NATURAL SCIENCES TRIPOS Part III

Monday 14 January 2008 9:00am to 10:30am

EXPERIMENTAL AND THEORETICAL PHYSICS (1)
Advanced Quantum Condensed Matter Physics

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
20 Page Answer Book
Rough Work Pad
Metric Graph Paper

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 Write brief notes on **two** of the following:

(a) the electronic structure and optical absorption of a two-dimensional semiconductor quantum well; [15]

(b) the scattering mechanisms that determine elastic and inelastic mean free paths, and the requirements for observing quantum transport phenomena in solids; [15]

(c) Fermi liquid theory. [15]

2 Answer all parts.

(a) Give three examples of phenomena that are manifestations of electron-electron interactions in solids and discuss the extent to which Hartree–Fock theory takes into account the correlations between electrons. [12]

(b) Consider a homogeneous system of spin-$\frac{1}{2}$ fermions, in the absence of a periodic potential, in which the fermions interact with each other through a contact interaction potential $U_{\mathrm{int}}(\boldsymbol{r}-\boldsymbol{r}') = g \times \delta(\boldsymbol{r}-\boldsymbol{r}')$ as opposed to the normal Coulomb potential. Show, for a state in which the number of fermions with spin up is $N_\uparrow$ and the number of fermions with spin down is $N_\downarrow$, that the Hartree–Fock ground-state energy is given by: [12]

$$E = \frac{3}{5}\frac{\hbar^2}{2m_{\mathrm{e}}}\left(\frac{6\pi^2}{V}\right)^{\frac{2}{3}}\left(N_\uparrow^{\frac{5}{3}} + N_\downarrow^{\frac{5}{3}}\right) + \frac{g}{V}N_\uparrow N_\downarrow,$$

where V is the volume; undefined symbols have their usual meanings.

[*Start with the general expression for the Hartree–Fock ground-state energy*

$$E = \sum_{i=1}^{N}\left\{\int \mathrm{d}^3\boldsymbol{r}\,\phi_i^*(\boldsymbol{r})\left(-\frac{\hbar^2}{2m_e}\nabla^2\right)\phi_i(\boldsymbol{r})\right\}$$

$$+\frac{1}{2}\sum_{i,j}\left\{\int \mathrm{d}^3\boldsymbol{r}'\int \mathrm{d}^3\boldsymbol{r}\,|\phi_j(\boldsymbol{r})|^2|\phi_i(\boldsymbol{r}')|^2 U_{\mathrm{int}}(\boldsymbol{r}-\boldsymbol{r}')\right\}$$

$$-\frac{1}{2}\sum_{i,j}\left\{\int \mathrm{d}^3\boldsymbol{r}'\int \mathrm{d}^3\boldsymbol{r}\,\phi_i^*(\boldsymbol{r})\phi_j^*(\boldsymbol{r}')U_{\mathrm{int}}(\boldsymbol{r}-\boldsymbol{r}')\phi_j(\boldsymbol{r})\phi_i(\boldsymbol{r}')\right\}$$

and assume that the single-particle Hartree–Fock orbitals are plane waves.]

(c) Determine the carrier concentration for which the ferromagnetic state with $N_\uparrow = N$ and $N_\downarrow = 0$ becomes energetically favoured compared to the paramagnetic state in which $N_\uparrow = N_\downarrow = \frac{N}{2}$. [6]

3 Answer all parts.

(a) Explain the origin of exchange interactions giving at least two examples of exchange mechanisms that are important in solids. [12]

(b) Discuss the meaning of the symbols in the expression below for the Heisenberg Hamiltonian

$$H = -\sum_{i,j} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j = -\sum_{i,j} J_{ij}[S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)]$$

and explain how the sum over the indices i and j is to be conducted. [6]

(c) Consider the ferromagnetic Heisenberg Hamiltonian for a square, two-dimensional lattice with a total of N lattice sites and lattice constant a. Assume that only nearest-neighbour exchange interactions need to be considered with $J_{ij} = J =$ const. Show that the spin-wave states $|\boldsymbol{q}\rangle = \frac{1}{\sqrt{N}} \sum_j \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}_j} |j\rangle$ with $|j\rangle = |\uparrow\uparrow \dots \downarrow_j \dots \uparrow\uparrow\rangle$ diagonalize the Hamiltonian, and calculate their energy-momentum dispersion relation. [12]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Monday 14 January 2008 11:30am to 1:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
Soft Matter

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains FOUR sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

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1 Consider a coexisting melt of polystyrene (PS) and poly(methyl methacrylate) (PMMA) with an index of polymerisation of $N = 1000$ and a Kuhn step length of $b = 0.67\,\text{nm}$. The Flory–Huggins interaction parameter, χ, is given by $\chi = -0.032 + 48.2/T$, where T is the temperature in Kelvin. The number of monomers between entanglements for both polymers is $N_e = 180$.

(a) Calculate the critical temperature, T_c, of the blend. Does the blend have an upper or lower critical temperature? Explain your reasoning. [3]

(b) Calculate the width of a PS-PMMA interface at $T = 20\,°\text{C}$, justifying any choice of equation you make. [3]

(c) Calculate the probability that interfacial chains are entangled. What are the consequences of your finding? [3]

(d) To what temperature does the melt have to be heated to ensure that any strand of polymer crossing the interface is entangled with the other species? Comment on the feasibility of this. [4]

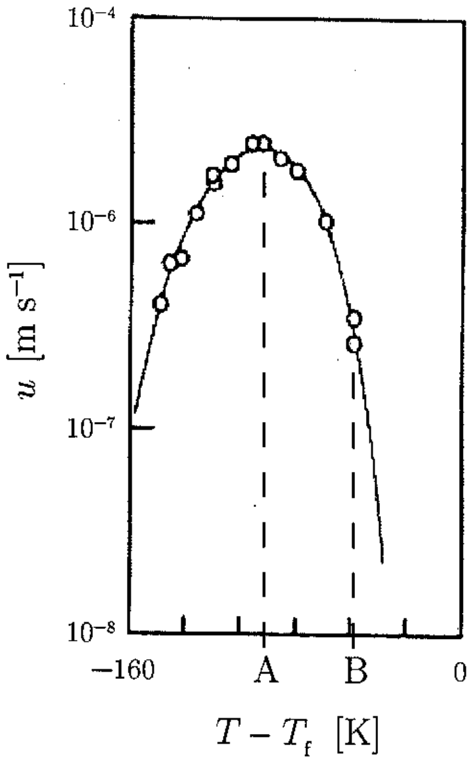
(e) To provide a good interfacial strength, PS–PMMA diblock copolymers are employed, consisting of a long PMMA block and varying PS block lengths. What is the optimal PMMA block length, N_{PMMA}, to ensure good bonding on the PMMA side? Explain your reasoning. [3]

(f) Two sets of copolymers were synthesized, each with the PMMA block length from (e) and PS block lengths $N_{\text{PS}} = 220$ and $N_{\text{PS}} = 700$. This system has a pull-out force per monomer $f_{\text{mono}} = 6 \times 10^{-12}\,\text{N}$, a bond strength $f_b = 2 \times 10^{-9}\,\text{N}$, and a critical stress of crazing $\sigma_c = 55\,\text{MPa}$ for PS, and $\sigma_c = 80\,\text{MPa}$ for PMMA. Calculate the failure mechanisms for interfacial block-copolymer coverages, Σ, of $\Sigma = 0.02$ chains/nm^2 and $\Sigma = 0.06$ chains/nm^2 for both copolymer types. Calculate the value of Σ, Σ^*, at which the cross-over between the two failure mechanisms occurs for both copolymers. [9]

(g) Based on the calculations in (e) and (f), what are the values for optimal interfacial strengthening of N_{PS}, N_{PMMA}, and Σ? Why is it unfavourable to go to very large values of these three parameters? [5]

2 Consider a partially crystalline polymeric material with a freezing (melting) temperature T_f.

(a) Describe the morphology of the material, making reference to the structure on all relevant length scales. Explain the composite nature of the material, and give an estimate for the smallest relevant length scale in terms of the structure of the polymer. [4]



(b) The figure shows the growth rate, u, of the crystalline lamellae of spherulites of Nylon 6. Write down an equation giving the functional form of the curve and discuss the physical origin of the two limits in which u goes to zero. [4]

(c) For samples that are quenched to the temperatures indicated by A and B in the figure, sketch the qualitative variation of the volume fraction of crystalline material, ϕ_c, as a function of time. Describe the different regimes of the crystallization process. [6]

(d) For the long-time limit in (c), describe the similarities and differences in the morphologies of the two samples. [2]

(e) Thermodynamically, what is the difference between the material's melting (freezing) temperature and its glass transition temperature? Give a qualitative description for the molecular process that underlies the glass transition. [4]

(f) A diffusion experiment was carried out in a glass-forming polymer (long-chain polystyrene) above its glass transition temperature $T_g = 100\,^{\circ}$C. Two sets of samples were annealed at two different temperatures. At $T = 120\,^{\circ}$C, diffusion distances of 3 nm, 5 nm, and 8.5 nm were found for annealing times of 100 min, 1000 min, and 10000 min, respectively. For an annealing temperature of $T = 140\,^{\circ}$C, the diffusion distances were 8.5 nm, 15 nm, 35 nm, and 50 nm for annealing times of 100 min, 1000 min, 5000 min,

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and and 10000 min, respectively. Use the WLF equation $\log a_{\mathrm{T}} = -9.06(T - 120\,°\mathrm{C})/(69.8 + T - 120\,°\mathrm{C})$ to plot all diffusion data for a reference temperature of $T = 120\,°\mathrm{C}$ in a double-logarithmic plot. Describe the diffusion at short and long times in terms of the molecular transport mechanisms of the chains. Locate and label the characteristic cross-over time. Why was the experiment not performed at a single temperature? [10]

3 Polystyrene chains with an index of polymerization $N = 400$ and a Kuhn step length of $b = 0.67$ nm are irreversibly grafted onto a solid surface.

(a) The sample is put into cyclohexane under conditions in which it is a θ-solvent for PS. What is the conformation of tethered chains at grafting densities, σ, of 4×10^{-3} chains/nm^2 and 1×10^{-2} chains/nm^2? Calculate the value of the grafting density, σ^*, at which a cross-over is expected between the two different conformations of the chains. [4]

(b) Using an Alexander–de Gennes model, derive the parameters that describe the grafted polymer layer for $\sigma > \sigma^*$ (where it forms a brush). What are the values of these parameters when $\sigma = 1 \times 10^{-2}$ chains/nm^2? [6]

(c) What are the shortcomings of the Alexander–de Gennes model? Give a qualitative description of a model that is more appropriate to stretched polymer brushes. In this model, what is the functional dependence of the composition profile as a function of the distance from the surface for a stretched brush in a θ-solvent. Why are the scaling relations of the Alexander–de Gennes model commonly used despite its shortcomings? [4]

(d) The brush is now put into contact with polystyrene homopolymer chains of index of polymerisation $P = 480$. Describe qualitatively the difference between the behaviour of the brush in this polymeric solvent, and in the low-molecular-weight solvent in (a). What is the conformation of the brush for $\sigma = 3 \times 10^{-3}$ chains/nm^2, $\sigma = 0.05$ chains/nm^2, and $\sigma = 0.2$ chains/nm^2? Calculate the cross-over between the grafting-density regimes. Why is it difficult to achieve grafting densities of $\sigma = 0.2$ chains/nm^2? [6]

(e) The brush with $\sigma = 0.05$ chains/nm^2 is now put into water. Describe the composition profile. [2]

(f) Before immersion into water, the brush from (e) is sulphonated and carries an effective charge of $+e$ per three Kuhn monomers. Using a Rubinstein-type model, describe chain conformation on the different relevant length scales in the absence of salt. [4]

(g) Give a qualitative account of why such a charged brush has a friction-reducing effect, and describe a biological system for which this effect may be significant. [4]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Monday 14 January 2008 2:30pm to 4:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (3)
Astrophysics and Cosmology

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1 The second cosmological field equation can be written as

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} = -\frac{kc^2}{R^2}.$$

Explain briefly the meaning of the quantities in this expression. [2]

Show that for a spatially-flat universe

$$1 - \Omega_{\rm m} - \Omega_\Lambda = 0,$$

where $\Omega_{\rm m} = \rho/(3H^2/8\pi G)$ and $\Omega_\Lambda = \Lambda/(3H^2)$. [4]

Show also that in a matter-dominated flat universe

$$\dot{a} = H_0 \left(\Omega_{{\rm m},0} a^{-1} \ + \ \Omega_{\Lambda,0} a^2\right)^{1/2},$$

where $a = R/R_0$ is a dimensionless scale factor, and the subscript 0 indicates the current epoch. [6]

Find approximate solutions for the dimensionless scale factor as a function of cosmic time that are valid at short and long times. [10]

Hence show that the universe must go from a decelerating to accelerating phase and find the scale factor at which this transition occurs. Comment on the results. [8]

2 Answer both parts.

(a) The virial equilibrium of a self-gravitating cloud of radius r_0 subject to an external pressure p_0 can be written as

$$3\int \frac{p}{\rho} {\rm d}M \ + \ \Omega = 4\pi r_0^3 p_0,$$

where Ω is the gravitational potential energy of the cloud and the other symbols have their usual meanings.

The singular isothermal sphere has a density given by $\rho(r) = a_T^2/(2\pi G r^2)$, where a_T is the isothermal sound speed. Find an expression for the gravitational potential energy in this case. [4]

Using the virial equation, find an expression for the maximum external pressure to which the cloud can be subjected before collapse. [7]

By comparing this pressure to the surface pressure of the cloud comment on the stability of the singular isothermal sphere. [4]

(b) In a matter-dominated expanding universe, the growth of instabilities in the linear regime is described by the equation

$$\frac{{\rm d}^2\Delta}{{\rm d}t^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{{\rm d}\Delta}{{\rm d}t} = (4\pi G\bar{\rho} - k^2 c_s^2)\Delta,$$

where Δ is the density contrast, k the wavenumber and c_s the sound speed, and the other symbols have their usual meanings.

In a non-expanding universe, what is the condition for instability, and how do the instabilities grow with time? [6]

For the case when the pressure can be neglected, show that instabilities in a matter-dominated, Einstein–de Sitter universe grow algebraically with time. [6]

Discuss, without mathematical detail, why these results present a challenge to the simplest model for galaxy formation. [3]

3 Write brief accounts of **three** of the following topics:

(a) accretion onto a compact object; [10]

(b) radio pulsars; [10]

(c) black hole feedback in galaxies; [10]

(d) gamma-ray bursts; [10]

(e) gravitational lensing by clusters of galaxies. [10]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Tuesday 15 January 2008 9:00am to 10:30am

EXPERIMENTAL AND THEORETICAL PHYSICS (4)
Particle Physics

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains FOUR sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

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Mathematical Formulae Handbook

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 Draw the lowest-order Standard Model Feynman diagrams for the process $e^+e^- \to \mu^+\mu^-$ and the additional diagram(s) for $e^+e^- \to e^+e^-$. Discuss the relative importance of the different diagrams at $\sqrt{s} = m_Z$. [5]

The forward-backward asymmetry is defined as $A_{FB} = (\sigma_F - \sigma_B)/(\sigma_F + \sigma_B)$, where $\sigma_F = \sigma(\cos\theta > 0)$ and $\sigma_B = \sigma(\cos\theta < 0)$. Explain:

(a) why A_{FB} for $e^+e^- \to e^+e^-$ is different from that for $e^+e^- \to \mu^+\mu^-$;

(b) why, for centre-of-mass energies in the range $\sqrt{s} = m_Z \pm \Gamma_Z$, A_{FB} for $e^+e^- \to \mu^+\mu^-$ depends strongly on $\sqrt{s}$. [3]

(Quantities not otherwise defined have their usual meanings.)

The matrix elements for the process $e^+e^- \to Z \to \mu^+\mu^-$ at $\sqrt{s} = m_Z$ are:

$$|M_{RR}|^2 = \kappa(c_R^e)^2(c_R^\mu)^2(1+\cos\theta)^2, \qquad |M_{LL}|^2 = \kappa(c_L^e)^2(c_L^\mu)^2(1+\cos\theta)^2,$$
$$\text{and } |M_{RL}|^2 = \kappa(c_R^e)^2(c_L^\mu)^2(1-\cos\theta)^2, \qquad |M_{LR}|^2 = \kappa(c_L^e)^2(c_R^\mu)^2(1-\cos\theta)^2,$$

where $\kappa = g_Z^4 m_Z^2/\Gamma_Z^2$, and $g_Z\, c_L$ and $g_Z\, c_R$ are the coupling strengths of the Z to left- and right-handed particles. Draw diagrams indicating the helicities of the initial- and final-state particles for the matrix elements M_{RR} and M_{RL} and explain clearly why only four of the possible sixteen helicity combinations give non-zero matrix elements. [5]

For unpolarised electrons and positrons, the differential cross section for $e^+e^- \to Z \to \mu^+\mu^-$ can be written in the form $d\sigma/d\Omega = A(1+\cos^2\theta) + B\cos\theta$. Using $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle|M|^2\rangle$, find expressions for A and B in terms of the Z couplings to left- and right-handed particles and show that for $\sqrt{s} = m_Z$

$$\sigma(e^+e^- \to Z \to \mu^+\mu^-) = \frac{g_Z^4}{48\pi\Gamma_Z^2}\left[(c_R^e)^2 + (c_L^e)^2\right]\left[(c_R^\mu)^2 + (c_L^\mu)^2\right].$$

Assuming lepton universality, $c_L^e = c_L^\mu = c_L$ and $c_R^e = c_R^\mu = c_R$, use the measurement of $\sigma(e^+e^- \to Z \to \mu^+\mu^-) = 2.00 \times 10^{-37}\,\mathrm{m}^2$ to obtain a value for $c_R^2 + c_L^2$. [7]

For the process $e^+e^- \to Z \to \mu^+\mu^-$ obtain an expression for A_{FB} in terms of c_L and c_R. Taking $A_{FB} = 0.017$ and the result you obtained for $c_R^2 + c_L^2$, determine values for $|c_L|$ and $|c_R|$. [8]

Discuss briefly how $|c_L|$ and $|c_R|$ are determined for the different lepton flavours when universality is not assumed. [2]

$\left[\Gamma_Z = 2.49\,\mathrm{GeV}, \quad g_Z = 0.75, \quad \textit{and} \quad \hbar c = 0.197\,\mathrm{GeV\,fm}. \right]$

2 Draw Feynman diagrams for the possible $\overline{\nu}_\mu$ charged-current weak interactions with the constituents of the proton, assuming that only u, d, $\overline{\mathrm{u}}$, and $\overline{\mathrm{d}}$ are present. [3]

The differential cross sections for the charged-current weak interactions of high energy $\overline{\nu}_\mu$ with quarks/anti-quarks are:

$$\frac{\mathrm{d}\sigma_{\overline{\nu}\mathrm{q}}}{\mathrm{d}\Omega^*} = \frac{G_\mathrm{F}^2 \widehat{s}}{16\pi^2}(1+\cos\theta^*)^2 \quad \text{and} \quad \frac{\mathrm{d}\sigma_{\overline{\nu}\overline{\mathrm{q}}}}{\mathrm{d}\Omega^*} = \frac{G_\mathrm{F}^2 \widehat{s}}{4\pi^2},$$

where θ^* is the polar angle of the final-state μ^+ in the centre-of-mass frame and $\sqrt{\widehat{s}}$ is the centre-of-mass energy of the anti-neutrino quark system. Explain the angular dependences of these cross sections. [4]

In $\overline{\nu}_\mu$ deep-inelastic scattering, y is defined as $y \equiv (p_2 \cdot q)/(p_2 \cdot p_1)$, where p_1 and p_2 are the four-momenta of the $\overline{\nu}_\mu$ and the struck quark respectively, and q is the four-momentum of the virtual W–boson. Neglecting particle masses, show that

$$y = \frac{1}{2}(1-\cos\theta^*), \qquad \frac{\mathrm{d}\sigma_{\overline{\nu}_\mu \mathrm{q}}}{\mathrm{d}y} = \frac{G_\mathrm{F}^2 \widehat{s}}{\pi}(1-y)^2, \qquad \text{and} \qquad \frac{\mathrm{d}\sigma_{\overline{\nu}_\mu \overline{\mathrm{q}}}}{\mathrm{d}y} = \frac{G_\mathrm{F}^2 \widehat{s}}{\pi}.$$ [6]

Many neutrino experiments employ detectors made of iron which contains an equal number of neutrons and protons. By considering the $\overline{\nu}_\mu$ interactions with protons and neutrons in terms of the parton distribution functions for the *proton*, $u(x), d(x), \overline{u}(x)$ and $\overline{d}(x)$, show that

$$\frac{\mathrm{d}\sigma_{\overline{\nu}_\mu N}}{\mathrm{d}y} \equiv \frac{1}{2}\left(\frac{\mathrm{d}\sigma_{\overline{\nu}_\mu \mathrm{n}}}{\mathrm{d}y} + \frac{\mathrm{d}\sigma_{\overline{\nu}_\mu \mathrm{p}}}{\mathrm{d}y}\right) = \frac{G_\mathrm{F}^2}{2\pi} s\left[f_{\overline{\mathrm{q}}} + (1-y)^2 f_\mathrm{q}\right],$$

where $\sqrt{s}$ is the centre-of-mass energy of the neutrino–nucleon system ($\widehat{s} = xs$), and f_q and $f_{\overline{\mathrm{q}}}$ are the fractions of the momentum of the nucleon carried by the quarks and anti-quarks respectively. [9]

For a $\overline{\nu}_\mu$ beam with energy 100 GeV in the laboratory frame, the total $\overline{\nu}_\mu$ charged-current deep-inelastic nucleon cross section is measured to be

$$\sigma_{\overline{\nu}_\mu N} = \frac{1}{2}(\sigma_{\overline{\nu}_\mu \mathrm{p}} + \sigma_{\overline{\nu}_\mu \mathrm{n}}) = 3.4 \times 10^{-41}\,\mathrm{m}^2$$

and the mean value of y is measured to be 0.34. Use these results to determine f_q and $f_{\overline{\mathrm{q}}}$, and comment on your answer. [8]

$$\left[G_\mathrm{F} = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}, \quad \hbar c = 0.197\,\mathrm{GeV\,fm}, \quad m_\mathrm{p} = m_\mathrm{n} = 0.94\,\mathrm{GeV}. \right]$$

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3 Write brief notes on **three** of the following:

(a) Electron–proton *elastic* scattering; [10]

(b) The proton wave-function. You should include a discussion of the reasons for the symmetries of the different parts of the wave-function; [10]

(c) The differences in the methods for detecting for $\overline{\nu}_{\mathrm{e}}$ from nuclear reactors, ν_{e} from the sun, and atmospheric ν_{μ}. You should include a brief discussion of relevant energy thresholds for the different reactions; [10]

(d) CP violation in the Standard Model. [10]

Answers in the form of a logically-ordered bullet-pointed list are acceptable. Diagrams and simple calculations should be included where appropriate.

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Tuesday 15 January 2008 11:30am to 1:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (5)
Physics of the Earth as a Planet

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

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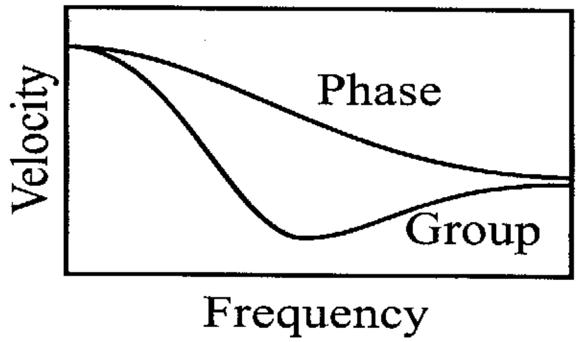
1 Answer all parts.

(a) Discuss the effects of a low-velocity zone on the propagation of seismic waves. Include diagrams showing ray trajectories as a function of ray parameter and travel time. [10]

(b) How can surface waves be used to determine the seismic structure of the Earth? Draw a labelled surface-wave seismogram corresponding to the dispersion curve shown in the figure. [10]



(c) Describe briefly how the thickness of oceanic crust is measured. Why is the crust found to be of nearly uniform thickness of 7±1 km across all ocean basins and for a wide range of spreading rates (30–160 mm/yr)? Is the crust at ultra-slow spreading ridges (spreading rate <20 mm/yr) thicker or thinner than this, and why? [10]

2 Answer both parts.

(a) Magnetic anomalies show that the South American and African plates are moving in opposite directions with equal speeds v. The movement is caused by the generation of new oceanic plate at the Mid-Atlantic Ridge. The thermal structure of the ridge and plates can be described using the plate model. Ignoring horizontal conduction, the temperature T satisfies

$$\kappa \frac{\partial^2 T}{\partial z^2} = v \frac{\partial T}{\partial x},$$

where κ is the thermal diffusivity and z is positive downwards. Solve the differential equation and show that the temperature is given by

$$T = T_{\mathrm{m}} \left(\frac{z}{L} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi z}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right)\right),$$

where $t = x/v$, L is the thickness of the lithosphere, and T_{m} the mantle temperature. [20]

(b) The generation of the African plate is balanced by its subduction under the Eurasian plate. How may the results of (a) be used to describe the thermal structure of the subducting plate? Discuss the characteristics, showing appropriate sketches, without actually solving the equations. [10]

3 Answer both parts.

(a) Answer **one** of the following:

(i) Sketch the fault, indicating the motion, and the fault-plane solution ('beachball') for:
an earthquake along the San Andreas fault in California;
an earthquake in the extending backarc region of the Aegean in Greece;
an earthquake resulting from a caldera collapse in Iceland;
an earthquake on the boundary fault between India and Tibet in the Himalaya. [15]

(ii) Write short notes on the elastic rebound model for earthquakes. [15]

(b) Write short notes on **one** of the following:

(i) post-glacial uplift; [15]

(ii) mapping patterns of mantle convection. [15]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Tuesday 15 January 2008 2:30pm to 4:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (6)
Quantum Condensed Matter Field Theory

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F

1 Defining Bose creation and annihilation operators $a^\dagger$ and a, show that the Holstein–Primakoff transformation [8]

$$\widehat{S}^z = -S + a^\dagger a \ , \ \ \widehat{S}^+ = a^\dagger(2S - a^\dagger a)^{1/2} \ , \ \ \widehat{S}^- = (2S - a^\dagger a)^{1/2} a$$

satisfies the commutation relation $[\widehat{S}^+, \widehat{S}^-] = 2\widehat{S}^z$.

A spin-boson Hamiltonian $\widehat{H}$ couples a large spin with total spin quantum number S to a degenerate boson ϕ:

$$\widehat{H} = \epsilon \widehat{S}^z + \frac{g}{(2S)^{1/2}}[\widehat{S}^+\phi + \phi^\dagger \widehat{S}^-] + \epsilon\phi^\dagger\phi$$

Assuming $S \gg 1$, use the Holstein–Primakoff transformation to expand $\widehat{H}$ to order $1/S^{1/2}$ (keeping the leading terms that represent interactions between bosons). [6]

By diagonalising the quadratic piece of $\widehat{H}$, and keeping the lowest-energy eigenstate, show that the low-energy excitations of the theory can be written in terms of a new mode $\alpha = (a - \phi)/\sqrt{2}$, with an effective Hamiltonian [8]

$$H_{\text{eff}} - \mu\widehat{n}_\alpha = -E_0 + (\epsilon_\alpha - \mu)\alpha^\dagger\alpha + V\alpha^\dagger\alpha^\dagger\alpha\alpha \ ,$$

and determine E_0, ϵ_α, and V as a function of the parameters ϵ, g and S.

Using a variational wavefunction of a coherent state $|\psi\rangle = \exp[\psi\alpha^\dagger]\,|0\rangle$, or otherwise, compute the semi-classical expectation value $\langle\alpha\rangle$ as a function of the chemical potential μ, and comment on the physical interpretation of the result. [8]

2 Write detailed notes on **one** of the following:

(a) the coherent-state formulation of the path integral; [30]

(b) the dilute Bose gas; [30]

(c) the connection of the Feynman path integral to classical statistical physics. [30]

3 A Hamiltonian for non-interacting fermions has eigenstates labeled by α with eigenvalues ϵ_α. Write down the quantum partition function Z in the continuous-time representation, explaining the meaning of all of the terms. [4]

Transforming to the frequency (Matsubara) representation, show that [4]

$$Z = \int D(\bar{\psi}, \psi)e^{-S[\bar{\psi},\psi]} \ ,$$

where

$$S = \sum_\alpha \sum_n \bar{\psi}_{\alpha n}(-\mathrm{i}\omega_n + \epsilon_\alpha - \mu)\psi_{\alpha n} \ ,$$

$\omega_n = (2n+1)\pi/\beta$ and $\beta = 1/k_B T$.

The Hamiltonian for a free Fermi gas whose spins couple to a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ is

$$\widehat{H} = \sum_{k,\sigma\sigma'} a^\dagger_{k\sigma}\left[\epsilon_k \delta_{\sigma\sigma'} - \frac{\mu_0 B}{2}(\sigma_z)_{\sigma\sigma'}\right] a_{k\sigma'} \ ,$$

where $\epsilon_k = \hbar^2 k^2/2m$ is the free particle energy, and σ_z the Pauli matrix.

Using this Hamiltonian, write down the frequency representation of the action, and by integrating over the Grassmann fields, show that [8]

$$Z = C\prod_{k,n}\left[(-\mathrm{i}\omega_n + \epsilon_k - \mu)^2 - \left(\frac{\mu_0 B}{2}\right)^2\right] .$$

where C is a constant.

Hence show that the spin contribution to the magnetic susceptibility $\chi = -\left.\frac{\partial^2 F}{\partial B^2}\right|_{B=0}$ is given by [4]

$$\chi = -\frac{\mu_0^2}{2\beta}\sum_{k,n}\frac{1}{(-\mathrm{i}\omega_n + \epsilon_k - \mu)^2} .$$

Perform the Matsubara summation to express the susceptibility as a sum over momentum eigenstates, and demonstrate that at sufficiently low temperatures [10]

$$\chi \approx \frac{\mu_0^2}{2}\rho(\mu) \ ,$$

where $\rho(\mu) = \sum_k \delta(\mu - \epsilon_k)$ is the density of states. Comment on the physical interpretation of your result.

You may wish to make use of the formula

$$I = \sum_n h(\omega_n) = -\frac{1}{2\pi\mathrm{i}}\int_C h(-\mathrm{i}z)g(z)\mathrm{d}z = \sum_\alpha \mathrm{Res}\left[h(-\mathrm{i}z)g(z)\right]_{z=z_\alpha} = \sum_\alpha g'(z_\alpha) .$$

for a function $h(-\mathrm{i}z)$ which has poles of the second order at $z = z_\alpha$, and where $g(z) = \beta/(e^{\beta z} + 1)$. C is a contour enclosing the imaginary axis.

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Wednesday 16 January 2008 9:00am to 10:30am

EXPERIMENTAL AND THEORETICAL PHYSICS (7)
Classical Field Theory and Gravitation

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
20 Page Answer Book
Rough Work Pad
Metric Graph Paper

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 A cosmic string may be considered as a straight, non-rotating, infinitely long, axisymmetric configuration of matter that is invariant to translations and Lorentz boosts along its axis of symmetry, and to reflections perpendicular to this axis.

Assuming the string to lie along the z-axis of a cylindrical coordinate system (ct, r, ϕ, z), show that the line–element for the spacetime, valid both inside and outside the string, can be written in the form [5]

$$ds^2 = c^2\,dt^2 - dr^2 - f^2(r)\,d\phi^2 - dz^2, \qquad (*)$$

where $f(r)$ is an arbitrary function.

If the matter of which the string is composed has a proper density $\rho(r)$, show that a self-consistent solution to the Einstein field equations may be obtained if the energy–momentum tensor of the matter in the (ct, r, ϕ, z) coordinate system is [7]

$$T_{\mu\nu} = \mathrm{diag}(\rho c^2, 0, 0, -\rho c^2)$$

and the function $f(r)$ in the cosmic string line-element satisfies

$$\frac{f''}{f} = -\frac{8\pi G}{c^2}\rho.$$

Suppose that the cosmic string has a finite coordinate radius r_0 and a uniform proper density $\rho(r) = \rho_0$. By demanding that $f(r) \to r$ as $r \to 0$, show that [7]

$$f(r) = \begin{cases} \dfrac{1}{\lambda}\sin\lambda r & \text{for } r \le r_0, \\ \alpha + \beta r & \text{for } r > r_0, \end{cases}$$

and obtain expressions for the constants λ, α and β.

For the case in which $\lambda r_0 \ll 1$, show that for $r \gg r_0$ [5]

$$ds^2 = c^2\,dt^2 - dr^2 - \left(1 - \frac{8G\mu}{c^2}\right) r^2\,d\phi^2 - dz^2,$$

where $\mu = \pi r_0^2 \rho_0$ is the 'mass per unit length' of the string, and briefly interpret this line-element geometrically.

Hence show that, in this case, the trajectory of a free particle moving in a plane perpendicular to the string has the form [6]

$$\frac{1}{r} = \frac{1}{b}\sin\left[\left(1 - \frac{4G\mu}{c^2}\right)(\phi - \phi_0)\right],$$

where b and ϕ_0 are constants, and make a sketch illustrating the main features of this trajectory.

[*You may assume that the only non-zero components of the Ricci tensor for the cosmic string line-element* $(*)$ *are* $R_{11} = f''/f$ *and* $R_{22} = f''f$, *where a prime denotes* $\mathrm{d}/\mathrm{d}r$.]

2 The Lagrangian density for a free Dirac spinor field ψ of mass m is

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi,$$

where ψ and $\bar{\psi} \equiv \psi^\dagger\gamma^0$ are considered as independent 4-spinor fields.

Obtain the equations of motion for the fields ψ and $\bar{\psi}$. [2]

Use Noether's theorem (see end of question) to show that invariance of the corresponding action under the global phase transformation $\psi'(x) = \mathrm{e}^{-\mathrm{i}q\alpha}\psi(x)$, with q and α real constants, implies a conserved total charge

$$Q = q\int \psi^\dagger\psi \; \mathrm{d}^3x,$$

where the integral extends over some 3-dimensional spacelike hypersurface. [4]

Similarly show that invariance of the corresponding action under infinitesimal translations $x'^\mu = x^\mu + a^\mu$ implies a conserved field 4-momentum P^ν given by

$$P^\nu = \mathrm{i}\int \psi^\dagger \partial^\nu \psi \; \mathrm{d}^3x.$$

[8]

Show further that invariance of the action under infinitesimal homogeneous Lorentz transformations $x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$, with $\omega_{\mu\nu} = -\omega_{\nu\mu}$, implies a conserved field angular momentum tensor $M^{\nu\lambda}$ given by

$$M^{\nu\lambda} = \mathrm{i}\int \psi^\dagger(x^\nu\partial^\lambda - x^\lambda\partial^\nu + \mathrm{S}^{\nu\lambda})\psi \; \mathrm{d}^3x,$$

where $\mathrm{S}^{\nu\lambda}$ are the Lorentz group generator matrices for Dirac spinors. [8]

Show that the Dirac Lagrangian density $\mathcal{L}$ is not, in general, a real number, but that the real Lagrangian density $\widehat{\mathcal{L}} \equiv \frac{1}{2}(\mathcal{L} + \mathcal{L}^\dagger)$ is equivalent to $\mathcal{L}$. [8]

[*Noether's theorem states that if an action is invariant under the simultaneous infinitesimal transformations $\Phi'^a(x) = \Phi^a(x) + \delta\Phi^a(x)$ and $x'^\mu = x^\mu + \xi^\mu(x)$ of the fields and coordinates respectively, then $\partial_\mu j^\mu = 0$, where*

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\delta\Phi^a + \mathcal{L}\xi^\mu.$$

]

3 Write brief notes on **three** of the following:

(a) representations of the Lorentz group; [10]

(b) interactions between fields and the gauge principle; [10]

(c) the equivalence principle and local inertial coordinates; [10]

(d) Kruskal coordinates and the maximally-extended Schwarzschild solution. [10]

END OF PAPER

NATURAL SCIENCES TRIPOS Part III

Wednesday 16 January 2008 11:30am to 1:00pm

EXPERIMENTAL AND THEORETICAL PHYSICS (8)
Quantum Field Theory

Answer **two** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right hand margin where appropriate. The paper contains THREE sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

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SPECIAL REQUIREMENTS
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1 A real scalar field, ϕ, is governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2,$$

where symbols have their usual meanings.

(a) Write down the canonical momentum $\pi(\boldsymbol{x})$. After quantization, what commutations relations should one impose on $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ in the Schrödinger picture? [4]

(b) In the Heisenberg picture, the mode expansion of $\phi(x)$ and $\pi(x)$ is given by

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}\,\mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger_{\boldsymbol{p}}\,\mathrm{e}^{+\mathrm{i}p\cdot x}\right)$$

$$\pi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}(-\mathrm{i})\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} - a^\dagger_{\boldsymbol{p}}\mathrm{e}^{+\mathrm{i}p\cdot x}\right)$$

where $p^2 = m^2$ and $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. Show that $\phi(x)$ satisfies the Klein–Gordon equation. [4]

(c) The creation and annihilation operators $a^\dagger_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}$ satisfy the commutation relations

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \quad , \quad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = [a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = 0.$$

Show that these imply equal-time commutation relations for $\phi(\boldsymbol{x},t)$ and $\pi(\boldsymbol{x},t)$. [6]

(d) Express the normal-ordered Hamiltonian in terms of creation and annihilation operators, and show that

$$[H, a_{\boldsymbol{p}}] = -E_{\boldsymbol{p}}a_{\boldsymbol{p}} \quad \text{and} \quad [H, a^\dagger_{\boldsymbol{p}}] = E_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}.$$

[10]

(e) Define the vacuum and explain the particle interpretation of the theory. Why are the particles bosons? [6]

2 The Dirac Lagrangian density is given by

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi$$

where the gamma matrices obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,1_4$, and other symbols have their usual meanings.

(a) Derive the Dirac equation from this Lagrangian, and show that each individual component of the Dirac spinor obeys the Klein–Gordon equation. [4]

(b) Explain briefly the importance of the matrices $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. [2]

(c) Define γ^5. Show how this may be used to define projection operators onto left-handed and right-handed spinors for a general basis of gamma matrices, and explain why this projection is Lorentz invariant. [8]

(d) A particular representation of the gamma matrices, known as the Majorana representation, is given by:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad ; \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix};$$
$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad ; \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}.$$

What are the projection operators in this basis? [3]

(e) The action of parity on a Dirac spinor is given by

$$P : \psi(\boldsymbol{x}, t) \to \gamma^0 \psi(-\boldsymbol{x}, t).$$

Show that parity exchanges left-handed and right-handed spinors. How do the following fermi bilinears transform under parity:

(i) $\bar{\psi}\psi(\boldsymbol{x}, t)$;
(ii) $\bar{\psi}\gamma^5\psi(\boldsymbol{x}, t)$;
(iii) $\bar{\psi}\gamma^\mu\psi(\boldsymbol{x}, t)$;
(iv) $\bar{\psi}\gamma^5\gamma^\mu\psi(\boldsymbol{x}, t)$. [6]

(f) Show that if $\psi(\boldsymbol{x}, t)$ obeys the Dirac equation, then so does the parity-transformed spinor $\gamma^0\psi(-\boldsymbol{x}, t)$. [7]

3 Write an essay on anti-particles in quantum field theory, including a discussion of the difference between relativistic and non-relativistic fields, the difference between real and complex scalar fields, and a comparison of quantum field theory with Dirac's hole interpretation. [30]

END OF PAPER