UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics
NATURAL SCIENCES TRIPOS: Part III Astrophysics
MASTER OF ADVANCED STUDY IN PHYSICS
MASTER OF ADVANCED STUDY IN ASTROPHYSICS

Tuesday 14th January 2020: 14:00 to 16:00

MAJOR TOPICS
Paper 1/PP (Particle Physics)

*Answer **two** questions only. The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate. The paper contains four sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

*You should use a **separate Answer Book** for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2x20-page answer books	Mathematical Formulae Handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

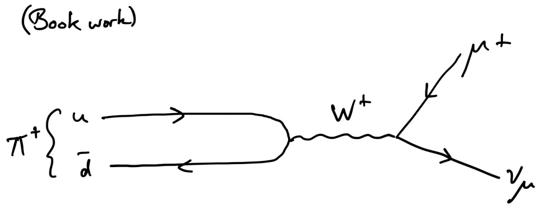
(7th January 2020)

1 Consider the decay $\pi^+ \to \mu^+\nu_\mu$ where the π^+ has quark content $u\bar{d}$.

This is a partial re-use of a question I created in 2014 for the 2015 examination. It is a very 'standard' question in the sense that the $\pi^+ \to \mu^+\nu_\mu$ decay is the 'classic' one used in both the lectures and the examples sheets to illustrate the weak interaction charged current vertex and direct consequences of its preference for left chiral partices and right chiral anti=particles. Much of this question is thereofore bookwork, the exceptions being much of (c) and a lot of (e).

(a) Draw a Feynman diagram for this decay. [3]

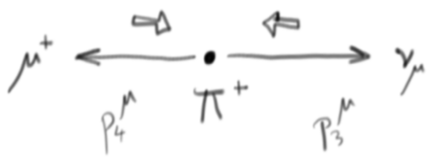
(Book work)



(b) Draw a diagram showing the momentum and spin directions of the outgoing particles in the centre-of-mass frame, explaining clearly the reasons for your choice of spin state. [3]

Bookwork

The π^+ is a spin-0 meson, si in its rest frame we have a equal and opposite muon and neutrino momenta, and they must have equal and opposite helicities. The neutrino is a a particle, and so is produced in a left handed (LH) chiral state by the W-boson. As an effectively massless particle, the neutrino's LH chiral state is co-incident with a LH helicity state and therefore to conserve total spin the anti-muon must also be in a LH helicity state:



(c) The lepton current for the final state can be written as

$$\bar{u}(p_3)\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right)v(p_4)$$

where p_3 is the four-momentum of the ν_μ and p_4 is that of the μ^+. Forms for the γ-matrices and spinors can be found at the end of the question. Show that the magnitude of the lepton current is proportional to

$$\frac{(E+m-p)\sqrt{p}}{\sqrt{E+m}}$$

where E, m and p are the energy, the mass and the magnitude of the three-momentum of the μ^+. You are not required to compute the components of the current. [6]

Unseen in this form, though similar to lectures

$$J^\mu = \bar{u}_\downarrow(p_3)\gamma^\mu \frac{1}{2}(1-\gamma^5)v_\downarrow(p_4) \tag{1}$$

By aligning the z-axis with the neutrino direction, we can take $\theta = 0$ for the neutrino, $\theta = \pi$ for the anti-muon, and $\phi = 0$ for both. Accordingly:

$$v_\downarrow(p_4) = \sqrt{E+m}\begin{pmatrix} 0 \\ \alpha \\ 0 \\ 1 \end{pmatrix}$$

and

$$u_\downarrow(p_3) = \sqrt{p}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

in which p is the magnitude of the three momentum of either of the daughters of the pion in its rest frame, m is the muon mass, $E = \sqrt{m^2 + p^2}$, and $\alpha = \frac{p}{E+m}$.

Using the supplied gamma matrices, the students can compute that

$$1 - \gamma^5 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

and hence

$$(1-\gamma^5)v_\downarrow(p_4) = \sqrt{(E+m)}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

$$= \sqrt{E+m}\begin{pmatrix} 0 \\ \alpha - 1 \\ 0 \\ -\alpha + 1 \end{pmatrix} \tag{3}$$

$$\propto \sqrt{E+m}(1-\alpha). \tag{4}$$

Therefore

$$J^\mu \propto \sqrt{p}\sqrt{E+m}(1-\alpha). \tag{5}$$

We therefore see that

$$J^\mu \propto \sqrt{p}\sqrt{E+m}(1-\alpha) \tag{6}$$

$$= \sqrt{p}\sqrt{E+m}(1 - \frac{p}{E+m}) \tag{7}$$

$$= \sqrt{p}\sqrt{E+m}\frac{E+m-p}{E+m} \tag{8}$$

$$= \frac{\sqrt{p}(E+m-p)}{\sqrt{E+m}} \tag{9}$$

as required.

(d) Given that the two-body decay rate is

$$\Gamma = \frac{p}{8\pi m_\pi^2}\left\langle |M|^2 \right\rangle,$$

where m_π is the mass of the pion and M is the matrix element for the decay, estimate the ratio

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)}.$$

You may assume that the momentum of the muon emitted in the pion decay is 30 MeV while that of the electron is 70 MeV. Their respective masses are 106 MeV and 0.511 MeV. [6]

(7th January 2020)

Unseen in this form, but similar to lectures:

From the previous result,

$$(J^\mu \pi_\mu)^2 \propto \frac{p(E+m-p)^2}{E+m} \tag{10}$$
$$= \frac{p(E^2+m^2+p^2+2Em-2Ep-2mp)}{E+m} \tag{11}$$
$$= \frac{p(2E^2+2Em-2Ep-2mp)}{E+m} \tag{12}$$
$$= \frac{2p(E-p)(E+m)}{E+m} \tag{13}$$
$$\propto p(E-p). \tag{14}$$

Using the supplied two-body decay-rate together with the result just proved,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \left(\frac{p_e}{p_\mu}\right)\frac{p_e(E_e-p_e)}{p_\mu(E_\mu-p_\mu)} \tag{15}$$

which works out to be about 1.273×10^{-4} using the data supplied in the question.

(e) Explain how measurements of the decay

$$^{60}\mathrm{Co} \to {}^{60}\mathrm{Ni}^\star + e^- + \bar{\nu}_e$$

can be used to show that the laws of nature are not symmetric under parity. [8]

Bookwork:

Here it is expected that an answer will include a description of Madame Wu's experiment – including a diagram, the cobalt spin alignment, the preference for electron emission antiparallel to the magnetic field, the parity odd nature of the emitted electron momentum (vector) contrasted with the parity even nature of the magnetic field direction (pseudovector). It will note that this experiment unambiguously determined that this process did not respect parity as a symmetry of nature, since the experimental data observed (electrons departing preferentially antiparallel to the spin direction) would not have been invariant under a parity transformation on a virtual representation of the experiment.

(f) To what extent do forward-backward asymmetries measured at LEP test for the presence (or absence) of parity as a symmetry of the Standard Model? [4]

Answers will describe the forward backward asymmetry of the Z-boson at LEP. A first class answer should conclude by noting that, though the forward-backward asymmetry of the Z is a consequence of the parity volation in the weak interation, the forward backward asymmetry of the Z-boson does not, in itself, show that the Standard Model violates parity. This is because a parity inversion on LEP would result in the forward direction being mapped to the forward direction, and the backward direction being mapped to the backward direction (since forward means μ^- goes in same direction that e^- was going, and both of these directions invert themselves under parity). The FB asmmetry, therefore, would be invariant under a parity transofmration, and so provides no direct evidence for parity violation.

[*The gamma matrix and spinor conventions used in the lecture course were:*

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$u_\uparrow = N \begin{pmatrix} \hat{c} \\ e^{i\phi}\hat{s} \\ \alpha\hat{c} \\ \alpha e^{i\phi}\hat{s} \end{pmatrix}, u_\downarrow = N \begin{pmatrix} -\hat{s} \\ e^{i\phi}\hat{c} \\ \alpha\hat{s} \\ -\alpha e^{i\phi}\hat{c} \end{pmatrix}, v_\uparrow = N \begin{pmatrix} \alpha\hat{s} \\ -\alpha e^{i\phi}\hat{c} \\ -\hat{s} \\ e^{i\phi}\hat{c} \end{pmatrix}, v_\downarrow = N \begin{pmatrix} \alpha\hat{c} \\ \alpha e^{i\phi}\hat{s} \\ \hat{c} \\ e^{i\phi}\hat{s} \end{pmatrix}$$

where $N = \sqrt{E+m}$, $\hat{c} = \cos\left(\frac{\theta}{2}\right)$, $\hat{s} = \sin\left(\frac{\theta}{2}\right)$, $\alpha = \frac{|\boldsymbol{p}|}{E+m}$, *and where* θ *and* ϕ *are the usual spherical angles (polar and azimuthal respectively) and where* E, $\boldsymbol{p}$ *and* m *are the energy, momentum and mass of the particle (or antiparticle).*]

2 Suppose that after Brexit the UK takes back control of the laws of physics from Europe by abolishing the use of the colour green in QCD. This change makes post-Brexit QCD a two-colour $SU(2)$ symmetry based on red (not white) and blue, rather than on the three-colour $SU(3)$ symmetry which the UK used throughout its membership of the European Union.

(a) Determine which 'Brexit mesons' and 'Brexit baryons' (or their nearest equivalents) could exist by constructing any important colour, flavour and spin wave-functions. Categorise the expected 'Brexit hadrons' by type (meson/baryon), spin, and the multiplets they inhabit. Compare the main similarities and differences between the pre- and post-Brexit structure of hadrons. [*Each of the three instructions above (there is one per sentence) carries approximately eight marks. It is not necessary to address the issues in the order listed. You need only consider light quark types: u, d and s.*] [24]

(b) Are any four or five quark-and/or-antiquark states allowed by the pre-Brexit Standard Model? What sort of four or five quark-and/or-antiquark states could be seen after Brexit? [6]

This question (based on a similar one with a different part (b) written in Nov 2015) tries to test candidates understanding of the arguments made in their notes in the meson and baryon part of the lecture course. Mere photographic recall of the lecture notes (without understanding) will not help the candidate, but a candidate who is able to recall the kinds of things said, and can make reasonable educated guesses about how to adapt them from $SU(3)$ colour to $SU(2)$ colour will be rewarded based on the clarity, completeness, and level of understanding of the $SU(3)$ colour theory demonstrated in the nature of their answer(s). Since some parts of $SU(3)$ QCD are not fully understood (e.g. colour confinement, and much of the non-perturbative part of the theory) it would be possible for two equally good candidates to come up with mutually incompatible answers that could both be, on physical grounds, plausible. In that sense there cannot be any 'model' answer, and for this reason marking will always give credit where it is due, even if there is not conformity to the suggested form of the answer below.

BOOKWORK[The bookwork components of this question consist of all the places where the candidate can legitimately bring in a description of the processes used to derive/describe hadron structure in our own three-colour universe to motivate a generalisation to the two-colour case. There are many such places.]

This question provides many more discursive opportunities than the other questions in the paper, yet good answers will still necessarily involve a bit of serious mathematical argumentation. The completely new section (b) asks for specific information about multiquark states, given that pentaquark discoveries from LHCb have solidified in recent years, and attention was specifically drawn to them in the lectures. This year therefore seemed to be a good idea to draw attention to them.

The question deliberately avoids breaking (a) up into three parts (i), (ii) and (iii) with 8 marks for each, since to do so would almost certainly cause every student to tackle (i) first, then (ii), then (iii) in an effort to get the marks in each section. This would heavily constrain the way they generate their arguments in an undesirable way. For most of this question there are links between the mesons and the baryons that make it simpler to give the students freedom to advance their ideas in whichever ways flow most naturally to them, whether that be by considering (say) colour wavefunctions before flavour, or vice versa. Nontheless, an approximate allocation of marks with the 24 as 8+8+8 is given in the question's hint, so that there are nonetheless some boundaries set. A second reason that (a) remains as one undivided 24 mark unit is becase a more prescriptive mark scheme would start to remove some parts of the problem altogether: for example, I see a significant part of the problem is actually identifying that there even are Brexit baryons at all. The Brexit measons' existence is not too difficult to show as they are almost the same as those in our own universe. However the Brexit baryons no longer have three quarks – and there is some effort in finding that. If the question starts to break things down into a tiny mark here and a tiny one there, it is in great danger of becoming (in my view) overly prescritive and gives the game away of how much of each thing there is to find. The question is (I hope) clear in its lists of what is required to be delivered. Thuogh a mark scheme has been provided, mark schemes always need to be adapted in the light of what the candidates write – and indeed our rubrics make clear that mark-schemes are only ever approximate. The overriding principle when marking this sort of question will be to reward appropriate demonstrations of relevant knowledge and understanding displayed by each candidate.

(a) A key fact the candidate should bring to the table here is that the SU(2) colour theory will require a

$$\frac{1}{\sqrt{2}}\left(r\bar{r}+b\bar{b}\right)$$

equivalent of the SU(3)

$$\frac{1}{\sqrt{3}}\left(r\bar{r}+g\bar{g}+b\bar{b}\right)$$

colour-anticolour singlet thereby permitting mesons to exist for most of the same reasons they can in the real universe. A poor answer would omit this altogether. A medium answer would mention it without proof merely appealing to its plausibility and connection to colour confinement hypothesis. A good answer might demonstrate that this really is a singlet by consideration of the action of properly defined ladder operators on it, etc. It might even go on to question whether the colour confinement hypothesis would still be important in the Brexit universe.

Answers will hopefully reproduce the potential spin wavefunctions of the ‘real’ mesons, noting those in the ‘Brexit’ universe could be identical.

A good answer would hopefully re-capitulate the flavour part of the notes (that coveres the meson nonets) noting that, as in ‘real’-space, the Brexit universe allows any spin combinations with any flavour combinations since the lack of any identical fermions in the mesons leads no need to have antisymmetry of the overall wavefunction.

The spetra of excited mesonic states would presumably differ in the real universe from that in the Brexit, as the different colour potential would space excitations differently. [8]

For quark-only states (i.e. baryons) the key fact is that the three colour singlet of $SU(3)$

$$\frac{1}{\sqrt{6}}\left(rgb-rbg+gbr-brb+brg-bgr\right)$$

is replaced in the Brexit universe by the

$$\frac{1}{\sqrt{2}}\left(rb-br\right)$$

two-colour singlet of $SU(2)$, meaning that the colour confinement hypothesis (if still needed!) would permit two-quark baryons and forbid three-quark baryons. Again, a poor answer would neglect to mention this at all. A medium answer would just state it. A good answer would argue the case clearly. [8]

(7th January 2020)

The disappearance of one colour would not change the approximate (u,d)-isopin SU(2) flavour or (u,d,s)-isospin SU(3) flavour symmetries available to nature – but the need for only two quark states would require us now to consider only the $3 \otimes 3 = 6 \oplus \bar{3}$ not the $3 \otimes 3 \otimes 3 = 10 + 8 + 8 + 1$ version of before. A good answer would work out that the 6 is symmetric in the two quark flavours, while the $\bar{3}$ is antisymmetric.

What flavour/spin/colour combinations would be allowed? Assuming the lowest angular momentum states would have $L = 0$ making them even parity, and given that the colour singlet is already antisymmetric, we'd need flavour x spin to be symmetric. We would need to combine the 6 with a symmetric $S = 1$ spin-triplet, or the antisymmetric $\bar{3}$ with an antisymmetric $S = 0$ spin-singlet.

The Brexit (u,d,s)-baryons would therefore be expected to come in a $S = 1$ hextet of and a $S = 0$ triplet of di-quark states.

Note that the electric charges of the quarks remained as thirds, then these Brexit baryons would have non-integer charges: the lightest three (uu, ud, dd) having charges $\frac{4}{3}$, $\frac{1}{3}$ and $-\frac{2}{3}$ respectively. This could have important consequences for the stability of atoms and nearly all chemistry – which could present problems for the presevation of chlorinated chicken. If (as one external examiner pointed out) the charges were instead quantised on halves, this would not be the case. Whichever argument the candidate provides could be valid, so long as it makes clear its assumptions. [8]

(b) The possiblilty of $q\bar{q}q\bar{q}$ states would be present in both Brexit and Real universes. But whereas the real universe forbids $qqqq$ and allows $qqqq\bar{q}$ states, the Brexit would allow $qqqq$ and forbid $qqqq\bar{q}$ due to the change in which contains a colour singleton. A bad answer would not realise the above. A better answer would say that there are 'at least' the above possibilities since qq or $q\bar{q}$ states could be stuck together. The best possible answers would attempt to identify how many $qqqq$ states there are, e.g. by multiplying together the $SU(2)$ doublet states as mentioned in the group-theory part of the course. E.g. the $qqqq$ state could be worked out (post-brexit) from $2 \otimes 2 = 1 \oplus 3$ and thus
$2 \otimes 2 \otimes 2 \otimes 2 = (2 \otimes 2) \otimes (2 \otimes 2) = (1 \oplus 3) \otimes (1 \oplus 3) =$
$(1 \otimes 1) \oplus (1 \otimes 3) \oplus (3 \otimes 1) \oplus (3 \otimes 3) = 1 \oplus 3 \oplus 3 \oplus (1 \oplus 3 \oplus 5) = 1 \oplus 1 \oplus 3 \oplus 3 \oplus 3 \oplus 5$
and so the candidate could note that there are two (different) ways of getting colour singlets in four-quark states. [6]

3

(a) How do cosmic rays form in the upper atmosphere? [2]

This ought to be bookwork. Answers should make clear the difference between the progenitor particles in space and the showers they make in the atmosphere.

(b) Comment on the relative sizes of the sea-level fluxes of down-going (↓) and up-going (↑) atmospheric neutrinos and anti-neutrinos of all flavours. Make clear your assumptions. [6]

A good answer should contain some evidence of understanding that at generation we have approximately $N(\nu_\mu + \bar{\nu}_\mu) = 2N(\nu_e + \bar{\nu}_e)$ as a result of weak pion decay, and that there will be little preference for anti-neutrinos over neutrinos save for small effects caused by differential atmospheric shielding. Production of tau neutrinos (and anti-neutrinos) should be a lot smaller as pions cannot decay into them – though it will happen at some level from rays of sufficiently high energy. The down-going rates should be listed as being largely the same as the production rates, since the thickness of the atmosphere is not large enough for appreciable oscillation, but that the up-going rates will be reduced in muon rates on account of muon neutrinos having converted to taus (which Superkamiokande cannot see).

(c) Describe the Super-Kamiokande experiment and explain which flavours of atmospheric neutrino (and antineutrino) you might expect it to be able to see. Justify your answer with kinematic calculations where necessary. [8]

Bookwork - recapitulation of sections of the lecture notes on SuperK, together with some of the notes on kinematic thresholds. Answer would probably including the following points:

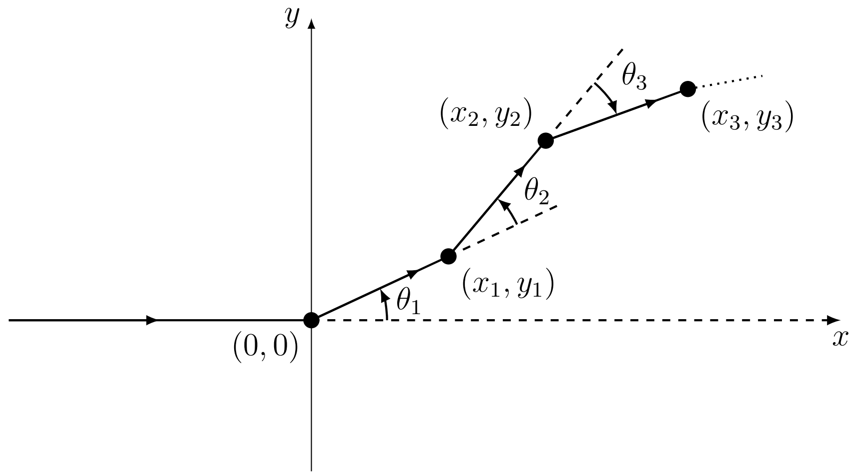
- Water cherenkov, sees light from charged particles going through at faster than speed of light in water.
- Of primary relevance to us are charged leptons from charged current interactions from incident (above kinematic threshold) neutrinos striking electrons in water.
- Conservation of momentum in the lab frame easily shows that (neglecting mass of neutrino) that threshold neutrino energy for $\nu_l e \to l \nu_e$ is $E_{\nu_l} \geq \left(\frac{m_l^2}{m_e^2} - 1\right) \frac{m_e}{2}$.
- The above gives which is $E_{\nu_l} = 0$ for ν_e but 11 GeV and 3090 GeV for ν_μ and ν_τ respectively. The latter two energies do not need to be reported accurately, but it does need to be demonstrated that the canidates are aware that (i) the muon and tau thresholds are well beyond the energies relevant to solar neutrinos, (ii) that GeV scale energies (or energies well above solar neutrinos) are kinematically possible for cosmic rays, relatively easily.
- The mechanism of cosmic ray neutrino prodiction, however, (having been discussed in (b)) should then lead to the candidate noting that the dominant production processes do not produce tau neutrinos in large numbers – that it is primarily electron and muon neutrinos (and anti-neutrinos) that are made. Therefore, the majority of cosmic events that will be **seen** will be of those types.

- SuperK can has directional capability based on the timing of the PM hits and the shape of the Cherenkov light cone where it hits the wall.
- Electron Cherenkov rings are 'cleaner' (less scatter) than muon rings, so there is good particle ID too.

(d) What have Super-Kamiokande atmospheric neutrino results told us about the neutrinos of the Standard Model? [2]

Some set of statements broadly equivalent to $|\delta m^2_{\text{atmos}}| \sim 0.0025eV^2$ and $\sin^2(2\theta_{atmos}) \sim 1$ should be made.

Now consider a high energy cosmic ray muon in a two-dimensional universe which is filled with concrete in the region $x \geq 0$ and contains vacuum elsewhere. The muon starts on the negative x-axis and enters the concrete at the origin where it immediately scatters through a random angle θ_1 as illustrated below. It then travels for a distance δl before scattering again through another random angle θ_2. This process repeats for a total of N scatterings, over which time the muon will have travelled a path-length $D = N\delta l$ in the concrete. The first three scatterings are shown in the diagram. Every angle θ_i may be assumed to be independent of the others, and each may be assumed to be drawn from a Normal Distribution with mean 0 and variance K. You may assume that the individual scattering angles and the net scattering angle are all small ($\ll 1$). The position of the comsic ray after it has travelled a distance δl beyond the n-th scatter is denoted (x_n, y_n).



(e) Show that

$$\text{Var}(y_N) = \alpha \frac{D^3}{\delta l}\left(1 + O\left(\frac{\delta l}{D}\right)\right)$$

for some constant α, which you should determine. [*You may use, without* *proof, the identity* $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$*, and may also assume that for independent random variables* $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$*.*] [6]

$y_N = \delta l \sin(\theta_1) + \delta l \sin(\theta_1+\theta_2) + \delta l \sin(\theta_1+\theta_2+\theta_3) + \cdots$ which by the small angle approximation $\sin\theta \sim \theta$ gives $y_N = \delta l \sum_{k=1}^{N} k\theta_{N+1-k}$ and so $\mathrm{Var}(y_N) = \mathrm{Var}(\delta l \sum_{k=1}^{N} k\theta_{N+1-k}) = \delta l^2 \mathrm{Var}(\sum_{k=1}^{N} k\theta_{N+1-k}) = \delta l^2 \sum_{k=1}^{N} \mathrm{Var}(k\theta_{N+1-k}) = \delta l^2 \sum_{k=1}^{N} k^2 \mathrm{Var}(\theta_{N+1-k}) = \delta l^2 \sum_{k=1}^{N} k^2 K = K\delta l^2 \sum_{k=1}^{N} k^2 = \frac{1}{6}K\delta l^2 N(N+1)(2N+1)$ (using the supplied hint) and so $\mathrm{Var}(y_N) = \frac{1}{3}K\delta l^2 N^3(1+O(1/N)) = \frac{1}{3}K\frac{D^3}{\delta l}(1+O(\frac{\delta l}{D}))$ as required.

(f) Without lengthy calculations, describe how (if at all) you would expect the above answer to change in a three-dimensional universe. [2]

If each scattering still had mean 0 and varance K, then the main difference caused by 3D over 2D is that there is an extra azumuthal angle which is (presumably) distributed uniformly about the axis of propogation of the particle at any one time. On account of this randomly distributed azimuth, the variance of an individual scattering event in (say) the y-direction would be only half of what it was in the 2D case. To get full marks here, however, I would be entirely happy to see the candidates simply state that the variance in any given projection (e.g. y) would be SMALLER (by some order-one factor) on account of the fact that movement now only has (on average) a reduced projection in the y-direction. I will not require them to determine that the reduction factor is precisely one-half! [Were that factor of one half needed, it could be justified as follows: A scatter in 2D has $\mathrm{Var}(y_1) = \mathrm{Var}(\delta l \sin\theta_1) \approx \mathrm{Var}(\delta l\theta_1) = \delta l^2 \mathrm{Var}(\theta_1) = \delta l^2 K$ whereas a scatter in 3D has $\mathrm{Var}(y_1) = \mathrm{Var}(\delta l \sin\theta_1 \sin\phi) \approx \mathrm{Var}((\delta l\theta_1)(\sin\phi))$ which may be shown to equal the desired result $\frac{1}{2}\delta l^2 K$ by using $\mathrm{Var}(XY) = \mathrm{Var}(X)\mathrm{Var}(Y) + \mathrm{Var}(X)E(Y)^2 + \mathrm{Var}(Y)E(X)^2$ and $\mathrm{Var}(\sin\phi) = \int_0^{2\pi} \sin^2\phi \frac{1}{2\pi} d\phi = \frac{1}{2}$ as $\mathrm{Var}(y_1) = = \delta l^2 K$ together with the result from 2D.] The final statement, therefore, would be that the functional form of the result worked out in 2D will still work for 3D, though to re-use it the 'effective' value of K would be smaller in 3D than in 2D if the scattering angle distribution remained the same.

(g) What message might the last two results send to an engineer hoping to use cosmic ray muons for high-resolution concrete tomography? [2]

The last two results would tell an engineer (whether in 2D or 3D) that muons scatter transversely by an amount that grows as the depth D to the power of $\frac{3}{2}$. (That power is $\frac{3}{2}$ not 3 since the D^3 dependence in the answer to (e) referred to the VARIANCE of y_n, not to its standard deviation.) If, therefore, a scanner (perhaps one which measure the entrance and exit locations of muons traversing a block of concrete very carefully) were to be built which tries to infer the density distribution within the concrete, then uncertainty in the location of the muon

WITHIN the concrete, due to multiple scattering, would would set an unavoidable limit to the accuracy that any scanner could ever achieve. That resolution limit would be worse at greater depths, and would grow as the three-halves power of depth.

(h) How might such an engineer exploit the fact that not all cosmic ray muons have the same energy to improve the resolution of a concrete scanner? [2]

Although the resolution is constrained to be $\sigma \sim \pm\sqrt{\frac{K}{3}}D^{\frac{3}{2}}$ this statement is only true at fixed K. The exam candidates should be able to think back to their muon scattering examples in lectures and recall (or realise) that at higher and higher energies the muons will be more and more forward peaked in scattering, i.e. the effective value of K will decrease at high energy. Therefore, higher energy muons will have straighter tracks, and so to get high resolutions in the centre of the contrete it would be beneficial to make special use of high energy event. (E.g. they could have larger weights in reconstruction as they carry more accurate information.) Since high energy events carry more inforamtion, the scanner ought, therefore, to be built with some mechanisms to determine not only WHERE the muons entered and left the scanner, but also what energy they had.
