$\sqrt{\left(\frac{\chi}{2}\right)^2 + k^2} = \frac{M}{2}$$

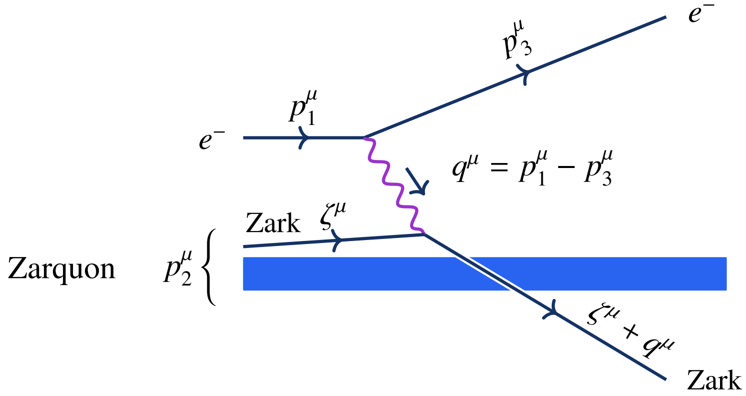
or

$$k^2 = \left(\frac{M}{2}\right)^2 - \left(\frac{\chi}{2}\right)^2.$$

The above shows that $\chi \in [0, M]$ and

$$|\boldsymbol{k}| = \frac{1}{2}\sqrt{M^2 - \chi^2}.$$

It is planned to investigate the Zark content of the Zarquon by a series of fixed-target deep inelastic scattering experiments in which a beam of electrons is fired at a Zarquon target as shown:



(TURN OVER

The probe electron has four-momentum p_1^μ when incoming and p_3^μ when outgoing. The Zarquon has initial four-momentum p_2^μ. The struck Zark has momentum ζ^μ before and $\zeta^\mu + q^\mu$ after the interaction. The rest masses of the Zark and electron are unaffected by their interaction. Assume that the electron mass can be neglected and in the lab frame the momenta p_1^μ, p_2^μ, p_3^μ and ζ^μ take the form:

$$p_1^\mu = \begin{pmatrix} p \\ p \\ 0 \\ 0 \end{pmatrix}, \qquad p_2^\mu = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad p_3^\mu = \begin{pmatrix} E \\ E\cos\theta \\ 0 \\ E\sin\theta \end{pmatrix} \quad \text{and} \quad \zeta^\mu = \begin{pmatrix} \sqrt{m^2+a^2} \\ -a\cos\alpha \\ a\sin\alpha\cos\delta \\ a\sin\alpha\sin\delta \end{pmatrix}$$

where $p > 0$, $a \geq 0$, $0 \leq \alpha \leq \pi$, $0 \leq \delta < 2\pi$, $0 \leq \theta \leq \pi$ and $E \geq 0$. The value m will be equal to either $\chi/2$ or $\chi/4$ depending on whether the struck object was a Fat Zark or a Ferret Zark.

(b) How will the value of the parameter a depend on whether the struck object was a Fat Zark or a Ferret Zark? What geometrical interpretation can be given to the quantities α and δ ? How is α distributed? [4]

Bookwork

BOOKWORK[Testing recall of spherical polar coordinartes – α being the polar angle and δ being the azimuthal angle. It should be recalled that $\cos\alpha$ will be distributed uniformly between -1 and +1 given the isotropic distribution of $\boldsymbol{k}$ reported at the beginning of the question.]

The answer should report that a is equal to $|\boldsymbol{k}|$ for a Fat Zark, but is equal to $|\boldsymbol{k}|/2$ for a Ferret Zark.

For the scattering process described, the quantities in the set S

$$S = \{p, M, m, a, \alpha, \delta, \theta\}$$

are not independent of E.

(c) Write down (but do not yet solve) an equation which, if solved, would fix E in terms of the quantities in S. Explain the physical meaning of this constraint. [2]

Straightforward

The question says that the masses of the Zark and the probe electron are not changed by their being struck. This is self-evident for the latter, as both p_1^μ and p_3^μ have zero invariant mass. The invariant mass of $\zeta^\mu + q^\mu$ is not, however, as parameterised, necessarily still equal to its value before being struck (m). The constraint that could be solved therefore is (in words) 'mass of Zark after being struck = m = mass of Zark before being struck'. Algebraically this could be written as:

$$(\zeta + q)^2 = m^2$$

or, eliminating q to gain explicit E dependence could be written

$$(\zeta + p_1 - p_3)^2 = m^2.$$

Full marks could be obtained by saying the above in words and writing down the equation above in some form.

(d) By solving the constraint just written down (or otherwise) find an expression for E in terms of the quantities in S. [4]

Straightforward

Multiplying out the square, and noting that $\zeta^2 = m^2$ and $p_1^2 = p_3^2 = 0$, the constraint could also be written as

$$\zeta.p_1 = \zeta.p_3 + p_1.p_3.$$

This RHS of the above is linear in p_3 and may be solved for E. Specifically: $p_3^\mu = Ek^\mu$ where

$$k^\mu = \begin{pmatrix} 1 \\ \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$$

and so

$$\zeta.p_1 = Ek.(\zeta - p_1)$$

and hence

$$E = \frac{\zeta.p_1}{k.(\zeta + p_1)}.$$

Putting in the components gives

$$E = \frac{p\left(\sqrt{m^2 + a^2} + a\cos\alpha\right)}{p(1 - \cos\theta) + \sqrt{m^2 + a^2} + a\cos\theta\cos\alpha - a\sin\theta\sin\alpha\sin\delta}. \qquad (2)$$

(TURN OVER

(e) Suggest two physical reasons why an experiment might be unable to make measurements all the way out to $\cos\theta = \pm 1$. [4]

Straightforward

It is impossible to have detection equipment all the way down to $\cos\theta = -1$ as these detectors would clash with the beam injection hardware. Making measurements at $\cos\theta = +1$ could be hard because the unscattered beam (and/or secondaries from the input beam line) could form intense sources of background radiation.

In the rest of this question you may assume that the scattering experiments are conducted only in the regime in which $|\cos\theta| \le 0.9$ and $10M \ll p$ and that you may simplify expressions by neglecting terms accordingly.

(f) Explain why it is the case that, in the scattering regime just described, your answer for E obtained in part (d) simplifies to

$$E \approx \frac{\sqrt{m^2 + a^2} + a\cos\alpha}{1 - \cos\theta}.$$ [4]

Unseen

If $|\cos\theta| \le 0.9$ and $10M \ll p$ then $M \ll p(1 - \cos\theta)$. However, from earlier parts of the question we also know that $m \le \chi/2 \le \chi \le M$ and $a \le |\boldsymbol{k}| \le M/2 < M$. We are therefore able to deduce that $a \ll p(1 - \cos\theta)$ and $m \ll p(1 - \cos\theta)$. These two results allow us to lose all the terms on the denominator of our original E expression (equation (2)) other than the leading $p(1 - \cos\theta)$ term.

The 'Bjorken x' variable is *defined* by the equation $x = -q^2/(2p_2 \cdot q)$.

(g) To the level of approximation permitted in this scattering regime, show that the 'Bjorken x' variable satisfies

$$x \approx \frac{\sqrt{m^2 + a^2} + a\cos\alpha}{M}.$$ [4]

Unseen

$$\begin{aligned}
x &= \frac{-(p_1 - p_3)^2}{2p_2 \cdot (p_1 - p_3)} && \text{by definition} \\
&= \frac{-p_1^2 - p_2^2 + 2p_1 \cdot p_3}{2p_2 \cdot (p_1 - p_3)} \\
&= \frac{p_1 \cdot p_3}{p_2 \cdot (p_1 - p_3)} && \text{as } m_e = 0 \\
&= \frac{pE(1 - \cos\theta)}{M(p - E)} \\
&= \frac{E(1 - \cos\theta)}{M} \left(1 - \frac{E}{p}\right)^{-1}
\end{aligned}$$

but the result of part (f) shows us that

$$\frac{E}{p} \approx \frac{\sqrt{m^2 + a^2} + a\cos\alpha}{p(1 - \cos\theta)}$$

while working in (f) noted that $a \ll p(1 - \cos\theta)$ and $m \ll p(1 - \cos\theta)$. We therefore see that

$$\frac{E}{p} \ll 1$$

and so we may drop the $\left(1 - \frac{E}{p}\right)^{-1}$ term in our expression for x:

$$\begin{aligned}
x &= \frac{E(1 - \cos\theta)}{M} \left(1 - \frac{E}{p}\right)^{-1} \\
&\approx \frac{E(1 - \cos\theta)}{M} \\
&\approx \frac{\sqrt{m^2 + a^2} + a\cos\alpha}{M} && \text{using the answer to (f) again.}
\end{aligned}$$

(h) Determine and then sketch the shape of the parton distribution function $F(x)$ of the Fat Zark. Make sure to show how the key features of $F(x)$ depend on χ and M. [2]

Unseen

In the answer to part (b) the candidates should have stated that $\cos\alpha$ is distributed uniformly in $[-1, 1]$. Everything else in x is constant (for a given type of Zark). Therefore x is uniformly distributed between its max and min values, i.e. the values it would take when $\cos\alpha = \pm 1$. This means that the shape of $F(x)$ is a top-hat function. The width of this top hat will generically be wid $= 2a/M$. For the Fat Zark we have $a = |\boldsymbol{k}| = \frac{1}{2}\sqrt{M^2 - \chi^2}$ from an earlier part of the question, and hence the width of the Fat Zark's top hat is $\sqrt{1 - \chi^2/M^2}$ which we will call L:

$$L = \sqrt{1 - \chi^2/M^2}$$

(TURN OVER

The centre x_{mid} of the top-hat is generically at $x_{\text{mid}} = \sqrt{m^2 + a^2}/M$. For the Fat Zark this is specifically at $x_{\text{mid}}^{\text{Fat}} = \sqrt{(\frac{\chi}{2})^2 + \frac{1}{4}(M^2 - \chi^2)}/M = \frac{1}{2}$.

The height of the top-hat should be the reciprocal of its width, since the integral $\int f(x)\mathrm{d}x$ should equal 1 as there is only one Fat Zark in each Zarquon.

In summary: $F(x)$ is a top hat of width L and height $1/L$ centred on $x = 1/2$.

(i) Determine and then sketch the shape of the parton distribution function $f(x)$ of the Ferret Zark. Make sure to show how the key features of $f(x)$ depend on χ and M. [2]

Unseen

The Ferret Zark is similar to the Fat Zark, the only differences are that (i) its a and its m are half those of the Ferret Zark, and (ii) there are two of them instead of one per Zarquon. Since the generic top-hat width and location expressions given in the last answer (wid $= 2a/M$ and $x_{\text{mid}} = \sqrt{m^2 + a^2}/K$) scale linearly with coupled changes to m and a, the PDF for the Ferret Zark is still a top hat but its width is half what it was previously and it will be centred on $1/4$ instead of on $1/3$. The area must this time be 2 however (since there are two Ferret Zarks per Zarquon) and so the height must be four times what it was previously. In summary:

$f(x)$ is a top-hat centred on $1/4$ with width $L/2$ and height $4/L$.

(j) What value would you expect the integral $\int_0^1 (xF(x) + xf(x))\,\mathrm{d}x$ to take and why? [You are not required to evaluate the above integral explicitly.] [1]

Straightforward

The expression $\int_0^1 xF(x)\mathrm{d}x$ should evaluate total fraction of the momentum of the Zarquon that is carried by the Fat Zarks, in the infinite momentum frame. Likewise the expression $\int_0^1 xf(x)\mathrm{d}x$ should evaluate total fraction of the momentum of the Zarquon that is carried by the Ferret Zarks, in the infinite momentum frame. Since they are the only things in the Zarquon, their total should sum to 1. The intention of asking this question is that a student will realise (if they have not already done so) that they can use this as a check that their answer to the preceding part(s) is correct. I.e. a candidate who wishes to make a check can evaluate this integral explicitly and use any deviation from 1 to find and fix numerical errors or slips they have made earlier. The integral is not hard to do explicitly:

$$
\begin{aligned}
\int_0^1 xF(x) + .xf(x)\mathrm{d}x &= \int_{\frac{1}{2}(1-L)}^{\frac{1}{2}(1+L)} x\frac{1}{L}\mathrm{d}x + \int_{\frac{1}{4}(1-L)}^{\frac{1}{4}(1+L)} x\frac{4}{L}\mathrm{d}x \\
&= \frac{1}{2L}\left[x^2\right]_{\frac{1}{2}(1-L)}^{\frac{1}{2}(1+L)} + \frac{2}{L}\left[x^2\right]_{\frac{1}{4}(1-L)}^{\frac{1}{4}(1+L)} \\
&= \frac{1}{8L}\left[x^2\right]_{(1-L)}^{(1+L)} + \frac{1}{8L}\left[x^2\right]_{(1-L)}^{(1+L)} \\
&= \frac{1}{4L}\left[x^2\right]_{(1-L)}^{(1+L)} \\
&= \frac{1}{4L}(4L) \\
&= 1.
\end{aligned}
$$

END OF PAPER