UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics
MASTER OF ADVANCED STUDY IN PHYSICS
NATURAL SCIENCES TRIPOS: Part III Astrophysics

Tuesday 16 January 2018: 14:00 to 16:00

MAJOR TOPICS
Paper 1/PP (Particle Physics)

Answer ***two*** *questions only. The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate. The paper contains four sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

You should use a ***separate Answer Book*** *for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2x20-page answer books	Mathematical Formulae Handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.4

A

1 Suppose that in an extension of the Standard Model there is a fourth generation of heavy leptons comprising a charged lepon, L^-, with a mass of 3.0 TeV, and a heavy neutrino, ν_L, with a mass of 500 GeV. You may assume that these fourth generation leptons have the same couplings to the W, Z and γ as the first three generations.

(a) The L^- would decay via the weak interaction, i.e. $L^- \to e^- \nu_L \bar{\nu}_e$. Draw the leading order Feynman diagram(s) for this decay and, by considering the other possible decays of the W-boson, estimate the branching fraction for $L^- \to \nu_L$ + hadrons. [6]

The requested diagram would have an L^- coming in from the left, splitting into a ν_L and a W^-, with the W^- then itself decaying to $e^- \bar{\nu}_e$. Since the W^- will have all of its usual decay modes open to it, the branching fraction of $L^- \to \nu_L$ + hadrons should be the same as the branching fraction for $W^- \to$ hadrons. **BOOKWORK[In lectures and handouts the students saw an explanation for why $BR(W \to q\bar{q}) = 6BR(W \to e\nu)$. They may either state this, or re-prove it.]** From that they can go on to show that since $BR(W \to e\nu) = BR(W \to \mu\nu) = BR(W \to \tau\nu)$ then the branching fraction for $W^- \to$ hadrons must be $\frac{6}{6+3} = \frac{2}{3}$.

(b) Heavy charged leptons could be produced in proton-proton collisions via the Drell-Yan process, i.e. the production of L^+L^- through the annihilation of a quark and anti-quark into a photon. Draw the leading order Feynman diagram(s) for this interaction and explain why the cross section is non-zero for proton-proton collisions. [3]

This diagram should show $q\bar{q} \to \gamma \to L^+L^-$ with it somehow clearly indicated that the $\bar{q}$ must be some kind of sea-quark within the proton. The non-absence of sea-quarks in the proton is why this cross section is non-zero for proton-proton collisions.

(c) If the squared centre-of-mass energy of the proton-proton collision is s, show that the squared centre-of-mass energy of the qq system is $\hat{s} = x_1 x_2 s$, where x_1 and x_2 are the fractional momenta carried by the partons involved in the collision. [3]

$$\hat{s} = (x_1 p_1 + x_2 p_2)^2 \tag{1}$$
$$= 2x_1 x_2 p_1^\mu p_{2\mu} + x_1^2 p_1^\mu p_{1\mu} + x_2^2 p_2^\mu p_{2\mu} \tag{2}$$
$$= 2x_1 x_2 p_1^\mu p_{2\mu} \qquad \text{(since quarks have negligible mass here)} \tag{3}$$
$$= x_1 x_2 \left(2p_1^\mu p_{2\mu} + 0 + 0\right) \tag{4}$$
$$= x_1 x_2 \left(2p_1^\mu p_{2\mu} + p_1^\mu p_{1\mu} + p_2^\mu p_{2\mu}\right) \tag{5}$$
$$= x_1 x_2 (p_1 + p_2)^2 \tag{6}$$
$$= x_1 x_2 s \qquad \text{Q.E.D..} \tag{7}$$

(d) The QED cross section for $qq \to l^+l^-$, where l is a charged lepton, is given by

$$\sigma = \frac{4\pi}{3}\frac{\alpha^2}{s}e_q^2 \times f(s, m_l),$$

where e_q is the quark charge (i.e. $e_u = +2/3$ and $e_d = -1/3$) and $f(s, m_l)$ is a kinematic factor that depends on the lepton mass. Assuming that the $\bar{u}$ and $\bar{d}$ parton distribution functions can be described by a single function, $S(x)$, and neglecting the strange quark contribution, show that the parton model prediction for the $pp \to L^+L^-X$ differential cross section can be written

$$\frac{d^2\sigma}{dx_1 dx_2} = f(sx_1x_2, m_L)\frac{2\pi\alpha^2}{81 x_1 x_2 s}\left[9u_V(x_1)S(x_2) + 9u_V(x_2)S(x_1) + 20S(x_1)S(x_2)\right],$$

where $u_V(x)$ is the valence up-quark parton distribution function. Clearly state any assumptions you have made. [10]

The parton model allows us to re-purpose the QED cross section given above as a partonic cross section, so long as we (a) replace s with the partonic centre of mass energy $\hat{s} = x_1x_2s$, (b) multiply it by an appropriate parton density function (times $p(x_1)dx_1$ and $p(x_2)dx_2$) to account for how many of each parton species is in the proton, and (c) sum the resulting quantity over each type of parton species we wish to assume the proton contains. The resulting answer is expressed as a differential cross section by taking the dx_1dx_2 to the denominator of the LHS.

The proton has two valence up quarks and one valence down quark, each in one of three colours. We will assume that $u_v(x) = 2d_v(x)$, i.e. that at any value of x there are always twice as many valence u quarks as d quarks. Since photons carry no colour, we only get interactions when red meets anti-red or green meets anti-green, etc.. Colour therefore, contributes another factor of $\frac{1}{3}$ to the above the QED form of the cross section, as this is the change that a quark and an anti-quark, chosen at random, have the right colours to annihilate into a photon. Furthermore, when we take a valence quark from one proton we will need a sea-quark to provide the anti-quark on the other side. We will also get sea-anti-sea collisions. Since the photon is neutral and cannot change flavour, we will need always u opposite $\bar{u}$ and d opposite $\bar{d}$.

Putting all the above together, we therfore expect that

$$\frac{d^2\sigma}{dx_1 dx_2} = \frac{1}{3} \times \frac{4\pi}{3}\frac{\alpha^2}{x_1 x_2 s} f(x_1 x_2 s, m_L) \left\{ e_q^2 \times \text{pdf products for:} \begin{pmatrix} \text{(i) valence } u \text{ sea } \bar{u} \\ \text{(ii) valence } d \text{ sea } \bar{d} \\ \text{(iii) sea-anti-sea } u\bar{u} \\ \text{(iv) sea-anti-sea } d\bar{d} \end{pmatrix} \right\} \quad (8)$$

$$= \frac{4\pi}{9}\frac{\alpha^2}{x_1 x_2 s} f(x_1 x_2 s, m_L) \begin{pmatrix} (\frac{2}{3})^2 u_v(x_1)S(x_2) + (\frac{2}{3})^2 u_v(x_2)S(x_1) + \\ (\frac{1}{3})^2 d_v(x_1)S(x_2) + (\frac{1}{3})^2 d_v(x_2)S(x_1) + \\ (\frac{2}{3})^2 S(x_1)S(x_2) + (\frac{2}{3})^2 S(x_2)S(x_1) + \\ (\frac{1}{3})^2 S(x_1)S(x_2) + (\frac{1}{3})^2 S(x_2)S(x_1) \end{pmatrix} \quad (9)$$

$$= \frac{4\pi}{9}\frac{\alpha^2}{x_1 x_2 s} f(x_1 x_2 s, m_L) \begin{pmatrix} (\frac{2}{3})^2 u_v(x_1)S(x_2) + (\frac{2}{3})^2 u_v(x_2)S(x_1) + \\ ((\frac{1}{3})^2 u_v(x_1)S(x_2) + (\frac{1}{3})^2 u_v(x_2)S(x_1))/2 + \\ (\frac{2}{3})^2 S(x_1)S(x_2) + (\frac{2}{3})^2 S(x_2)S(x_1) + \\ (\frac{1}{3})^2 S(x_1)S(x_2) + (\frac{1}{3})^2 S(x_2)S(x_1) \end{pmatrix} \quad (10)$$

$$= \frac{4\pi}{81}\frac{\alpha^2}{x_1 x_2 s} f(x_1 x_2 s, m_L) \begin{pmatrix} 4u_v(x_1)S(x_2) + 4u_v(x_2)S(x_1) + \\ (u_v(x_1)S(x_2) + u_v(x_2)S(x_1))/2 + \\ 4S(x_1)S(x_2) + 4S(x_2)S(x_1) + \\ S(x_1)S(x_2) + S(x_2)S(x_1) \end{pmatrix} \quad (11)$$

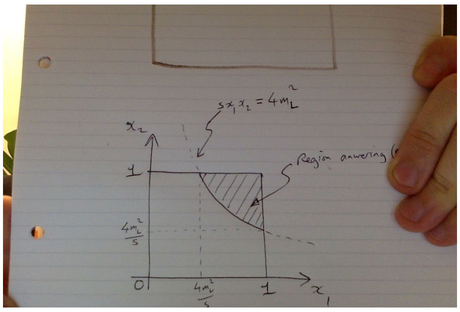
$$= \frac{2\pi}{81}\frac{\alpha^2}{x_1 x_2 s} f(x_1 x_2 s, m_L) \begin{pmatrix} 8u_v(x_1)S(x_2) + 8u_v(x_2)S(x_1) + \\ u_v(x_1)S(x_2) + u_v(x_2)S(x_1) + \\ 8S(x_1)S(x_2) + 8S(x_2)S(x_1) + \\ 2S(x_1)S(x_2) + 2S(x_2)S(x_1) \end{pmatrix} \quad (12)$$

$$= \frac{2\pi}{81}\frac{\alpha^2}{x_1 x_2 s} f(x_1 x_2 s, m_L) \{9u_v(x_1)S(x_2) + 9u_v(x_2)S(x_1) + 20S(x_1)S(x_2)\} \quad (13)$$

as required.

(e) Draw a diagram showing the region of x_1 versus x_2 that contributes to the cross section for $pp \to L^+L^-X$ at the LHC operating at $\sqrt{s} = 13$ TeV. [2]

The region of x_1 and x_2 that contributes to $pp \to L^+L^-X$ production will be that subset of the unit square in which the partonic centre of mass energy exceeds $2m_L$ as this is the kinematic boundary for production of two L particles. This constraint can be written as $\hat{s} \geq (2m_L)^2$ or equivalently $sx_1x_2 \geq 4m_L^2$. The allowed region has a hyperbolic boundary, symmetric under $x_1 \leftrightarrow x_2$ and passing through $x_1 = x_2 = 2m_L/\sqrt{s}$. Despite the existence of the last two sentences, I have been asked to include an actual picture of the diagram just described. Look about and it will be somewhere near here, wherever LaTeX deigns to place it.



In principle, any physical values of x_1 and x_2 on the correct (upper) side of that hyperbola can contribute, however in practice it is likely that only values of x near the kinematic boundary will be significant, as pdfs will fall steeply as either x value increases. Those up near $x_1 = 1$ or $x_2 = 1$ are unlikely to be used much.

(f) Explaining your reasoning, which of the terms $20S(x_1)S(x_2)$ or $[9u_V(x_1)S(x_2) + 9u_V(x_2)S(x_1)]$, would you expect to dominate in this region? [2]

Sea quark distributions peak at low values of x. For LLX production we need the product of x_1 and x_2 must be large, so it is unlikely that $20S(x_1)S(x_2)$ will contribute significantly. The majority of our LLX production will come from the valence-quark meets anti-sea production term: $9u_V(x_1)S(x_2) + 9u_V(x_2)S(x_1)$.

(g) In the relevant regions of x, the parton distribution functions can be taken to have the approximate forms, $u_V(x) \approx ax^{-\lambda}$ and $S(x) \approx bx^{-\lambda}$. Taking $f(sx_1x_2, m_L) = 1$, and by performing the appropriate integration over x_1 and x_2, obtain an approximate expression for the Drell-Yan cross section for heavy lepton production in terms of α, a, b, λ, m_L and s. [4]

$$\sigma = \int_{sx_1x_2 \geq 4m_L^2, 0 \leq x_1, x_2 \leq 1} \frac{2\pi\alpha^2}{81 x_1 x_2 s} \left(9ax_1^{-\lambda} bx_2^{-\lambda} + 9ax_2^{-\lambda} bx_1^{-\lambda} + 20bx_1^{-\lambda} bx_2^{-\lambda}\right) dx_1 dx_2 \tag{14}$$

$$= (18a + 20b) \frac{2\pi\alpha^2 b}{81 s} \int_{sx_1x_2 \geq 4m_L^2, 0 \leq x_1, x_2 \leq 1} \frac{1}{x_1 x_2} (x_1 x_2)^{-\lambda} dx_1 dx_2 \tag{15}$$

$$= \left(18ab + 20b^2\right) \frac{2\pi\alpha^2}{81 s} \int_{sx_1x_2 \geq 4m_L^2, 0 \leq x_1, x_2 \leq 1} (x_1 x_2)^{-\lambda-1} dx_1 dx_2 \tag{16}$$

$$= \left(18ab + 20b^2\right) \frac{2\pi\alpha^2}{81 s} \int_{x_1 = \frac{4m_L^2}{s}}^{1} \int_{x_2 = \frac{4m_L^2}{sx_1}}^{1} (x_1 x_2)^{-\lambda-1} dx_1 dx_2 \tag{17}$$

$$= \left(18ab + 20b^2\right) \frac{2\pi\alpha^2}{81 s} \frac{1}{\lambda^2} \left(1 - \left(\frac{s}{4m_L^2}\right)^{\lambda} \left(1 + \lambda \log\left(\frac{4m_L^2}{s}\right)\right)\right). \tag{18}$$

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2 Write detailed notes on **one** of the following topics:

(a) parton distribution functions, **or** [30]

(b) SU(3) colour symmetry. [30]

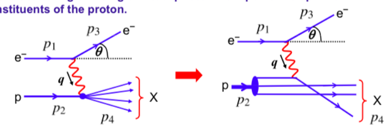
There are many things a candidate could write about parton distribution functions. This exam does not intend to be prescriptive on exactly what topics should or should not be included. The main goal of the examiner, on reading the answers provided, will be to gauge the degree to which each candidate appears to understand and communicate his/her understanding of the topic mentoned. The bullet points listed below are not specimen answers. They are not a minimal set of topics that need to be covered, nor are they a maximal set of topics outlining the scope of the question. Instead they are a list of topics that the examiner has been required to provide, as part of the examination review process. for reasons that are not entirely clear to the examiner. Somewhat grudgingly, with those caveats, here are some lists of topics:

(a) Parton distribution functions,

- A parton distribution function (pdf) is often denoted $p(x)$ for short, though full specficiation would require saying what sort of parton is being distributed (e.g. up-quark vs gluon) within what sort of hadron (e.g. proton or neutron).
- By definition, $p(x)dx$ is number of partons of some type, in some kind of hadron, with Bjorken x between x and $x + dx$.
- In the case of electron-proton scattering, Bjorken x is $Q^2/(2p_2.q)$ with $Q^2 = -q^2$ and q^μ being the being the momentum transfer $q = p_3 - p_1$, with p_1 and p_3 the incoming and outgoing electron momenta (respectively) and p_2 the initial proton momentum.
- Bjorken x is also the fraction of the momentum that is carried by the struct object, if computed in the infinite momentum frame.
- Isospin symmetry between the neutron and the proton (or between the u and d quarks) is expressable in terms of pdfs as an approximate equivalence between the manitude of the up pdf of a proton and the down pdf of a neutron, etc. $u_p(x) \approx d_n(x)$.
- Our knowledge of pdfs comes exclusively from experiments and techniques that have been used to measure them, and demonstrate self-consistency within the parton model. It's therefore relevant to talk about the aspects of deep inelastic scattering that are relevant ... e.g.
- Hera and its measurements
- The relevant parts of the Quark Parton Model (see images below)
- Differences between neutrino scattering and electron scattering – e.g. how one provides direct access to the quark vs antiquark content of hadrons, while the other provides access to the up and down fractions
- Necessity of Use in LHC production.

The Quark-Parton Model

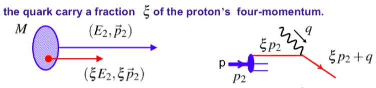
- Before quarks and gluons were generally accepted Feynman proposed that the proton was made up of point-like constituents "partons"
- Both Bjorken Scaling and the Callan-Gross relationship can be explained by assuming that Deep Inelastic Scattering is dominated by the scattering of a single virtual photon from point-like spin-half constituents of the proton.



Scattering from a proton with structure functions ➡ Scattering from a point-like quark within the proton

★ How do these two pictures of the interaction relate to each other?

- In the parton model the basic interaction is ELASTIC scattering from a "quasi-free" spin-½ quark in the proton, i.e. treat the quark as a free particle!
- The parton model is most easily formulated in a frame where the proton has very high energy, often referred to as the "infinite momentum frame", where we can neglect the proton mass and $p_2 = (E_2, 0, 0, E_2)$
- In this frame can also neglect the mass of the quark and any momentum transverse to the direction of the proton.
- Let the quark carry a fraction ξ of the proton's four-momentum.



- After the interaction the struck quark's four-momentum is $\xi p_2 + q$

$$(\xi p_2 + q)^2 = m_q^2 \approx 0 \quad \Rightarrow \quad \xi^2 p_2^2 + q^2 + 2\xi p_2.q = 0 \qquad (\xi^2 p_2^2 = m_q^2 \approx 0)$$

$$\Rightarrow \quad \xi = \frac{Q^2}{2p_2.q} = x$$

Bjorken x can be identified as the fraction of the proton momentum carried by the struck quark (in a frame where the proton has very high energy)

- In terms of the proton momentum

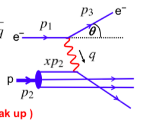
$$s = (p_1 + p_2)^2 \simeq 2p_1.p_2 \qquad y = \frac{p_2.q}{p_2.p_1} \qquad x = \frac{Q^2}{2p_2.q}$$

- But for the underlying quark interaction

$$s^q = (p_1 + xp_2)^2 = 2xp_1.p_2 = xs$$

$$y_q = \frac{p_q.q}{p_q.p_1} = \frac{xp_2.q}{xp_2.p_1} = y$$

$x_q = 1$ (elastic, i.e. assume quark does not break up)



- Previously derived the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit (handout 4 + Q10 on examples sheet).

Now apply this to $e^-q \to e^-q$

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2 e_q^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s_q}\right)^2\right]$$

e_q is quark charge, i.e. $e_u = +2/3; \quad e_d = -1/3$

- Using $-q^2 = Q^2 = (s_q - m^2)x_q y_q \quad \Rightarrow \quad \frac{q^2}{s_q} = -y_q = -y$

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1 + (1-y)^2\right]$$

(where the last two expression assume the massless limit m=0)

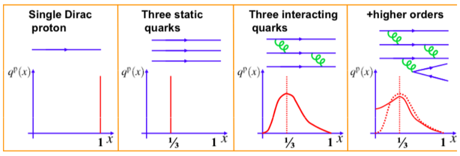
$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2 e_q^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right] \qquad (3)$$

★ This is the expression for the differential cross-section for elastic e^-q scattering from a quark carrying a fraction x of the proton momentum.

- Now need to account for distribution of quark momenta within proton

★ Introduce parton distribution functions such that $q^p(x)dx$ is the number of quarks of type q within a proton with momenta between $x \to x + dx$

● Expected form of the parton distribution function ?



★ The cross section for scattering from a particular quark type within the proton which in the range $x \to x + dx$ is

$$\frac{d^2\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right] \times e_q^2 q^p(x)dx$$

★ Summing over all types of quark within the proton gives the expression for the electron-proton scattering cross section

$$\frac{d^2\sigma^{ep}}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]\sum_q e_q^2 q^p(x) \qquad (5)$$

★ Compare with the electron-proton scattering cross section in terms of structure functions (equation (2)):

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \qquad (6)$$

★ By comparing (5) and (6) obtain the parton model prediction for the structure functions in the general L.I. form for the differential cross section

$$F_2^p(x,Q^2) = 2xF_1^p(x,Q^2) = x\sum_q e_q^2 q^p(x)$$

⇨ Can relate measured structure functions to the underlying quark distributions

The parton model predicts:

- Bjorken Scaling $\quad F_1(x,Q^2) \to F_1(x) \qquad F_2(x,Q^2) \to F_2(x)$
  - ✶ Due to scattering from point-like particles within the proton
- Callan-Gross Relation $\quad F_2(x) = 2xF_1(x)$
  - ✶ Due to scattering from spin half Dirac particles where the magnetic moment is directly related to the charge; hence the "electro-magnetic" and "pure magnetic" terms are fixed with respect to each other.

★ At present parton distributions cannot be calculated from QCD

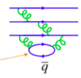
- Can't use perturbation theory due to large coupling constant

★ Measurements of the structure functions enable us to determine the parton distribution functions !

★ For electron-proton scattering we have:

$$F_2^p(x) = x\sum_q e_q^2 q^p(x)$$

- Due to higher orders, the proton contains not only up and down quarks but also anti-up and anti-down quarks (will neglect the small contributions from heavier quarks)



- For electron-proton scattering have:

$$F_2^{ep}(x) = x\sum_q e_q^2 q^p(x) = x\left(\frac{4}{9}u^p(x) + \frac{1}{9}d^p(x) + \frac{4}{9}\bar{u}^p(x) + \frac{1}{9}\bar{d}^p(x)\right)$$

- For electron-neutron scattering have:

$$F_2^{en}(x) = x\sum_q e_q^2 q^n(x) = x\left(\frac{4}{9}u^n(x) + \frac{1}{9}d^n(x) + \frac{4}{9}\bar{u}^n(x) + \frac{1}{9}\bar{d}^n(x)\right)$$

★ Now assume "isospin symmetry", i.e. that the neutron (ddu) is the same as a proton (uud) with up and down quarks interchanged, i.e.

$$d^n(x) = u^p(x); \qquad u^n(x) = d^p(x)$$

and define the neutron distributions functions in terms of those of the proton

$$u(x) \equiv u^p(x) = d^n(x); \qquad d(x) \equiv d^p(x) = u^n(x)$$

$$\bar{u}(x) \equiv \bar{u}^p(x) = \bar{d}^n(x); \qquad \bar{d}(x) \equiv \bar{d}^p(x) = \bar{u}^n(x)$$

giving:

$$F_2^{ep}(x) = 2xF_1^{ep}(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\bar{u}(x) + \frac{1}{9}\bar{d}(x)\right) \qquad (7)$$

$$F_2^{en}(x) = 2xF_1^{en}(x) = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\bar{d}(x) + \frac{1}{9}\bar{u}(x)\right) \qquad (8)$$

- Integrating (7) and (8) :

$$\int_0^1 F_2^{ep}(x)dx = \int_0^1 x\left(\frac{4}{9}[u(x) + \bar{u}(x)] + \frac{1}{9}[d(x) + \bar{d}(x)]\right)dx = \frac{4}{9}f_u + \frac{1}{9}f_d$$

$$\int_0^1 F_2^{en}(x)dx = \int_0^1 x\left(\frac{4}{9}[d(x) + \bar{d}(x)] + \frac{1}{9}[u(x) + \bar{u}(x)]\right)dx = \frac{4}{9}f_d + \frac{1}{9}f_u$$

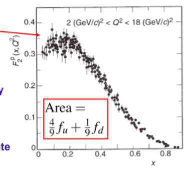
★ $f_u = \int_0^1 [xu(x) + x\bar{u}(x)]dx$ is the fraction of the proton momentum carried by the up and anti-up quarks

Experimentally

$$\int F_2^{ep}(x)dx \approx 0.18$$

$$\int F_2^{en}(x)dx \approx 0.12$$

$$\Rightarrow \quad f_u \approx 0.36 \qquad f_d \approx 0.18$$

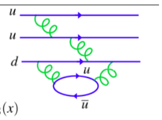


✶ In the proton, as expected, the up quarks carry twice the momentum of the down quarks

✶ The quarks carry just over 50 % of the total proton momentum. The rest is carried by gluons (which being neutral doesn't contribute to electron-nucleon scattering).

Valence and Sea Quarks

- As we are beginning to see the proton is complex…
- The parton distribution function $u^p(x) = u(x)$ includes contributions from the "valence" quarks and the virtual quarks produced by gluons: the "sea"



- Resolving into valence and sea contributions:

$$u(x) = u_V(x) + u_S(x) \qquad d(x) = d_V(x) + d_S(x)$$

$$\bar{u}(x) = \bar{u}_S(x) \qquad \bar{d}(x) = \bar{d}_S(x)$$

- The proton contains two valence up quarks and one valence down quark and would expect:

$$\int_0^1 u_V(x)dx = 2 \qquad \int_0^1 d_V(x)dx = 1$$

- But no *a priori* expectation for the total number of sea quarks !
- But sea quarks arise from gluon quark/anti-quark pair production and with $m_u = m_d$ it is reasonable to expect

$$u_S(x) = d_S(x) = \bar{u}_S(x) = \bar{d}_S(x) = S(x)$$

- With these relations (7) and (8) become

$$F_2^{ep}(x) = x\left(\frac{4}{9}u_V(x) + \frac{1}{9}d_V(x) + \frac{10}{9}S(x)\right) \qquad F_2^{en}(x) = x\left(\frac{4}{9}d_V(x) + \frac{1}{9}u_V(x) + \frac{10}{9}S(x)\right)$$

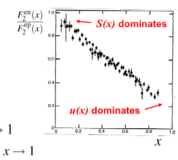
Giving the ratio

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} = \frac{4d_V(x) + u_V(x) + 10S(x)}{4u_V(x) + d_V(x) + 10S(x)}$$

- The sea component arises from processes such as $g \to \bar{u}u$. Due to the $1/q^2$ dependence of the gluon propagator, much more likely to produce low energy gluons. Expect the sea to comprise of low energy $q/\bar{q}$
- Therefore at low x expect the sea to dominate:

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \to 1 \quad \text{as} \quad x \to 0$$

Observed experimentally



- At high x expect the sea contribution to be small

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \to \frac{4d_V(x) + u_V(x)}{4u_V(x) + d_V(x)} \quad \text{as} \quad x \to 1$$

Note: $u_V = 2d_V$ would give ratio 2/3 as $x \to 1$

Experimentally $F_2^{en}(x)/F_2^{ep}(x) \to 1/4 \quad$ as $\quad x \to 1$

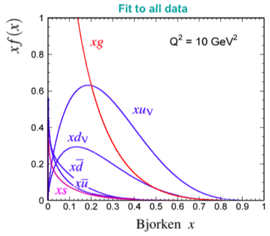
$$\Rightarrow \quad d(x)/u(x) \to 0 \quad \text{as} \quad x \to 1$$

This behaviour is not understood.

A

Parton Distribution Functions

- Ultimately the parton distribution functions are obtained from a fit to all experimental data including neutrino scattering (see handout 10)
- Hadron-hadron collisions give information on gluon pdf $g(x)$



Note:
- Apart from at large x: $u_V(x) \approx 2d_V(x)$
- For $x < 0.2$ gluons dominate
- In fits to data assume $u_s(x) = \bar{u}(x)$
- $\bar{d}(x) > \bar{u}(x)$ not understood – exclusion principle?
- Small strange quark component $s(x)$

(Try Question

★ For physics to remain unchanged – must have GAUGE INVARIANCE of the new field, i.e. physical predictions unchanged for $A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$

★ Hence the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (i.e. photon):

$$i\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\psi = 0$$

⇨ interaction vertex: $i\gamma^\mu qA_\mu$ (see p.111)

⇨ QED !

★ The local phase transformation of QED is a unitary U(1) transformation

$$\psi \to \psi' = \hat{U}\psi \quad \text{i.e.} \quad \psi \to \psi' = \psi e^{iq\chi(x)} \quad \text{with} \quad U^\dagger U = 1$$

Now extend this idea…

(b) SU(3) colour symmetry.

Dr Lester 245

From QED to QCD

★ Suppose there is another fundamental symmetry of the universe, say **"invariance under SU(3) local phase transformations"**

- i.e. require invariance under $\psi \to \psi' = \psi e^{ig\vec{\lambda}.\vec{\theta}(x)}$ where

  $\vec{\lambda}$ are the eight 3x3 Gell-Mann matrices introduced in handout 7

  $\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time

  ⇨ 8 spin-1 gauge bosons

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$$ wave function is now a vector in COLOUR SPACE

⇨ QCD !

★ QCD is fully specified by require invariance under SU(3) local phase transformations

Corresponds to rotating states in colour space about an axis whose direction is different at every space-time point

⇨ interaction vertex: $-\frac{1}{2}ig_s\lambda^a\gamma^\mu$

★ Predicts 8 massless gauge bosons – the gluons (one for each λ)

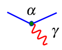
★ Also predicts exact form for interactions between gluons, i.e. the 3 and 4 gluon vertices – the details are beyond the level of this course

Dr Lester 246

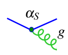
Colour in QCD

★ The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "colour" charges

In QED:
- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless "gauge boson" – the photon



In QCD:
- quarks carry colour charge: r, g, b
- anti-quarks carry anti-charge: $\bar{r}, \bar{g}, \bar{b}$
- The force is mediated by massless gluons



★ In QCD, the strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colours

⇨ SU(3) colour symmetry

- This is an exact symmetry, unlike the approximate uds flavour symmetry discussed previously.

Dr Lester 247

Colour Confinement

★ It is believed (although not yet proven) that all observed free particles are "colourless"
- i.e. never observe a free quark (which would carry colour charge)
- consequently quarks are always found in bound states colourless hadrons

★ **Colour Confinement Hypothesis:**

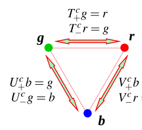
only colour singlet states can exist as free particles

★ All hadrons must be "colourless" i.e. colour singlets

★ To construct colour wave-functions for hadrons can apply results for SU(3) flavour symmetry to SU(3) colour with replacement

$$u \to r,\quad d \to g,\quad s \to b$$

★ just as for uds flavour symmetry can define colour ladder operators



Dr Lester 249

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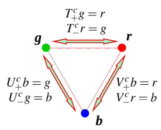


Dr Lester 249

Colour Singlets

★ It is important to understand what is meant by a singlet state

★ Consider spin states obtained from two spin 1/2 particles.
- Four spin combinations: $\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$
- Gives four eigenstates of $\hat{S}^2,\ \hat{S}_z$ $\quad (2\otimes 2 = 3\oplus 1)$

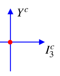
$$|1,+1\rangle = \uparrow\uparrow$$
$$|1,0\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$
$$|1,-1\rangle = \downarrow\downarrow$$

spin-1 triplet $\oplus$ $|0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ spin-0 singlet

★ The singlet state is "spinless": it has zero angular momentum, is invariant under SU(2) spin transformations and spin ladder operators yield zero

$$S_\pm|0,0\rangle = 0$$

★ In the same way COLOUR SINGLETS are "colourless" combinations:
- they have zero colour quantum numbers $I_3^c = 0,\ Y^c = 0$
- invariant under SU(3) colour transformations
- ladder operators $T_\pm,\ U_\pm,\ V_\pm$ all yield zero



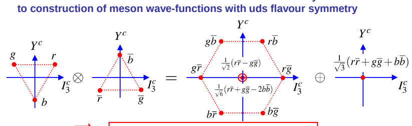
★ NOT sufficient to have $I_3^c = 0,\ Y^c = 0$: does not mean that state is a singlet

Dr Lester 250

Meson Colour Wave-function

★ Consider colour wave-functions for $q\bar{q}$

★ The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with uds flavour symmetry



➡ Coloured octet and a colourless singlet

- Colour confinement implies that hadrons only exist in colour singlet states so the colour wave-function for mesons is:

$$\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

★ Can we have a $qq\bar{q}$ state ? i.e. by adding a quark to the above octet can we form a state with $Y^c = 0;\ I_3^c = 0$. The answer is clear no.

➡ $qq\bar{q}$ bound states do not exist in nature.

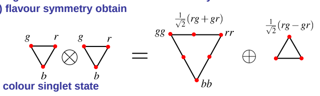
Dr Lester 251

V7.4

A

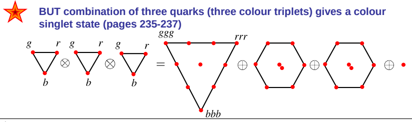
Baryon Colour Wave-function

- Do **qq** bound states exist ? This is equivalent to asking whether it possible to form a colour singlet from two colour triplets ?
- Following the discussion of construction of baryon wave-functions in SU(3) flavour symmetry obtain



- No **qq** colour singlet state
- Colour confinement ➡ bound states of **qq** do not exist

BUT combination of three quarks (three colour triplets) gives a colour singlet state (pages 235-237)



- The singlet colour wave-function is:

$$\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)$$

Check this is a colour singlet…
- It has $I_3^c = 0,\ Y^c = 0$: a necessary but not sufficient condition
- Apply ladder operators, e.g. T_+ (recall $T_+ g = r$)

$$T_+\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0$$

- Similarly $T_-\psi_c^{qqq} = 0;\ V_\pm\psi_c^{qqq} = 0;\ U_\pm\psi_c^{qqq} = 0;$

- Colourless singlet - therefore **qqq** bound states exist !

⇨ Anti-symmetric colour wave-function

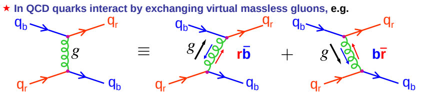
Allowed Hadrons i.e. the possible colour singlet states

- $q\bar{q},\ qqq$ — Mesons and Baryons
- $q\bar{q}q\bar{q},\ qqqq\bar{q}$ — Exotic states, e.g. pentaquarks

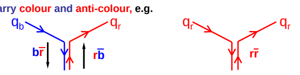
To date all confirmed hadrons are either **mesons** or **baryons**. However, some recent (but not entirely convincing) "evidence" for pentaquark states

Gluons

- In QCD quarks interact by exchanging virtual massless gluons, e.g.

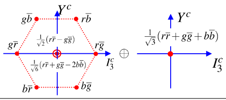


- Gluons carry colour and anti-colour, e.g.



- Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour)

⇨ OCTET + "COLOURLESS" SINGLET



- So we might expect 9 physical gluons:

OCTET: $r\bar{g},\ r\bar{b},\ g\bar{r},\ g\bar{b},\ b\bar{r},\ b\bar{g},\ \frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g}),\ \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

SINGLET: $\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$

- BUT, colour confinement hypothesis:

only colour singlet states can exist as free particles ⇨ Colour singlet gluon would be unconfined. It would behave like a strongly interacting photon ➡ infinite range Strong force.

- **Empirically**, the strong force is short range and therefore know that the physical gluons are confined. The colour singlet state does not exist in nature !

NOTE: this is not entirely ad hoc. In the context of gauge field theory (see minor option) the strong interaction arises from a fundamental SU(3) symmetry. The gluons arise from the generators of the symmetry group (the Gell-Mann λ matrices). There are 8 such matrices ➡ 8 gluons. Had nature "chosen" a U(3) symmetry, would have 9 gluons, the additional gluon would be the colour singlet state and QCD would be an unconfined long-range force.

NOTE: the "gauge symmetry" determines the exact nature of the interaction ➡ FEYNMAN RULES

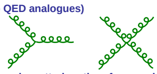
Gluon-Gluon Interactions

- In QED the photon does not carry the charge of the EM interaction (photons are electrically neutral)
- In contrast, in QCD the gluons do carry colour charge

⇨ Gluon Self-Interactions

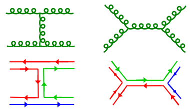
- Two new vertices (no QED analogues)

triple-gluon vertex



quartic-gluon vertex

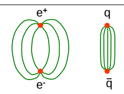
- In addition to quark-quark scattering, therefore can have gluon-gluon scattering



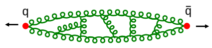
e.g. possible way of arranging the colour flow

Gluon self-Interactions and Confinement

- Gluon self-interactions are believed to give rise to colour confinement
- Qualitative picture:
  - Compare QED with QCD
  - In QCD "gluon self-interactions squeeze lines of force into a flux tube"



- What happens when try to separate two coloured objects e.g. $q\bar{q}$



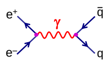
- Form a flux tube of interacting gluons of approximately constant energy density $\sim 1\,\mathrm{GeV/fm}$

➡ $V(r) \sim \lambda r$

- Require infinite energy to separate coloured objects to infinity
- **Coloured** quarks and gluons are always confined within colourless states
- In this way QCD provides a plausible explanation of confinement – but not yet proven (although there has been recent progress with Lattice QCD)

QCD and Colour in e⁺e⁻ Collisions

- e⁺e⁻ colliders are an excellent place to study QCD

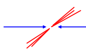


- Well defined production of quarks
  - QED process well-understood
  - no need to know parton structure functions
  - + experimentally very clean – no proton remnants

- In handout 5 obtained expressions for the $e^+e^- \to \mu^+\mu^-$ cross-section

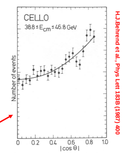
$$\sigma = \frac{4\pi\alpha^2}{3s} \qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

- In e⁺e⁻ collisions produce all quark flavours for which $\sqrt{s} > 2m_q$
- In general, i.e. unless producing a $q\bar{q}$ bound state, produce jets of hadrons
- Usually can't tell which jet came from the quark and came from anti-quark



- Angular distribution of jets $\propto (1+\cos^2\theta)$

➡ Quarks are spin ½



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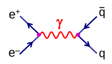
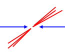
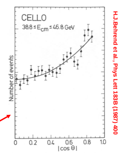






- Colour is conserved and quarks are produced as $r\bar{r},\ g\bar{g},\ b\bar{b}$
- For a single quark flavour and single colour

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{4\pi\alpha^2}{3s}Q_q^2$$

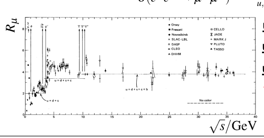
- Experimentally observe jets of hadrons:

$$\sigma(e^+e^- \to \text{hadrons}) = 3\sum_{u,d,s,\ldots}\frac{4\pi\alpha^2}{3s}Q_q^2$$

Factor 3 comes from colours

- Usual to express as ratio compared to $\sigma(e^+e^- \to \mu^+\mu^-)$

$$R_\mu = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{u,d,s,\ldots}Q_q^2$$



u,d,s: $R_\mu = 3\times(\frac{1}{9}+\frac{4}{9}+\frac{1}{9}) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

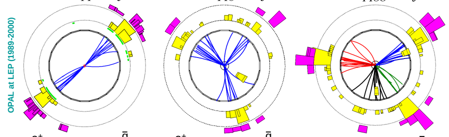
u,d,s,c,b: $R_\mu = \frac{11}{3}$

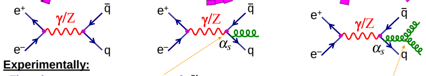
- Data consistent with expectation with factor 3 from colour

Jet production in e+e- Collisions

- e⁺e⁻ colliders are also a good place to study gluons

$e^+e^- \to q\bar{q} \to 2\text{jets}$ $\qquad e^+e^- \to q\bar{q}g \to 3\text{jets}$ $\qquad e^+e^- \to q\bar{q}gg \to 4\text{jets}$





Experimentally:
- Three jet rate ➡ measurement of α_s
- Angular distributions ➡ gluons are spin-1
- Four-jet rate and distributions ➡ QCD has an underlying SU(3) symmetry

V7.4 (TURN OVER

A

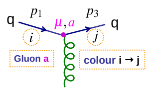
The Quark – Gluon Interaction

- Representing the colour part of the fermion wave-functions by:

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$
- The QCD qqg vertex is written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices
- Isolating the colour part:

$$c_j^\dagger\lambda^a c_i = c_j^\dagger\begin{pmatrix}\lambda^a_{1i}\\\lambda^a_{2i}\\\lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

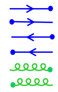
- Hence the fundamental quark - gluon QCD interaction can be written

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Feynman Rules for QCD

- **External Lines**

spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\overline{u}(p)$	
	incoming anti-quark	$\overline{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\varepsilon^\mu(p)$	
	outgoing gluon	$\varepsilon^\mu(p)^*$	



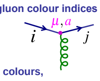
- **Internal Lines (propagators)**

spin 1 gluon $\quad \dfrac{-ig_{\mu\nu}}{q^2}\delta^{ab}$

a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark $\quad -ig_s\frac{1}{2}\lambda^a_{ji}\gamma^\mu$



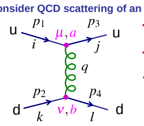
i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

- \+ 3 gluon and 4 gluon interaction vertices
- Matrix Element $-iM$ = product of all factors

Matrix Element for quark-quark scattering

★ Consider QCD scattering of an up and a down quark



- The incoming and out-going quark colours are labelled by $i,j,k,l = \{1,2,3\}$ (or $\{r,g,b\}$)
- In terms of colour this scattering is $ik \to jl$
- The 8 different gluons are accounted for by the colour indices $a, b = 1,2,\ldots,8$
- NOTE: the δ-function in the propagator ensures *a* = *b*, i.e. the gluon "emitted" at *a* is the same as that "absorbed" at *b*

★ Applying the Feynman rules:

$$-iM = [\overline{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u_u(p_1)]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\overline{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\}u_d(p_2)\right]$$

where summation over *a* and *b* (and μ and ν) is implied.

★ Summing over *a* and *b* using the δ-function gives:

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]$$

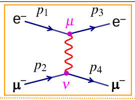
Sum over all 8 gluons (repeated indices)

QCD vs QED

QED

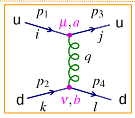
$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]$$



QCD

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]$$



★ QCD Matrix Element = QED Matrix Element with:

- $e^2 \to g_s^2$ or equivalently $\alpha = \dfrac{e^2}{4\pi} \to \alpha_s = \dfrac{g_s^2}{4\pi}$

\+ QCD Matrix Element includes an additional "colour factor"

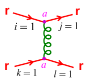
$$C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}$$

Evaluation of QCD Colour Factors

- QCD colour factors reflect the gluon states that are involved

$\lambda^1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$	$\lambda^4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$	$\lambda^6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\lambda^3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$
$\lambda^2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$	$\lambda^5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}$	$\lambda^7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$
Gluons: $r\overline{g}, g\overline{r}$	$r\overline{b}, b\overline{r}$	$g\overline{b}, b\overline{g}$	$\frac{1}{\sqrt{2}}(r\overline{r}-g\overline{g})$ $\frac{1}{\sqrt{6}}(r\overline{r}+g\overline{g}-2b\overline{b})$

❶ Configurations involving a single colour



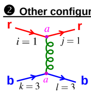
- Only matrices with non-zero entries in **11** position are involved

$$C(rr \to rr) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11}+\lambda^8_{11}\lambda^8_{11})$$

$$= \frac{1}{4}\left(1+\frac{1}{3}\right) = \frac{1}{3}$$

Similarly find $\quad C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \dfrac{1}{3}$

❷ Other configurations where quarks don't change colour e.g. $rb \to rb$



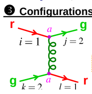
- Only matrices with non-zero entries in **11** and **33** position are involved

$$C(rb \to rb) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{33} = \frac{1}{4}(\lambda^8_{11}\lambda^8_{33})$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}.\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $\quad C(rb \to rb) = C(rg \to rg) = C(gr \to gr) = C(gb \to gb) = C(br \to br) = C(bg \to bg) = -\frac{1}{6}$

❸ Configurations where quarks swap colours e.g. $rg \to gr$



- Only matrices with non-zero entries in **12** and **21** position are involved

Gluons $r\overline{g}, g\overline{r}$

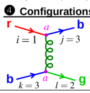
$$C(rg \to gr) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{21}\lambda^a_{12} = \frac{1}{4}(\lambda^1_{21}\lambda^1_{12}+\lambda^2_{21}\lambda^2_{12})$$

$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}_+^{(ij)}\hat{T}_-^{(kl)}$

$$C(rb \to br) = C(rg \to gr) = C(gr \to rg) = C(gb \to bg) = C(br \to rb) = C(bg \to gb) = \frac{1}{2}$$

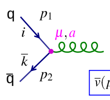
❹ Configurations involving 3 colours e.g. $rb \to bg$



- Only matrices with non-zero entries in the **13** and **32** position
- But none of the λ matrices have non-zero entries in the **13** and **32** positions. Hence the colour factor is zero

★ colour is conserved

★Finally we can consider the quark – anti-quark annihilation

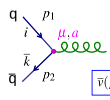


QCD vertex: $\quad \overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

with $\quad c_k^\dagger\lambda^a c_i = \lambda^a_{ki}$

$$\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)$$

★Finally we can consider the quark – anti-quark annihilation



QCD vertex: $\quad \overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

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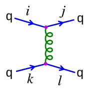
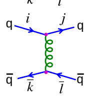
$$\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)$$

- Consequently the colour factors for the different diagrams are:



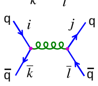
$$C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}$$

e.g.
$C(rr \to rr) = \frac{1}{3}$
$C(rg \to rg) = -\frac{1}{6}$
$C(rg \to gr) = \frac{1}{2}$



$$C(i\overline{k} \to j\overline{l}) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{kl}$$

$C(r\overline{r} \to r\overline{r}) = \frac{1}{3}$
$C(r\overline{g} \to r\overline{g}) = -\frac{1}{6}$
$C(r\overline{r} \to g\overline{g}) = \frac{1}{2}$



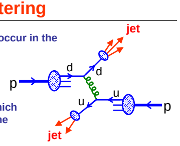
$$C(i\overline{k} \to j\overline{l}) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ki}\lambda^a_{jl}$$

$C(r\overline{r} \to r\overline{r}) = \frac{1}{3}$
$C(r\overline{g} \to r\overline{g}) = \frac{1}{2}$
$C(r\overline{r} \to g\overline{g}) = -\frac{1}{6}$

Colour index of adjoint spinor comes first

Quark-Quark Scattering

- Consider the process $u+d \to u+d$ which can occur in the high energy proton-proton scattering
- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states



$$\langle|M_{fi}|^2\rangle = \frac{1}{3}\cdot\frac{1}{3}\sum_{i,j,k,l=1}^{3}|M_{fi}(ij \to kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\langle|C|^2\rangle = \frac{1}{9}\sum_{i,j,k,l=1}^{3}|C(ij \to kl)|^2$$

- For $qq \to qq$ $\quad$ $rr \to rr$,.. $\quad$ $rb \to rb$,.. $\quad$ $rb \to br$,..

$$\langle|C|^2\rangle = \frac{1}{9}\left[3\times\left(\frac{1}{3}\right)^2 + 6\times\left(-\frac{1}{6}\right)^2 + 6\times\left(\frac{1}{2}\right)^2\right] = \frac{2}{9}$$

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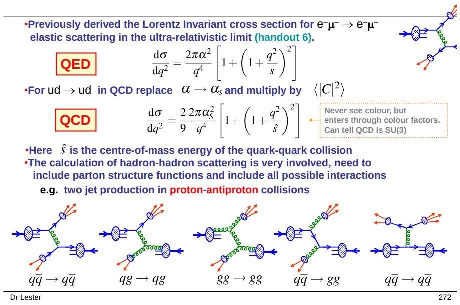

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3 The vertex factor for the interaction between a Higgs boson and a fermion is

$$-i\frac{g_W m_f}{2m_W},$$

where g_W is the weak decay constant, and m_W and m_f are the masses of the W-boson and the fermion f, respectively. Write down the matrix element for the decay $H \to f\bar{f}$. [3]

Up to an overall modulus-one conventional factor,

$$M_{ij} = -i\frac{g_W m_f}{2m_W}\bar{u}_i v_j \quad (19)$$

$$= -i\frac{g_W m_f}{2m_W}u_i^\dagger \gamma^0 v_j \quad (20)$$

$$= -i\frac{g_W m_f}{2m_W}\left(u_i^\dagger \gamma^0 v_j\right) \quad (21)$$

where i and j each take a label in $\{\uparrow, \downarrow\}$.

Consider the decay of the Higgs boson in its rest frame where the fermion is produced with polar angle θ and azimuthal angle ϕ. Assuming $m_H \gg m_f$, evaluate the $H \to f\bar{f}$ matrix elements for all four possible combinations of particle and anti-particle helicities and comment on your results. [8]

The supplied Dirac spinors are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}, u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ e^{i\phi}c \\ s \\ -e^{i\phi}c \end{pmatrix}, v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}, v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix},$$

however we will want our u and v spinors to use different values of θ and ϕ compared to each other. Specifically, if we let the 'vanilla' values of θ and ϕ refer to the direction of the particle in the Higgs rest frame, then the anti-particle will have $\theta' = \pi - \theta$ and $\phi' = \phi + \pi$. Accordingly

$$s' = \sin(\theta'/2) = \sin((\pi - \theta)/2) = \cos(\theta/2) = c$$

and

$$c' = \cos(\theta'/2) = \cos((\pi - \theta)/2) = sin(\theta/2) = s$$

while

$$e^{i\phi'} = e^{i\phi + i\pi} = -e^{i\phi}.$$

For this reason we must use the following spinors:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}, u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ e^{i\phi}c \\ s \\ -e^{i\phi}c \end{pmatrix}, v'_\uparrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ -c \\ -e^{i\phi}s \end{pmatrix}, v'_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -e^{i\phi}c \\ s \\ -e^{i\phi}c \end{pmatrix}.$$

Hermitian conjugating the u-spinors these gives:

$$u_\uparrow^\dagger = \sqrt{E}\left(c, e^{-i\phi}s, c, e^{-i\phi}s\right), u_\downarrow^\dagger = \sqrt{E}\left(-s, e^{-i\phi}c, s, -e^{-i\phi}c\right), v'_\uparrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ -c \\ -e^{i\phi}s \end{pmatrix}, v'_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -e^{i\phi}c \\ s \\ -e^{i\phi}c \end{pmatrix}$$

and pre-multiplying the v-spinors by γ^0 gives

$$u_\uparrow^\dagger = \sqrt{E}\left(c, e^{-i\phi}s, c, e^{-i\phi}s\right), u_\downarrow^\dagger = \sqrt{E}\left(-s, e^{-i\phi}c, s, -e^{-i\phi}c\right), \gamma^0 v'_\uparrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}, \gamma^0 v'_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}$$

from which, if we define $K = -i\frac{g_W m_f}{2m_W}$, we can read off:

$$M_{\uparrow\uparrow} = 2KE \tag{22}$$
$$M_{\uparrow\downarrow} = 0 \tag{23}$$
$$M_{\downarrow\uparrow} = 0 \tag{24}$$
$$M_{\downarrow\downarrow} = -2KE. \tag{25}$$

This answer makes a lot of sense: the higgs as a scalar particle has spin-0, and so must decay to a particle and an anti-particle each having the same helicity. Were this not the case, i.e. were opposite helicity possible, then the final state would have one plus or minus unit of spin in the direction parallel to the outgoing particles, contradicting the spin-0 of the Higgs. Furthermore, we should not be surprised that the magnitude of $M_{\uparrow\uparrow}$ and $M_{\downarrow\downarrow}$ agree since the Higgs vertex is not partiy-violating (dies not favour one helicity over another).

Given the expression for the decay rate:

$$\frac{d\Gamma}{d\Omega} = \frac{p^*}{32\pi^2 m_H^2}\left\langle |M_{fi}|^2 \right\rangle,$$

where p^* is the centre-of-mass momentum of either final state particle, show that the partial decay width for the Higgs boson to $\tau^+\tau^-$ is

$$\Gamma_\tau = \frac{G_F}{\sqrt{2}}\frac{m_\tau^2 m_H}{4\pi}.$$

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[4]

First let us calculate p^*. Since we are neglecting final state particla masses, $2p^* = m_H$ so $p^* = m_H/2$.

Second we need $\left\langle |M_{fi}|^2 \right\rangle$. This should be an average over possible input states of this Higgs and a sum over relevant final states. The Higgs has only one initial state – so we can ignore that averaging part. Instead we just sum over final states:

$$\left\langle |M_{fi}|^2 \right\rangle = |M_{\uparrow\uparrow}|^2 + |M_{\uparrow\downarrow}|^2 + |M_{\downarrow\uparrow}|^2 + |M_{\downarrow\downarrow}|^2 \tag{26}$$

$$= 4|K|^2 E^2 + 0 + 0 + 4|K|^2 E^2 \tag{27}$$

$$= 8|K|^2 E^2 \tag{28}$$

$$= 8|K|^2 (p^*)^2, \tag{29}$$

and so

$$\left\langle |M_{fi}|^2_\tau \right\rangle = 8 \frac{g_W^2 m_\tau^2}{4 m_W^2} \left(\frac{m_H}{2} \right)^2 \tag{30}$$

$$= \frac{g_W^2 m_\tau^2 m_H^2}{2 m_W^2}. \tag{31}$$

Putting these into the differential decay rate gives:

$$\frac{d\Gamma_\tau}{d\Omega} = \frac{m_H/2}{32\pi^2 m_H^2} \frac{g_W^2 m_\tau^2 m_H^2}{2 m_W^2} \tag{32}$$

$$= \frac{g_W^2 m_\tau^2 m_H}{128 \pi^2 m_W^2} \tag{33}$$

therefore

$$\Gamma_\tau = \int \frac{d\Gamma_\tau}{d\Omega} d\Omega \tag{34}$$

$$= 4\pi \frac{d\Gamma_\tau}{d\Omega} \tag{35}$$

$$= 4\pi \frac{g_W^2 m_\tau^2 m_H}{128 \pi^2 m_W^2} \tag{36}$$

$$= \frac{g_W^2}{8 m_W^2} \frac{\pi m_\tau^2 m_H}{4\pi^2} \tag{37}$$

$$= \frac{G_F}{\sqrt{2}} \frac{m_\tau^2 m_H}{4\pi} \tag{38}$$

as desired.

Assuming $m_H = 125$ GeV, and neglecting decays to $\nu\bar{\nu}$, e^+e^-, $\mu^+\mu^-$, $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$, obtain values for the total decay width of the Higgs boson, Γ_H and the branching fraction for $H \to b\bar{b}$. [5]

We have just found that

$$\Gamma_\tau = \frac{G_F}{\sqrt{2}}\frac{m_\tau^2 m_H}{4\pi}.$$

By the same argument

$$\Gamma_\mu = \frac{G_F}{\sqrt{2}}\frac{m_\mu^2 m_H}{4\pi},$$

$$\Gamma_c = 3\frac{G_F}{\sqrt{2}}\frac{m_c^2 m_H}{4\pi}$$

and

$$\Gamma_b = 3\frac{G_F}{\sqrt{2}}\frac{m_b^2 m_H}{4\pi}$$

where the 3 accounts or the number of colours. Γ_t is not something that needs to be considered as Higgs decay to $t\bar{t}$ is not kinematically possible. The total width (neglecting the requested modes) is therefore given by

$$\begin{aligned}
\Gamma &= \Gamma_\tau + \Gamma_c + \Gamma_b && (39)\\
&= \frac{G_F}{\sqrt{2}}\frac{m_H}{4\pi}\left(m_\tau^2 + 3m_c^2 + 3m_b^2\right) && (40)\\
&\approx \frac{1.66\times 10^{-5}}{\sqrt{2}}\frac{125}{4\pi}\left(1.8^2 + 3\times 1.5^2 + 3\times 4.5^2\right)\ \text{GeV} && (41)\\
&\approx 0.0082\ \text{GeV} && (42)
\end{aligned}$$

and furthermore

$$\begin{aligned}
\text{BR}(H\to\mu\bar{\mu}) &= \frac{\Gamma_\mu}{\Gamma} \approx \frac{m_\mu^2}{m_\tau^2 + 3m_c^2 + 3m_b^2} \approx 1.6\times 10^{-4} && (43)\\
\text{BR}(H\to\tau\bar{\tau}) &= \frac{\Gamma_\tau}{\Gamma} \approx \frac{m_\tau^2}{m_\tau^2 + 3m_c^2 + 3m_b^2} \approx 0.045 && (44)\\
\text{BR}(H\to c\bar{c}) &= \frac{\Gamma_c}{\Gamma} \approx \frac{3m_c^2}{m_\tau^2 + 3m_c^2 + 3m_b^2} \approx 0.10 && (45)\\
\text{BR}(H\to b\bar{b}) &= \frac{\Gamma_b}{\Gamma} \approx \frac{3m_b^2}{m_\tau^2 + 3m_c^2 + 3m_b^2} \approx 0.86. && (46)
\end{aligned}$$

(The question only requests the last of these branching ratios.)

At a future muon collider operating at the Higgs boson resonance (assumed to be $\sqrt{s} = 125$ GeV), compare the cross section for the process $\mu^+\mu^- \to H \to b\bar{b}$ to the cross section for the QED process $\mu^+\mu^- \to \gamma \to b\bar{b}$ which is $\sigma_{QED} = 4\pi\alpha^2/(9s)$. Comment on the possible advantages and disadvantages of a muon collider compared to an electron-positron collider. [6]

To estimate the cross section for $\mu^+\mu^- \to H \to b\bar{b}$ we can use the supplied relativistic Breit-Wigner formula for a resonance of mass m and spin J :

$$\sigma = \frac{4\pi(2J+1)}{m^2}\frac{s\Gamma_i\Gamma_f}{(s-m^2)^2+m^2\Gamma^2},$$

where Γ_i, Γ_f and Γ are the appropriate partial decay widths and the total decay width. Based on our earlier computations, and knowledge that $J = 0$ for the Higgs, we have:

$$\sigma^{s=m_H^2}_{\mu^+\mu^-\to H\to b\bar{b}} = \frac{4\pi}{m_H^2}\frac{m_H^2\Gamma_\mu\Gamma_b}{(m_H^2-m_H^2)^2+m_H^2\Gamma_H^2} \tag{47}$$

$$= \frac{4\pi}{m_H^2}\frac{\Gamma_\mu\Gamma_b}{\Gamma_H^2} \tag{48}$$

$$\approx \frac{4\pi}{s}\times(1.6\times10^{-4})\times(0.86) \tag{49}$$

$$\approx \frac{4\pi}{s}\times(3.8\times10^{-4}). \tag{50}$$

We can compare this to the supplied QED cross section of

$$\sigma_{QED} = \frac{4\pi}{s}\frac{\alpha^2}{9} \tag{51}$$

$$\approx \frac{4\pi}{s}\frac{1}{137\times137\times9} \tag{52}$$

$$\approx \frac{4\pi}{s}\times(5.9\times10^{-6}). \tag{53}$$

We thus note that the cross section for $b\bar{b}$ production via Higgs at this machine is nearly two orders of magnitude bigger ($380/5.9 \approx 58$) than for production via QCD.

For the same amount of synchrotron radiation, a circular muon collider should theoretically reach higher energy than a circular electron collider. Put another way, at the same energy a muon collider would have less synchroton losses than an electron collider and so would have less demand for powwer. There are strong disadvantages to muon colliders, however, namely that the source particles (muons) are unstabe and much harder than electrons to manufacture in large quantities – this means that muon beams would have much shorter lifetimes than electron beams (though time dilation at energy would suppress this) and correspondingly one might imagine large beam backgrounds from in-beam decays. For Higgs production specifically, one would expect far more Higgs bosons (by a factor of order $\frac{m_\mu^2}{m_e^2}$ at a muon collider than an electron collider, on account of the nature of the Higgs-fermion-fermion interaction discussed in this question. This would potentially lead to a much larger number of Higgs events available for analysis, if the cross section gain were not offset by the other disadvantages of a muon collider.

Assuming that a muon collider could operate with fully polarised beams where the helicities of the μ^+ and μ^- can be chosen, explain how one could distinguish between (i) a Higgs boson with a Standard Model scalar coupling to fermions, and (ii) an exotic Higgs boson with 'scalar minus pseudo-scalar' $(1-\gamma^5)$ couplings. Briefly discuss whether it would be possible to distinguish a Standard Model Higgs boson from one with a pure pseudo-scalar (γ^5) coupling. [4]

A standard model Higgs boson is spinless and so can only couple to spin zero initial states – e.g. it could be made from $\mu_L\bar{\mu}_L$ or $\mu_R\bar{\mu}_R$ but not from $\mu_L\bar{\mu}_R$ or $\mu_R\bar{\mu}_L$. These characteristics could be seen by running the collider in each of those states and then measuring the Higgs boson cross section in each case. If, instead, the higgs coupling took the form $(1-\gamma^5)$ then we recognise this as (up to a factor of 2) a Left projection operator P_L ... meaning that it would wipe out right handed spinor placed to its right. Consequently a scalar-minus-pseudoscalar Higgs would only couple to one of the two helicity combinations allowed to the standard model Higgs (and a scalar-plus-pseudoscalar Higgs would couple to the other). A pure pseudoscalar Higgs has a γ^5 coupling. We can write $\gamma^5 = (1+\gamma^5)/2 - (1-\gamma^5)/2 = P_R - P_L$ so both right and left spinors to the right of γ^5 are preseserved (though one gets a negative sign compared to the other). The negative sign for one helicity will disappear on squaring the matrix element for the leading order diagram, however, meaning that the pseudoscalar Higgs would look just like the scalar Higgs. An answer that stops here would be sufficient to gain full marks. This said, it would not hurt to remark that there ought, nonetheless, to exist ways of telling the difference between the two forms of Higgs – the relative sign that is introduced between the L and R amplitudes could have an observable effect if allowed to interfere with somehting else. In a sense, the change in signature that we saw when moving from scalar to scalar-minus-pseudoscalar Higgs can itself be thought of as a consequence of an interference between two diagrams that both achieve $f\bar{f} \to f\bar{f}$ but one via a scalar and the other by a pseudoscalar. Therefore one way to distinguish a pure scalar from a pure pseudoscalar Higgs would be to look at Higgs events an environment where the Higgs is not alone in mediating the process, i.e. in context where there is some level of significant interference from another well understood process with the same initial and final state.

You may make use of the following information:
In the limit $E \gg m$ the Dirac spinors for the helicity eigenstates are

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}, u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ e^{i\phi}c \\ s \\ -e^{i\phi}c \end{pmatrix}, v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}, v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix},$$

where $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$.
The Dirac matrices are given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$G_F/\sqrt{2} = g_W^2/(8m_W^2)$. *The second and third generation fermion masses are $m_\mu = 0.106$ GeV, $m_\tau = 1.777$ GeV, $m_s = 0.1$ GeV, $m_c = 1.5$ GeV, $m_b = 4.5$ GeV, and $m_t = 175$ GeV. The Fermi constant takes the numerical value $G_F = 1.166\times10^{-5}$ GeV^{-2}.*
The relativistic Breit-Wigner formula for a resonance of mass m and spin J is

$$\sigma = \frac{4\pi(2J+1)}{m^2}\frac{s\Gamma_i\Gamma_f}{(s-m^2)^2 + m^2\Gamma^2},$$

where Γ_i, Γ_f and Γ are the appropriate partial decay widths and the total decay width.