1 Things I forget to show the students

Dr C. G. Lester, Peterhouse 2011

1.1 Why you should be careful with your complex number manipulations

Complex numbers are well founded, and can always be manipulated safely. However, it is easy to be sloppy when manipulating them. For example, there is an error in the following argument but it is hard to spot:

$$\begin{aligned} 1 &= \sqrt{1} && \text{hopefully this is obvious} \\ &= \sqrt{(-1)(-1)} && \text{since } 1 = (-1)(-1) \\ &= \sqrt{-1}\sqrt{-1} && \text{since } (ab)^c = a^c b^c \\ &= ii && \text{by definition of } i \\ &= i^2 && \text{since } aa = a^2 \\ &= -1 && \text{since } i \text{ is the square root of } -1 \end{aligned} \tag{1}$$

Here is a variation on the same bad argument which avoids use of any explicit "i"s:

$$\begin{aligned} 1 &= \sqrt{1} && \text{hopefully this is obvious} \\ &= \sqrt{(-1)(-1)} && \text{since } 1 = (-1)(-1) \\ &= \sqrt{-1}\sqrt{-1} && \text{since } (ab)^c = a^c b^c \\ &= (\sqrt{-1})^2 && \text{since } aa = a^2 \\ &= ((-1)^{\frac{1}{2}})^2 && \text{since } \sqrt{a} = a^{\frac{1}{2}} \\ &= (-1)^1 && \text{since } (a^b)^c = a^{bc} \\ &= -1 && \text{since } a^1 = a \text{ for all } a. \end{aligned} \tag{2}$$

1.2 Why the difference between conditional- and absolute-convergence is important

Add up all the numbers in the table below and what do you get? It appears to be "0" or "-2" depending on whether you add up the column totals or the row totals! Why?

Row and column totals	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$\dots$	column totals sum to "-2"
0	-1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\dots$	
0	0	-1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\dots$	
0	0	0	-1	$\frac{1}{2}$	$\frac{1}{4}$	$\dots$	
0	0	0	0	-1	$\frac{1}{2}$	$\dots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
row totals sum to "0"							

(3)

1.3 Why you should be careful when changing the order of integration

Define $f(x, y)$ by

$$f(x,y) = \frac{2(x-y)}{((x-y)^2+1)^2} \tag{4}$$

and then compare

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty} f(x,y)dxdy = +\frac{\pi}{2} \tag{5}$$

with

$$\int_{x=0}^{\infty}\int_{y=0}^{\infty} f(x,y)dydx = -\frac{\pi}{2} \tag{6}$$

noting that the integrand is well behaved for all values of x and y.

For a more interesting example that proves that it's not just the sign that might change when you reverse integration limits, try

$$h(x,y) = \begin{cases} e^{-(y-x)} & \text{if } y > x \\ -2e^{2(y-x)} & \text{otherwise,} \end{cases} \tag{7}$$

and compare

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty} h(x,y)dxdy = -1 \tag{8}$$

with

$$\int_{x=0}^{\infty}\int_{y=0}^{\infty} h(x,y)dydx = \frac{1}{2}. \tag{9}$$

Determining the reason for the discrepancy is left as an excercise for the reader. [Hint: consider *convergence* versus *absolute convergence* for series.]

1.4 Second derivatives need not commute

The function $f(x,y)$ defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise,} \end{cases}$$

has perfectly valid (different!) mixed second derivatives

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = +1$$

and

$$\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = -1$$

which you may verify from first principles even though a naive calculation (only valid away from the origin) might suggest that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = A(x,y) \equiv \frac{(x^2-y^2)(x^4+10x^2y^2+y^4)}{(x^2+y^2)^3}$$

leading the careless mathematician to conclude falsely either that (a) mixed second derivatives of f always commute, being equal to $A(x,y)$ or (b) that mixed second derivatives of f do not exist at $(0,0)$ at all since $A(x,y)$ is not smooth there. [$\lim_{x\to 0} A(x,y) = -1$ and $\lim_{y\to 0} A(x,y) = +1$.]

See also the Wikipedia page on "Symmetry of second derivatives".

1.5 Why you should not extrapolate based on the first few terms of a sequence

$$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2} \tag{10}$$

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin \frac{x}{3}}{\frac{x}{3}} = \frac{\pi}{2} \tag{11}$$

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin \frac{x}{3}}{\frac{x}{3}} \cdot \frac{\sin \frac{x}{5}}{\frac{x}{5}} = \frac{\pi}{2} \tag{12}$$

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin \frac{x}{3}}{\frac{x}{3}} \cdot \frac{\sin \frac{x}{5}}{\frac{x}{5}} \cdot \frac{\sin \frac{x}{7}}{\frac{x}{7}} = \frac{\pi}{2} \tag{13}$$

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin \frac{x}{3}}{\frac{x}{3}} \cdot \frac{\sin \frac{x}{5}}{\frac{x}{5}} \cdot \frac{\sin \frac{x}{7}}{\frac{x}{7}} \cdot \frac{\sin \frac{x}{9}}{\frac{x}{9}} = \frac{\pi}{2} \tag{14}$$

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin \frac{x}{3}}{\frac{x}{3}} \cdot \frac{\sin \frac{x}{5}}{\frac{x}{5}} \cdot \frac{\sin \frac{x}{7}}{\frac{x}{7}} \cdot \frac{\sin \frac{x}{9}}{\frac{x}{9}} \cdot \frac{\sin \frac{x}{11}}{\frac{x}{11}} = \frac{\pi}{2} \tag{15}$$

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin \frac{x}{3}}{\frac{x}{3}} \cdot \frac{\sin \frac{x}{5}}{\frac{x}{5}} \cdot \frac{\sin \frac{x}{7}}{\frac{x}{7}} \cdot \frac{\sin \frac{x}{9}}{\frac{x}{9}} \cdot \frac{\sin \frac{x}{11}}{\frac{x}{11}} \cdot \frac{\sin \frac{x}{13}}{\frac{x}{13}} = \frac{\pi}{2} \tag{16}$$

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin \frac{x}{3}}{\frac{x}{3}} \cdot \frac{\sin \frac{x}{5}}{\frac{x}{5}} \cdot \frac{\sin \frac{x}{7}}{\frac{x}{7}} \cdot \frac{\sin \frac{x}{9}}{\frac{x}{9}} \cdot \frac{\sin \frac{x}{11}}{\frac{x}{11}} \cdot \frac{\sin \frac{x}{13}}{\frac{x}{13}} \cdot \frac{\sin \frac{x}{15}}{\frac{x}{15}} = \tag{17}$$

$$\frac{467807924713440738696537864469\pi}{935615849440640907310521750000} \tag{18}$$

All equalities above are *exact*. Note that the last fraction is *approximately*

$$0.99999999999852937186\frac{\pi}{2}.$$

1.6 Why you should be careful with Lagrange Multipliers

We can see that the extremum of

$$f = x^2 + y^2 + z^2 + mx$$

subject to the constraint

$$x^2 + y^2 = 0$$

is just zero, since the constraint simply implies that $x = y = 0$ and so f reduces to $f_c = z^2$ which has a global minimum of zero at $x = y = z = 0$. Note that an "unthinking" application of the method of Lagrange multipliers fails: If you define

$$\mathcal{L} = x^2 + y^2 + z^2 + mx - \lambda(x^2 + y^2)$$

then the Euler-Lagrange equation corresponding to x is:

$$2x + m - 2\lambda x = 0$$

and it is clear that this equation is NOT satisfied by the desired solution x = y = z = 0 (unless we happen to be in the special case where m = 0).[1] This is

[1]Note, however, that this argument presupposes λ is well defined. If the condition is perturbed my a small amount ϵ (see later) then λ may be seen to grow as an inverse power of ϵ – in effect saying that λ needs to go to infinity as the constraint becomes un-perturbed.

due to the gradient of the constraint function being a null vector at the place where the constraint is satisfied. It is not clear to me how in general one avoids getting caught in this trap when the maths is more obscure (as it frequently is in real problems). Is it sufficient to simply take grad of any constraint and check for inequality with the null vector at all places satisfying the constraint? Is it acceptable to perturb the constraint from null - eg by replacing the constraint with $x^2 + y^2 = \epsilon^2$?

1.7 Cauchy Riemann Equations

Strictly this section need not be here, since we never fail to get on to the Cauchy-Riemann equations at some point in MNST. However (without fail) this is one part of the course where I always manage to waste time by managing to derive them in a silly way. This section is to remind me what I really want to say, namely:

$$x = \frac{1}{2}(+z + \bar{z})$$

and

$$y = \frac{i}{2}(-z + \bar{z})$$

allowing us to note that:

$$\begin{aligned}
0 &= \frac{\partial f(z)}{\partial \bar{z}} && (19)\\
&= \frac{\partial (u+iv)}{\partial \bar{z}} && (20)\\
&= \frac{\partial (u+iv)}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial (u+iv)}{\partial y}\frac{\partial y}{\partial \bar{z}} && (21)\\
&= \frac{\partial (u+iv)}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial (u+iv)}{\partial y}\left(\frac{i}{2}\right) && (22)\\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) && (23)
\end{aligned}$$

and therefore, since u, v, x and y are all real, we must have

$$+\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}$$

and

$$+\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

1.8 L'Hopital's rule

If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = a$ and [$a = 0$ or $a = \pm\infty$] and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$.

Your attention is drawn to the last requirement above. It is important, as the example $f(x) = x + \sin x$ and $g(x) = x$ with $x \to \infty$ illustrates.