UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part III

January 2012

EXPERIMENTAL AND THEORETICAL PHYSICS
Major Topics: Number (Paper title)

ANSWERS

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1 **Bookwork:** The QCD Hamiltonian does not depend on flavour. Since $m_u \approx m_d$ the dynamic part of the overall Hamiltonian does not distinguish between flavours. The symmetry is only broken by the relatively small QED terms. The symmetry corresponds to SU(()2) rotations in isospin space where

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ [3]

Bookwork: The isospin states formed from three quarks can be obtained by adding an up or down quark to the two quark isospin singlet and triplet states. Of the six combinations formed from the two-quark triplet, the extreme states, ddd and uuu, can be immediately identified as the $I_3 = -3/2$ and $I_3 = +3/2$ states with total isospin $I = 3/2$. The $I_3 = -1/2$ can be obtained from

$$\begin{aligned} T_+\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle &= T_+(\mathrm{ddd}) = [T_+\mathrm{d}]\mathrm{dd} + \mathrm{d}[T_+\mathrm{d}]\mathrm{d} + \mathrm{dd}[T_+\mathrm{d}] \\ \sqrt{3}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle &= \mathrm{udd} + \mathrm{dud} + \mathrm{ddu} \end{aligned}$$

hence

$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}(\mathrm{udd} + \mathrm{dud} + \mathrm{ddu})\,.$$

The $I_3 = -1/2$ state with total isospin $I = 1/2$, is the linear combination of ddu and $\frac{1}{\sqrt{2}}(\mathrm{ud} + \mathrm{du})\mathrm{d}$ which is orthogonal to the $\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle$. The six states built from the qq triplet are

$$\begin{aligned} \left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle &= \mathrm{ddd} \\ \left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(\mathrm{udd} + \mathrm{dud} + \mathrm{ddu}) \\ \left|\tfrac{3}{2},+\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(\mathrm{uud} + \mathrm{udu} + \mathrm{duu}) \\ \left|\tfrac{3}{2},+\tfrac{3}{2}\right\rangle &= \mathrm{uuu} \\ \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_S &= -\tfrac{1}{\sqrt{6}}(2\mathrm{ddu} - \mathrm{udd} - \mathrm{dud}) \\ \left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_S &= \tfrac{1}{\sqrt{6}}(2\mathrm{uud} - \mathrm{udu} - \mathrm{duu}) \end{aligned}$$

and the two states from the qq isospin singlet are simply

$$\begin{aligned} \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_A &= \tfrac{1}{\sqrt{2}}(\mathrm{udd} - \mathrm{dud}) \\ \left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_A &= \tfrac{1}{\sqrt{2}}(\mathrm{udu} - \mathrm{duu})\,. \end{aligned}$$

[4]

Unseen problem: Since the overall wavefunction must be anti-symmetric and the spatial part of the wavefunction is symmetric ($L = 0$) then the spin × flavour part must be anti-symmetric. Since spin-half has the same $SU(2)$ algebra as isospin the possible spin-half states are a mixed symmetry doublet which is symmetric under $1 \leftrightarrow 2$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_S = -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$
$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_S = \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow),$$

and a mixed symmetry doublet which is anti-symmetric under $1 \leftrightarrow 2$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_A = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$
$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_A = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow).$$

An overall antisymmetric wave function can be obtained from a linear combination of $\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_S\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_A$ and $\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_A\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_S$, i.e.

$$\psi = \alpha\tfrac{1}{\sqrt{12}}(2\mathrm{uud} - \mathrm{udu} - \mathrm{duu})(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \beta\tfrac{1}{\sqrt{12}}(\mathrm{udu} - \mathrm{duu})(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$\begin{aligned}
\sqrt{12}\psi =& 2\alpha\,\mathrm{u}\!\uparrow\mathrm{u}\!\downarrow\mathrm{d}\!\uparrow - 2\alpha\mathrm{u}\!\downarrow\mathrm{u}\!\uparrow\mathrm{d}\!\uparrow - \alpha\,\mathrm{u}\!\uparrow\mathrm{d}\!\downarrow\mathrm{u}\!\uparrow + \alpha\,\mathrm{u}\!\downarrow\mathrm{d}\!\uparrow\mathrm{u}\!\uparrow - \alpha\,\mathrm{d}\!\uparrow\mathrm{u}\!\downarrow\mathrm{u}\!\uparrow + \alpha\,\mathrm{d}\!\downarrow\mathrm{u}\!\uparrow\mathrm{u}\!\uparrow\\
&+ 2\beta\,\mathrm{u}\!\uparrow\mathrm{d}\!\uparrow\mathrm{u}\!\downarrow - 2\beta\,\mathrm{d}\!\uparrow\mathrm{u}\!\uparrow\mathrm{u}\!\downarrow - \beta\,\mathrm{u}\!\uparrow\mathrm{d}\!\downarrow\mathrm{u}\!\uparrow + \beta\,\mathrm{d}\!\uparrow\mathrm{u}\!\downarrow\mathrm{u}\!\uparrow - \beta\,\mathrm{u}\!\downarrow\mathrm{d}\!\uparrow\mathrm{u}\!\uparrow + \beta\,\mathrm{d}\!\downarrow\mathrm{u}\!\uparrow\mathrm{u}\!\uparrow)\\
=& 2\alpha\,\mathrm{u}\!\uparrow\mathrm{u}\!\downarrow\mathrm{d}\!\uparrow - 2\alpha\mathrm{u}\!\downarrow\mathrm{u}\!\uparrow\mathrm{d}\!\uparrow - (\alpha+\beta)\,\mathrm{u}\!\uparrow\mathrm{d}\!\downarrow\mathrm{u}\!\uparrow + (\alpha+\beta)\,\mathrm{d}\!\downarrow\mathrm{u}\!\uparrow\mathrm{u}\!\uparrow\\
&+ 2\beta\,\mathrm{u}\!\uparrow\mathrm{d}\!\uparrow\mathrm{u}\!\downarrow - 2\beta\,\mathrm{d}\!\uparrow\mathrm{u}\!\uparrow\mathrm{u}\!\downarrow - (\alpha-\beta)\,\mathrm{d}\!\uparrow\mathrm{u}\!\downarrow\mathrm{u}\!\uparrow + (\alpha-\beta)\,\mathrm{u}\!\downarrow\mathrm{d}\!\uparrow\mathrm{u}\!\uparrow
\end{aligned}$$

There are many ways to see that $\beta = -\alpha$, e.g. this is the only way that the term with $\mathrm{d}\!\downarrow\mathrm{u}\!\uparrow\mathrm{u}\!\uparrow$ vanishes as it must for the wavefunction to be antisymmetric under $2 \leftrightarrow 3$. Hence

$$\psi = \tfrac{1}{\sqrt{6}}(\mathrm{u}\!\uparrow\mathrm{u}\!\downarrow\mathrm{d}\!\uparrow - \mathrm{u}\!\downarrow\mathrm{u}\!\uparrow\mathrm{d}\!\uparrow - \mathrm{u}\!\uparrow\mathrm{d}\!\uparrow\mathrm{u}\!\downarrow + \mathrm{d}\!\uparrow\mathrm{u}\!\uparrow\mathrm{u}\!\downarrow - \mathrm{d}\!\uparrow\mathrm{u}\!\downarrow\mathrm{u}\!\uparrow + \mathrm{u}\!\downarrow\mathrm{d}\!\uparrow\mathrm{u}\!\uparrow)$$

[7]

Unseen Problem: From above it is clear that a totally anti-symmetric octet can still be formed. In addition an overall anti-symmetric state can be constructed from the singlet flavour state and a totally symmetric spin state. Hence the observed states are a

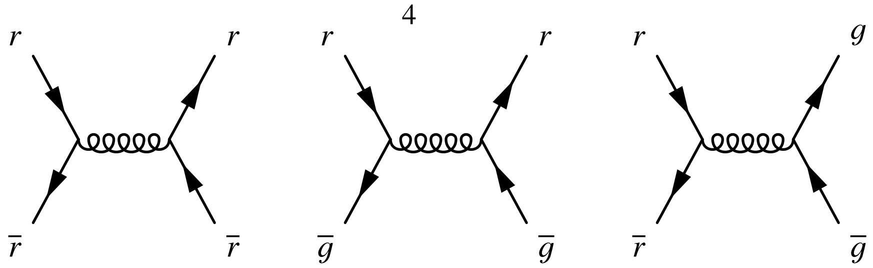
$$J^P = \frac{3}{2}^+ \quad \text{Singlet}$$

and a

$$J^P = \frac{1}{2}^+ \quad \text{Octet}$$

[4]

Unseen Problem (example of qq $\to$ qq **in lectures):**



[3]

Unseen Problem (example of qq $\to$ qq **in lectures):** For $r\overline{r} \to r\overline{r}$ the colour factor is

$$
\begin{aligned}
C(r\overline{r} \to r\overline{r}) &= \frac{1}{4}(\lambda^1_{11}\lambda^1_{11} + \lambda^3_{11}\lambda^3_{11}) \\
&= \frac{1}{4}\left(1 + \frac{1}{3}\right) \\
&= +\frac{1}{3}
\end{aligned}
$$

For $r\overline{g} \to r\overline{g}$ the colour factor is

$$
\begin{aligned}
C(r\overline{g} \to r\overline{g}) &= \frac{1}{4}(\lambda^3_{11}\lambda^3_{22} + \lambda^3_{11}\lambda^3_{22}) \\
&= \frac{1}{4}\left(-\frac{2}{3}\right) \\
&= -\frac{1}{6}
\end{aligned}
$$

For $r\overline{r} \to g\overline{g}$ the colour factor is

$$
\begin{aligned}
C(r\overline{r} \to g\overline{g}) &= \frac{1}{4}(\lambda^1_{21}\lambda^1_{12} + \lambda^2_{21}\lambda^2_{12}) \\
&= \frac{1}{4}(1 + 1) \\
&= +\frac{1}{2}
\end{aligned}
$$

Averaging over the nine intial state colours gives the overall colour factor

$$
\begin{aligned}
\langle |C|^2 \rangle &= \frac{1}{9}[3C(r\overline{r} \to r\overline{r})^2 + 6C(r\overline{g} \to r\overline{g})^2 + 6C(r\overline{r} \to g\overline{g})^2] \\
&= \frac{1}{9}\left(\frac{3}{9} + \frac{6}{36} + \frac{6}{4}\right) \\
&= +\frac{2}{9}
\end{aligned}
$$

[9]

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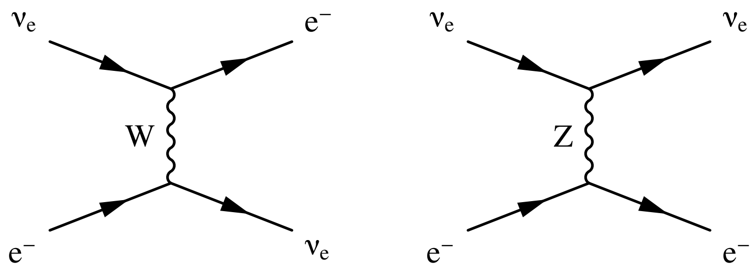
2 Answer 2

Bookwork



[2]

Bookwork: since $P_L = \frac{1}{2}(1-\gamma^5)$ projects out left-handed chiral states, which in the ultra-relativistic limit are equivalent to left-handed helicity states,

$$\overline{u}(p'_e)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(p_\nu) = \overline{u}(p'_e)\gamma^\mu u_\downarrow(p_\nu).$$

There are many ways to proceed. The most elegant is to insert P_L and move it onto the adjoint spinor

$$\begin{aligned} u^\dagger(p'_e)\gamma^0\gamma^\mu u_\downarrow(p_\nu) &= u^\dagger(p'_e)\gamma^0\gamma^\mu P_L u_\downarrow(p_\nu) \\ &= u^\dagger(p'_e)P_L\gamma^0\gamma^\mu u_\downarrow(p_\nu) \\ &= [P_L u(p'_e)]^\dagger\gamma^0\gamma^\mu[\tfrac{1}{2}(1+\gamma^5)]u_\downarrow(p_\nu) \\ &= \overline{u}_\downarrow(p'_e)\gamma^\mu u_\downarrow(p_\nu) \end{aligned}$$

[5]

Problem mostly covered in lectures In the centre-of-mass frame, the initial-state neutrino has $(\theta,\phi) = (0,0)$, the initial state electron has $(\theta,\phi) = (\pi,\pi)$, the final state electron has $(\theta,\phi) = (\theta,0)$ and the final state neutrino has $(\theta,\phi) = (\pi-\theta,\pi)$. Hence the spinors are

$$u_\downarrow(p_\nu) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, u_\downarrow(p_e) = \sqrt{E}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u_\downarrow(p'_e) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix} \quad \text{and} \quad u_\downarrow(p'_\nu) = \sqrt{E}\begin{pmatrix} -c \\ -s \\ c \\ s \end{pmatrix}.$$

Hence the two currents are

$$j_1^\mu = \overline{u}_\downarrow(p'_e)\gamma^\mu u_\downarrow(p_\nu) \quad \text{and} \quad j_2^\rho = \overline{u}_\downarrow(p'_\nu)\gamma^\rho u_\downarrow(p_e)\,.$$

Which, using the relations for spinor products, have components

$$\begin{aligned} j_1^0 &= E(c+c) = 2Ec \\ j_1^1 &= E(s+s) = 2Es \\ j_1^2 &= -iE(+s+s) = -2Eis \\ j_1^3 &= E(c+c) = 2Ec \end{aligned}$$

$$j_1 = 2E(c, s, -is, c)$$

and

$$\begin{aligned} j_2^0 &= E(c+c) = 2Ec \\ j_2^1 &= E(-s-s) = -2Es \\ j_2^2 &= -iE(s+s) = -2Eis \\ j_2^3 &= E(-c-c) = -2Ec \end{aligned}$$

$$j_2 = 2E(c, s, -is, -c)$$

Hence

$$j_1 \cdot j_2 = 4E^2(c^2 + s^2 + s^2 + c^2) = 2\sqrt{s}$$

and the matrix element

$$\begin{aligned} M_{fi} &= 2\sqrt{s}\frac{g_{\mathrm{W}}^2}{2m_{\mathrm{W}}^2} \\ \langle |M_{fi}|^2 \rangle &= \frac{1}{2}s^2\left(\frac{g_{\mathrm{W}}^2}{2m_{\mathrm{W}}^2}\right)^2 , \end{aligned}$$

where the half comes from the spin average over the two electron states.

Since there is no angular dependence

$$\begin{aligned} \sigma &= \frac{1}{16\pi s}\langle |M_{fi}|^2 \rangle = \frac{1}{32\pi s}s^2\left(\frac{g_{\mathrm{W}}^2}{2m_{\mathrm{W}}^2}\right)^2 , \\ &= \frac{s}{32\pi}\left(\frac{8G_{\mathrm{F}}}{\sqrt{2}}\right)^2 \\ &= \frac{G_{\mathrm{F}}^2 s}{\pi} \end{aligned}$$

[13]

Unseen Problem: The centre of mass energy $s = (E_\nu + m_{\mathrm{e}})^2 - E_\nu^2 \approx 2m_{\mathrm{e}}E_\nu$ giving a cross section

$$\begin{aligned}
\sigma &= \frac{2G_F^2 E_\nu m_e}{\pi} \\
&= 2 \times (1.166 \times 10^{-5})^2 \times 0.01 \times 5.11 \times 10^{-4}/\pi \\
&= 4.41 \times 10^{-16}\,\mathrm{GeV}^{-2} \\
&= 4.41 \times 10^{-16} \times \left(\frac{\hbar c}{1.6 \times 10^{-10}}\right)^2 \\
&= 1.74 \times 10^{-47}\,\mathrm{m}^2
\end{aligned}$$

[4]

Unseen Problem: Rate = flux × cross section × number of electrons

$$\text{Rate} = 1.74 \times 10^{-47} \times 5 \times 10^{10} \times N_e$$

The mass of SK water volume is 5×10^7 kg, of which a fraction 10/18 is in the form protons. Hence the total number of electrons, which equals the total number of protons is

$$N_e = \frac{10}{18} \times 5 \times 10^7/1.67 \times 10^{-27} \approx 2 \times 10^{34}\,.$$

Hence the overall rate is

$$\text{Rate} \approx 2 \times 10^{-47} \times 5 \times 10^{10} \times 2 \times 10^{34} \approx 0.01\,\mathrm{s}^{-1}$$

Hence the interaction rate is of order hundreds per day.

[4]

Description: Brief outline of Čerenkov radiation. Large background from β-decay. In centre-of-mass frame the electron is produced isotropically, when boosted back into the laboratory frame electron tends to be produced in the direction of the intitial-state neutrino, allows neutrino interactions to be distinguished from background.

[3]

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3 Answer 3

(a) Strangeness oscillations; [10]

Answer could include:

- Neutral kaon strong eigenstates $K^0(d\bar{s})$ and $\overline{K}^0(d\bar{d})$.
- mix because of weak box diagrams (expect Feynman diagrams).
- propagate as combined eigenstates of the strong and weak interaction.
- If CP is conserved propagate as CP eigenstates $K_1 = 1/\sqrt{2}(K^0 - \overline{K}^0)$ and $K_2 = 1/\sqrt{2}(K^0 + \overline{K}^0)$.
- If CP is conserved decays $K_1 \to \pi\pi$ and $K_2 \to \pi\pi\pi$.
- Phase space (+centrifugal barrier) leads to suppression of $\pi\pi\pi$ decays; K_1 relatively short lived
- Physical eigenstates are short/long lived K_S and K_L
- In absence of CP violation are equivalent to K_1 and K_2, if not mixture given by ϵ
- States produced in the strong interaction are K^0 and $\overline{K}^0$
- e.g. $\psi(t = 0) = K^0 = 1/\sqrt{2}(K_S + K_L)$
- time dependence has decaying part $\theta(t)_{S/L} = \exp(-im_{S/L}t - \Gamma_{S/L}t)$
- due to mass difference phase difference develops and $K^0 \to \overline{K}^0$ oscillations occur
- outline of derivation (with or without CP violation)
- oscillations damped by fast decay rate of K_S component, approximately same time as oscillation rate.
- oscillations observed by leptonic decays which tag K^0 or $\overline{K}^0$, e.g $K^0 \to \pi^-\mu^+\nu_\mu$.
- CPLEAR experiment measures production and decay

(b) Electron-proton deep inelastic scattering; [10]
Answer could include:

- •Elastic - proton remains intact
- •Virtual photon interacts with proton as a whole (i.e. coherently)
- •Only one independent variable - scattering angle fully determines kinematics, i.e. ($x = 1$)
- •Rutherford scattering is non-relativistic recoilless limit
- •Mott scattering electron relativistic, no recoil.
- •Both Mott and Rutherford scattering purely electric interaction
- •Charge distribution described by form factor
- •Form factor is FT of charge distribution
- •At relativistic energies with proton recoil Rosenbluth formula
- •Both electric term and magnetic term
- •Experimentally Ge and Gm show that magnetic and electric distributions are the same using anomalous magnetic moment of
- •Proton has rms radius of 1 fm
- •Discussion of experimental measurement of GM and GE from angular dependence
- •High energy measure GM
- •Due to form factor elastic scattering cross-section falls away rapidly with q^2.

(c) The PMNS matrix and its determination; [10]

Answer could include:

- Neutrino interact as weak eigenstates and propagate as mass eigenstates
- Related by unitary PMNS matrix
- Can be parameterised by four parameters, θ_{12}, θ_{13}, θ_{23} and the CP violating phase δ.
- Mass differences lead to neutrino oscillations which allow PMNS elements to be measured
- Oscillation equation
- Oscillations occur over two wavelengths, atmospheric and solar corresponding to $\Delta m^2 21$ and $\Delta m^2 32$.
- θ_{32} measured in atmospheric and beam neutrino oscillations (MINOS)
- Brief discussion of MINOS
- θ_{32} mainly $\nu_\mu \to \nu_\tau$. Due to threshold purely disappearance.
- θ_{21} measured from solar neutrino experiments and KAMLand
- Discussion of SNO experiment which gives strongest constraints on θ_{21} but interpretation complicated by MSW effect.
- θ_{13} hard to measure as small.
- Measured in reactor $\overline{\nu}_e \to \overline{\nu}_e$ experiments
- Brief discussion of CHOOZ
- CHOOZ results compatible with no oscillations and only limit set
- Hints from recent experiments of non-zero value
- δ not measurable in current generation of experiments need high intensity $P(\nu_\mu \to \nu_e)$ vs $P(\overline{\nu}_\mu \to \overline{\nu}_e)$
- measurements hard because θ_{13} is small.

(d) The Dirac equation and its solutions; [10]

Answer could include:

- •Need relativistic QM for particle physics
- •S.E. first order in time derivatives, second order in spatial derivatives and therefore not L.I.
- •K.G. equation leads negative energy solutions plus negative probability densities (ok in QFT)
- •Dirac proposed additional linear requirement $\frac{\partial\psi}{\partial t} = (\alpha.\nabla + m\beta)\psi$
- •Requires that the α and β be 4×4 matrices
- •Usually represented by gamma matrices
- •Gives positive and negative energy solutions
- •Predicts anti-particles
- •Feynman-Stuckleberg interpretation
- •Most useful basis states are helicity eigenstates
- •In ultra-relativistic limit helicity eigenstates correspond to chiral eigenstates, i.e. eigenstates of γ^5
- •Gauge principle determines the interaction between fermions and vector bosons.
- •Vector interaction described by four-vector current, $\overline{\phi}\gamma^{\mu}\psi$
- •In ultra-relativistic limit chiral nature of interaction term allows only certain combinations of helicities
- •C and P operators can be derived by considering transformation properties of the Dirac equation

END OF PAPER