Recent four- and five-loop results in QCD

Andreas Vogt (University of Liverpool)

with Joshua Davies* (UoL), Franz Herzog, Ben Ruijl°, Takahiro Ueda,
Jos Vermaseren (Nikhef) and Sven Moch (Hamburg University)

- Introduction: hard processes in perturbative QCD, diagram technology
- Five-loop results for the beta function and hadronic Higgs decays
- Four-loop calculations and structural features of splitting functions P
- Non-singlet: analytic (large-n_c) and numerical results for $P_{\rm ns}$ & $\gamma_{\rm cusp}$

* now at KIT, Karlsruhe ° now at ETH, Zurich

Four-loop References

Technology: Forcer & Form (latest version)

B. Ruijl, T. Ueda and J.A.M Vermaseren, arXiv:1704.06650 & arXiv:1707.06453

Results, up to now

B. Ruijl, T. Ueda, J.A.M. Vermaseren, J. Davies and A. Vogt,
First Forcer results on deep-inelastic scattering and related quantities,
arXiv:1605.08408, PoS LL 2016 (2016) 071

J. Davies, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt,
Large-n_f contributions to the four-loop splitting functions in QCD,
arXiv:1610.07477, Nucl. Phys. B915 (2017) 335

S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt,
Four-loop non-singlet splitting functions in the planar limit and beyond,
arXiv:1707.08315, JHEP 10 (2017) 041

Five-loop References

Technology: local R*-operation

F. Herzog and B. Ruijl, arXiv:1707.03776, JHEP 05 (2017) 037

Results, up to now

F. Herzog, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt,
The five-loop beta function of Yang-Mills theory with fermions,
arXiv:1701.01404, JHEP 02 (2017) 090

F. Herzog, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt,
On Higgs decays to hadrons and the R-ratio at N^{4}LO,
arXiv:1707.01044, JHEP 08 (2017) 113

More on the math. structure at four & five loops

Jamin, Miravitllas ('17),

J. Davies and A. Vogt
Absence of π^2 terms in physical anomalous dimensions in DIS
arXiv:1711.05267, Phys. Lett. B776 (2018) 189

Hard processes in perturbative QCD

Observables in ep and pp scattering, schematically, up to power corrections

$$O^{ep} = f_i \otimes c_i^{\mathrm{o}} \; , \quad O^{pp} = f_i \otimes f_k \otimes c_{ik}^{\mathrm{o}}$$

c^{o}: coefficient functions, scale $\mu \leftrightarrow$ physical hard scale, e.g., M_{Higgs}

Parton densities f_i : renormalization-group evolution ($\otimes$: Mellin convolution)

$$\frac{\partial}{\partial \ln \mu^2} f_i(\xi, \mu^2) \;=\; \left[P_{ik}(\alpha_{\mathsf{S}}(\mu^2)) \otimes f_k(\mu^2) \right](\xi)$$

Splitting functions ($\Leftrightarrow$ twist-2 anomalous dimensions) & coefficient funct's:

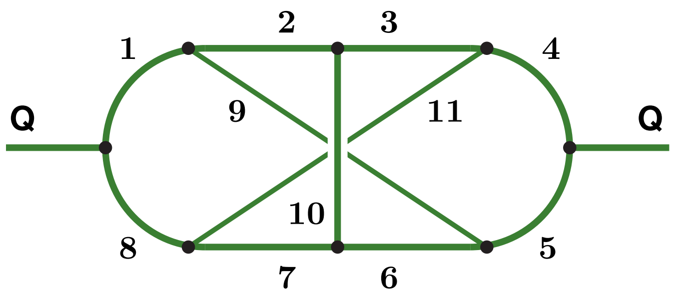
$$\begin{aligned} P \;&=\; \alpha_{\mathsf{S}}\, P^{(0)} + \alpha_{\mathsf{S}}^2\, P^{(1)} + \alpha_{\mathsf{S}}^3\, P^{(2)} + \alpha_{\mathsf{S}}^4\, P^{(3)} + \ldots \\ c_a \;&=\; \underbrace{\alpha_{\mathsf{S}}^{n_a} \Big[c_a^{(0)} + \alpha_{\mathsf{S}}\, c_a^{(1)} + \alpha_{\mathsf{S}}^2\, c_a^{(2)}}_{\text{NNLO: now the standard approximation for many processes}} + \alpha_{\mathsf{S}}^3\, c_a^{(3)} + \ldots \Big] \end{aligned}$$

N$^{n>2}$LO: for high precision (α_{S} from DIS); slow convergence (Higgs in pp)

. . . , Anzai et al ('15); Anastasiou, Duhr, Dulat, Herzog, Mistlberger ('15); Dreyer, Karlberg ('16); . . .

Diagram technology: FORCER

4-loop massless propagator integrals. Hard(est) example: topology no2



$$\int d^D p_1\, d^D p_2\, d^D p_3\, d^D p_4\, \frac{(2Q\cdot p_2)^{n_{12}}(2p_1\cdot p_4)^{n_{13}}(2Q\cdot p_3)^{n_{14}}}{(p_1^2)^{n_1}\ \ldots\ (p_{11}^2)^{n_{11}}}$$

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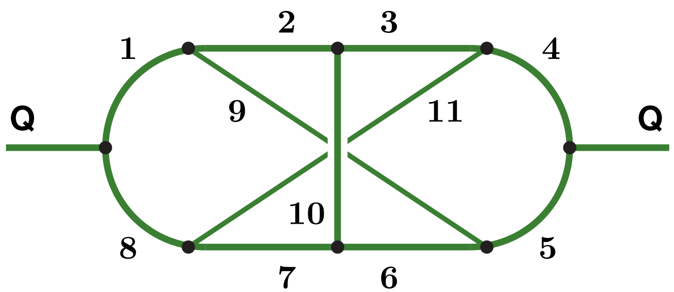


$$\int d^D p_1 \, d^D p_2 \, d^D p_3 \, d^D p_4 \, \frac{(2Q\cdot p_2)^{n_{12}} (2p_1\cdot p_4)^{n_{13}} (2Q\cdot p_3)^{n_{14}}}{(p_1^2)^{n_1} \, \ldots \, (p_{11}^2)^{n_{11}}}$$

Using (combinations of) IBP identities: symbolic lowering of $n_1, \ldots, n_{14}$
$\Rightarrow$ **simpler topologies or no2 master integral,** $n_{1,\ldots,11} = 1, \; n_{12,13,14} = 0$

438 topologies, 21 need such 'hand-built' reductions. Most of the code: generated using a PYTHON script. Exact in $\varepsilon = \frac{1}{2}(4-D)$ **up to masters**

Ruijl, Ueda, Vermaseren (mainly 2014/15)

Gorishnii et al. ('89), Larin, Tkachov, Vermaseren ('91), ...: MINCER, 3-loop analogue

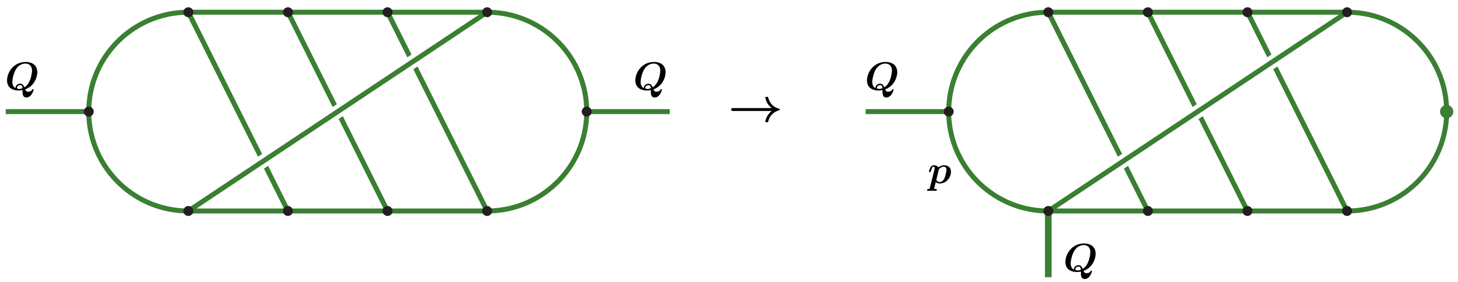
4-loop beta function, background field, $\xi = 0$**: < 3 min. with 16 modern cores**

Diagram technology: IRR, R* operation

Self-energy $1/\varepsilon^n$ pole parts K: overall UV divergence Δ + subdivergences

Log-divergent diagrams: Δ independent of external momenta and masses

$\Rightarrow$ infrared rearrangement (IRR) into simpler computable cases, e.g.,



Integral over p can be performed ('carpet rule') $\Rightarrow$ 4-loop diagrams, FORCER

Easy to say . . .

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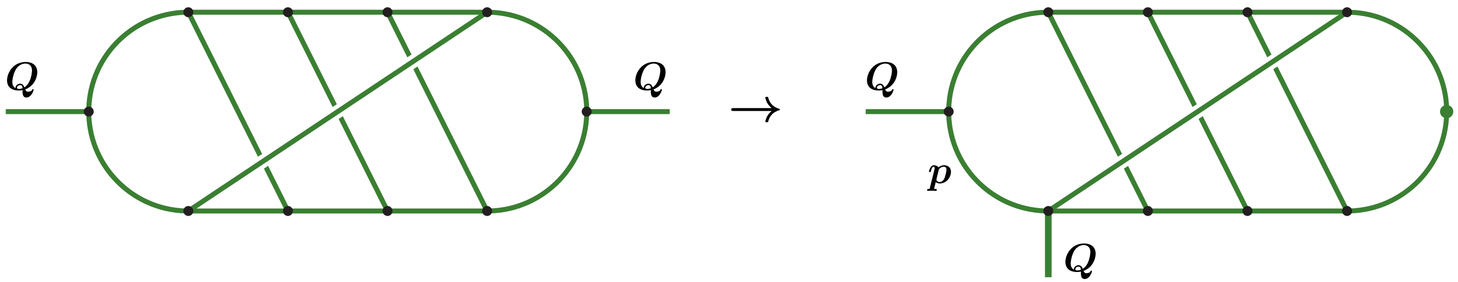

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Easy to say . . .

MANY nested subdivergences, high tensor integrals: recursive R* operation

Chetyrkin, Tkachov ('82); Chetyrkin, Smirnov ('84); . . . ; Herzog, Ruijl (2016/17)

$H \to gg$ complication: quartically divergent, 4$^{\rm th}$-order Taylor exp. at $Q = 0$
$\Rightarrow$ 'explosion' of terms, with complicated numerator structures

The five-loop beta function (I)

$$da_s/d\ln\mu^2 = -\beta_0\,a_s^2 - \beta_1\,a_s^3 - \beta_2\,a_s^4 - \beta_3\,a_s^5 - \beta_4\,a_s^6 - \ldots$$

. . . , N^{3}LO: van Ritbergen, Vermaseren, Larin ('97); Czakon ('04); . . .

FORCER validation ('15): 'all methods', all powers of ξ (and ε).

β_4, minimally: 5-loop to ε^{-1}, ξ^0, 4-loop to ε^0, ξ^1, 3-loop to ε^1, ξ^2 etc for h (q) and g propagator + hhg (qqg) vertex, or background-field g propagator

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$\widetilde{\beta} \equiv -\beta(a_s)/(a_s^2\beta_0)$ in QCD, numerically, re-expanded in $\alpha_s = 4\pi\, a_s$

$$\begin{aligned}
\widetilde{\beta}(\alpha_s, n_f=3) &= 1 + 0.565884\,\alpha_s + 0.453014\,\alpha_s^2 + 0.676967\,\alpha_s^3 + 0.580928\,\alpha_s^4 \\
\widetilde{\beta}(\alpha_s, n_f=4) &= 1 + 0.490197\,\alpha_s + 0.308790\,\alpha_s^2 + 0.485901\,\alpha_s^3 + 0.280601\,\alpha_s^4 \\
\widetilde{\beta}(\alpha_s, n_f=5) &= 1 + 0.401347\,\alpha_s + 0.149427\,\alpha_s^2 + 0.317223\,\alpha_s^3 + 0.080921\,\alpha_s^4 \\
\widetilde{\beta}(\alpha_s, n_f=6) &= 1 + 0.295573\,\alpha_s - 0.029401\,\alpha_s^2 + 0.177980\,\alpha_s^3 + 0.001555\,\alpha_s^4
\end{aligned}$$

SU(3): Baikov, Chetyrkin, Kühn ('16); general: HRUVV (01/17); Luthe et al & Chetyrkin et al (09/17)

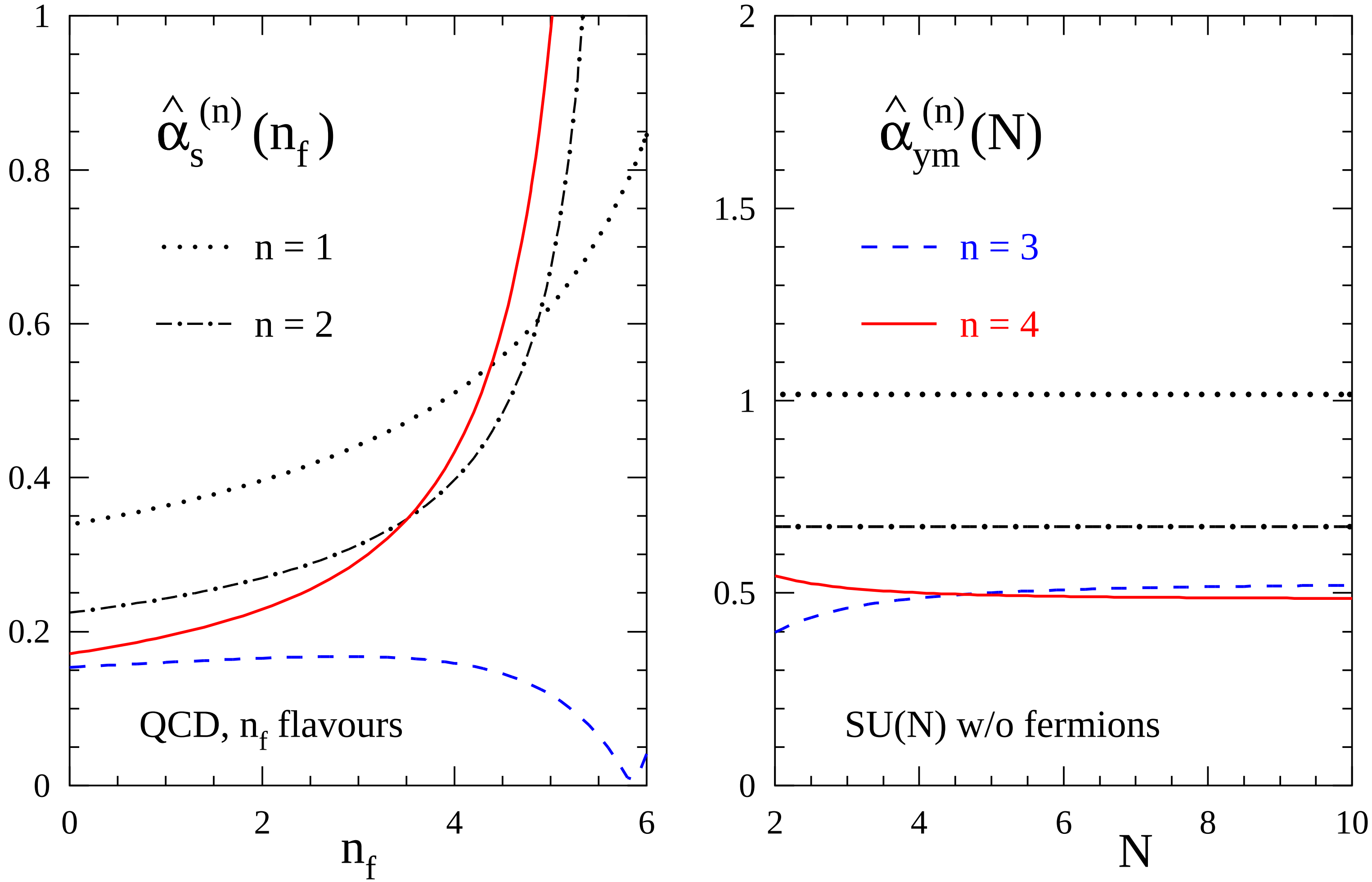
No sign so far of a possible divergence of the perturbation series for $\beta(\alpha_s)$

The five-loop beta function (II)

$$\begin{aligned}
\boldsymbol{\beta_4} \;=\;& C_A^5 \left[\frac{8296235}{3888} - \frac{1630}{81}\zeta_3 + \frac{121}{6}\zeta_4 - \frac{1045}{9}\zeta_5\right] - d_{AA}^{(4)}/N_A\, C_A \left[\frac{514}{3} - \frac{18716}{3}\zeta_3 + 968\zeta_4 + \frac{15400}{3}\zeta_5\right] \\
&- C_A^4 T_F n_f \left[\frac{5048959}{972} - \frac{10505}{81}\zeta_3 + \frac{583}{3}\zeta_4 - 1230\zeta_5\right] + C_A^3 C_F T_F n_f \left[\frac{8141995}{1944} + 146\zeta_3 + \frac{902}{3}\zeta_4 - \frac{8720}{3}\zeta_5\right] \\
&- C_A^2 C_F^2 T_F n_f \left[\frac{548732}{81} + \frac{50581}{27}\zeta_3 + \frac{484}{3}\zeta_4 - \frac{12820}{3}\zeta_5\right] + C_A C_F^3 T_F n_f \left[3717 + \frac{5696}{3}\zeta_3 - \frac{7480}{3}\zeta_5\right] \\
&- C_F^4 T_F n_f \left[\frac{4157}{6} + 128\zeta_3\right] + d_{AA}^{(4)}/N_A\, T_F n_f \left[\frac{904}{9} - \frac{20752}{9}\zeta_3 + 352\zeta_4 + \frac{4000}{9}\zeta_5\right] \\
&+ d_{RA}^{(4)}/N_A\, C_A n_f \left[\frac{11312}{9} - \frac{127736}{9}\zeta_3 + 2288\zeta_4 + \frac{67520}{9}\zeta_5\right] - d_{RA}^{(4)}/N_A\, C_F n_f \left[320 - \frac{1280}{3}\zeta_3 - \frac{6400}{3}\zeta_5\right] \\
&+ C_A^3 T_F^2 n_f^2 \left[\frac{843067}{486} + \frac{18446}{27}\zeta_3 - \frac{104}{3}\zeta_4 - \frac{2200}{3}\zeta_5\right] + C_A^2 C_F T_F^2 n_f^2 \left[\frac{5701}{162} + \frac{26452}{27}\zeta_3 - \frac{944}{3}\zeta_4 + \frac{1600}{3}\zeta_5\right] \\
&+ C_F^2 C_A T_F^2 n_f^2 \left[\frac{31583}{18} - \frac{28628}{27}\zeta_3 + \frac{1144}{3}\zeta_4 - \frac{4400}{3}\zeta_5\right] - C_F^3 T_F^2 n_f^2 \left[\frac{5018}{9} + \frac{2144}{3}\zeta_3 - \frac{4640}{3}\zeta_5\right] \\
&- d_{RA}^{(4)}/N_A\, T_F n_f^2 \left[\frac{3680}{9} - \frac{40160}{9}\zeta_3 + 832\zeta_4 + \frac{1280}{9}\zeta_5\right] + d_{RR}^{(4)}/N_A\, C_F n_f^2 \left[\frac{4160}{3} + \frac{5120}{3}\zeta_3 - \frac{12800}{3}\zeta_5\right] \\
&- d_{RR}^{(4)}/N_A\, C_A n_f^2 \left[\frac{7184}{3} - \frac{40336}{9}\zeta_3 + 704\zeta_4 - \frac{2240}{9}\zeta_5\right] - C_A^2 T_F^3 n_f^3 \left[\frac{2077}{27} + \frac{9736}{81}\zeta_3 - \frac{112}{3}\zeta_4 - \frac{320}{9}\zeta_5\right] \\
&- C_A C_F T_F^3 n_f^3 \left[\frac{736}{81} + \frac{5680}{27}\zeta_3 - \frac{224}{3}\zeta_4\right] + d_{RR}^{(4)}/N_A\, T_F n_f^3 \left[\frac{3520}{9} - \frac{2624}{3}\zeta_3 + 256\zeta_4 + \frac{1280}{3}\zeta_5\right] \\
&- C_F^2 T_F^3 n_f^3 \left[\frac{9922}{81} - \frac{7616}{27}\zeta_3 + \frac{352}{3}\zeta_4\right] + C_A T_F^4 n_f^4 \left[\frac{916}{243} - \frac{640}{81}\zeta_3\right] - C_F T_F^4 n_f^4 \left[\frac{856}{243} + \frac{128}{27}\zeta_3\right]
\end{aligned}$$

Fewer ζ_n values than in diagrams, as before. Terms with π^2 for the first time

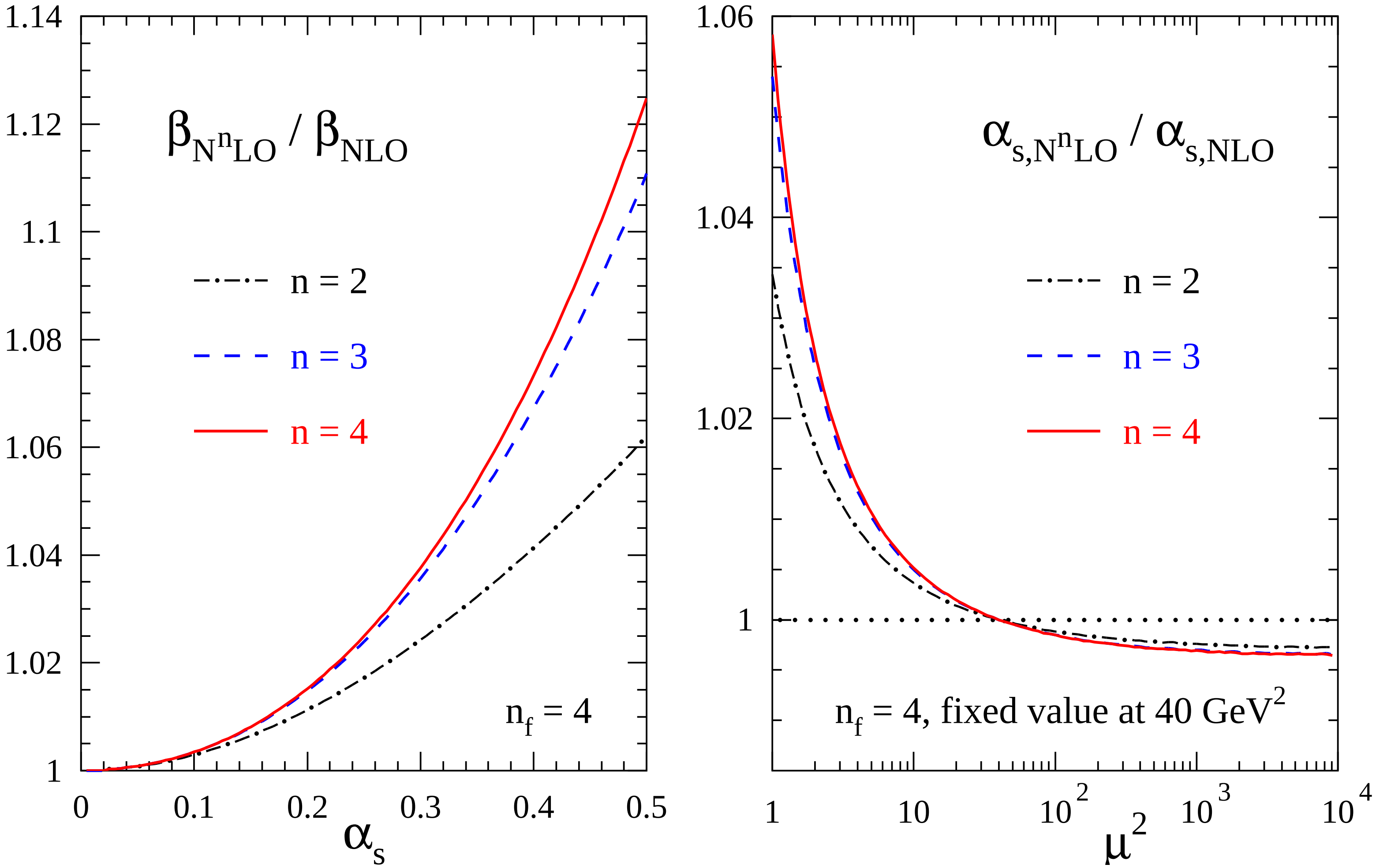
The five-loop beta function (III)



Values of α_s for which the n-th order corr. is 1/4 of that of the previous order

QCD: $\hat{\alpha}_s^{(n)}(n_f) \;=\; 4\pi \left| \frac{\beta_{n-1}(n_f)}{4\,\beta_n(n_f)} \right|$, **SU(N):** $\hat{\alpha}_{\mathrm{YM}}^{(n)}(N) \;=\; 4\pi N \left| \frac{\beta_{n-1}(N)}{4\,\beta_n(N)} \right|$

The five-loop beta function (IV)



N^4LO contributions to $\beta(\alpha_s)$: < 1% at $\alpha_s = 0.39\,(0.47)$ for $n_f = 3\,(4)$,

$n_f = 4$ effect on α_s: 0.08% (0.4%) at $\mu^2 = 3\,(1)$ GeV^2. N^3LO: 0.5% (2%)

Higgs decay to hadrons (I)

QCD, limit of large top-quark mass and n_f effectively massless flavours:

$$\Gamma_{H\to gg} \;=\; \frac{\sqrt{2}\,G_{\mathrm{F}}}{M_{\mathrm{H}}}\,|C_1|^2\,\mathrm{Im}\,\Pi^{GG}(-M_{\mathrm{H}}^2 - i\delta)$$

C_1**: Wilson coefficient (↔ decoupling of α_{S}), dep. on top mass (definition)**

. . . , N^{4}LO: Baikov, Chetyrkin, Kühn ('16)

Imaginary QCD part of Higgs-boson self energy Π^{GG} at L loops:

$$\mathrm{Im}\,\Pi(-q^2 - i\delta) \;=\; \mathrm{Im}\,e^{i\pi\varepsilon L}\,\Pi(q^2) \;=\; \sin(L\pi\varepsilon)\,\Pi(q^2)$$

⇒ **required finite part from $1/\varepsilon$ term of $\Pi^{GG}(q^2)$ at 5-loop via R^* + FORCER**

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K-factors, $\Gamma = K\Gamma_{\mathrm{Born}}$ at $\mu^2 = M_{\mathsf{H}}^2$, scale-invariant mass $\mu_{\mathsf{top}} = 164$ GeV

$$K_{\mathrm{SI}}(n_f=1) \;=\; 1 + 7.188498\,\alpha_{\mathsf{S}} + 32.65167\,\alpha_{\mathsf{S}}^2 + 112.015\,\alpha_{\mathsf{S}}^3 + 298.873\,\alpha_{\mathsf{S}}^4 + \ldots$$

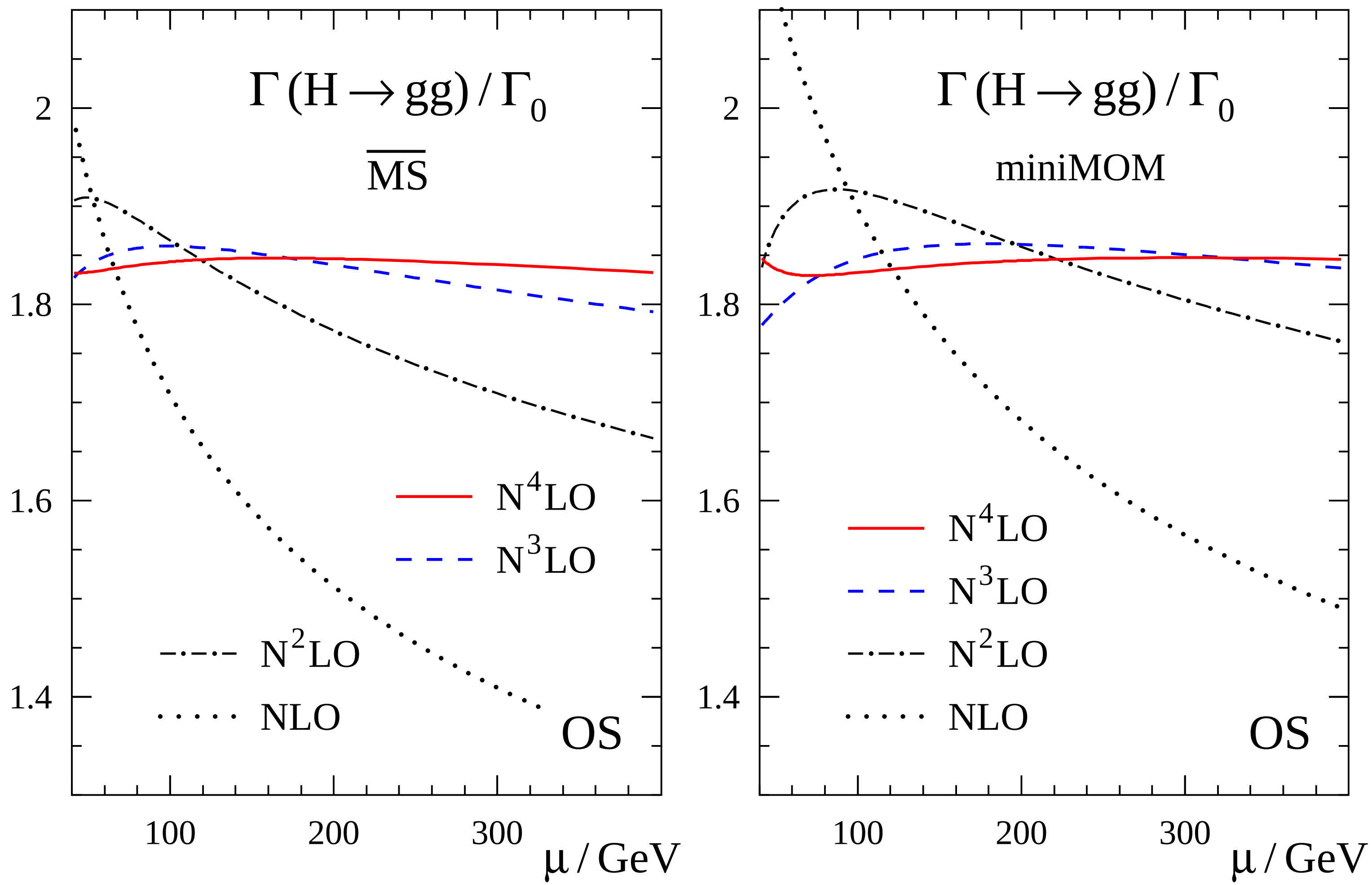
$$K_{\mathrm{SI}}(n_f=3) \;=\; 1 + 6.445775\,\alpha_{\mathsf{S}} + 23.74728\,\alpha_{\mathsf{S}}^2 + 56.0755\,\alpha_{\mathsf{S}}^3 + 62.4363\,\alpha_{\mathsf{S}}^4 + \ldots$$

$$K_{\mathrm{SI}}(n_f=5) \;=\; 1 + 5.703052\,\alpha_{\mathsf{S}} + 15.57384\,\alpha_{\mathsf{S}}^2 + 12.5520\,\alpha_{\mathsf{S}}^3 - 72.0916\,\alpha_{\mathsf{S}}^4 + \ldots$$

$$K_{\mathrm{SI}}(n_f=7) \;=\; 1 + 4.960329\,\alpha_{\mathsf{S}} + 8.131350\,\alpha_{\mathsf{S}}^2 - 19.3879\,\alpha_{\mathsf{S}}^3 - 123.853\,\alpha_{\mathsf{S}}^4 + \ldots$$

$$K_{\mathrm{SI}}(n_f=9) \;=\; 1 + 4.217606\,\alpha_{\mathsf{S}} + 1.419805\,\alpha_{\mathsf{S}}^2 - 40.5769\,\alpha_{\mathsf{S}}^3 - 110.998\,\alpha_{\mathsf{S}}^4 + \ldots$$

Higgs decay to hadrons (II)

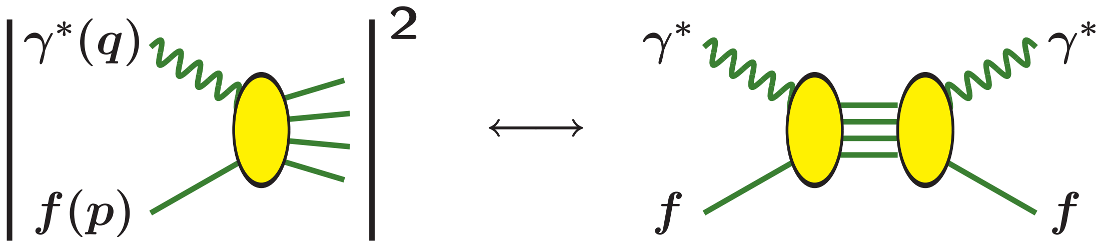


$\Gamma_0 = G_F M_H^3/(36\pi^3\sqrt{2})\,(\alpha_s(M_H))^2$. **For** $\alpha_s(M_H) = 0.11264$ **(**$\leftrightarrow 0.118@M_Z$**):**

$$\Gamma_{N^4LO}(H \to gg) = \Gamma_0\,(\,1.844 \pm 0.011_{\,\text{series}} \pm 0.045_{\,\alpha_s(M_Z),1\%})$$

Splitting functions via DIS calculations

Inclusive DIS: probe-parton total cross sections $\leftrightarrow$ forward amplitudes



$p^2 = 0, \; q^2 < 0$

Dispersion relation in x: coefficient of $(2p{\cdot}q)^N$ $\leftrightarrow$ N-th Mellin moment

$$A(N) \;=\; \int_0^1 dx \, x^{N-1} A(x)$$

Fixed N: harmonic projection $\rightarrow$ self-energy integrals $\rightarrow$ MINCER / FORCER

Projection on structure functions, $D = 4 - 2\varepsilon$ dimensions, mass factorization

ε^{-1} : splitting functions $P_{\rm ik}^{(n)}(N) = -\,\gamma_{\rm ik}^{(n)}(N)$, ϵ^0: n-loop coefficient fct's

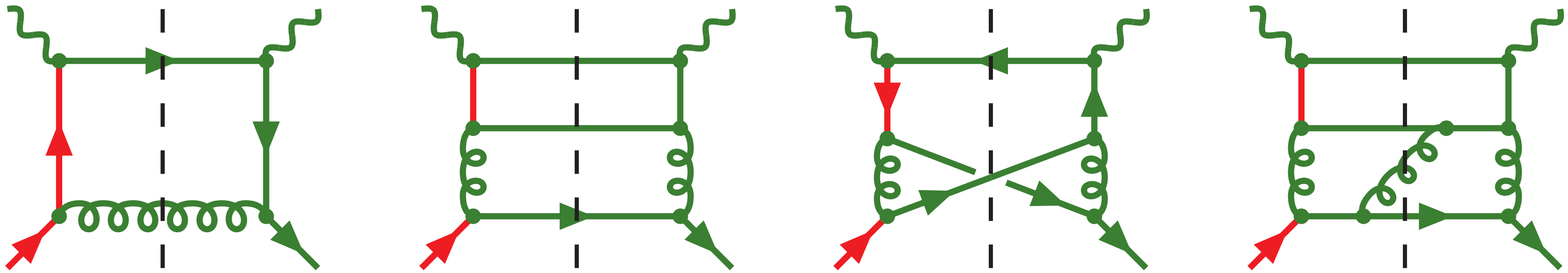
2 and 3 loops: Larin, Vermaseren ('91); Larin, van Ritbergen, Vermaseren [, Nogueira] ('93, '96); . . .

[D]RUVV ('16): 4 loops to $N \leq 6$ in general; up to $N > 40$ for high-n_f parts

Quark-quark splitting functions

General structure of the (anti-)quark (anti-)quark splitting functions

$$P_{\mathrm{q}_i\mathrm{q}_k} = P_{\bar{\mathrm{q}}_i\bar{\mathrm{q}}_k} = \delta_{ik} P^{\mathrm{v}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\mathrm{qq}} \,, \quad P_{\mathrm{q}_i\bar{\mathrm{q}}_k} = P_{\bar{\mathrm{q}}_i\mathrm{q}_k} = \delta_{ik} P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} + P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}}$$



$P^{\mathrm{v}}_{\mathrm{qq}} = \mathcal{O}(\alpha_{\mathrm{s}})$ | $P^{\mathrm{s}}_{\mathrm{qq}},\, P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} \;:\; \alpha_{\mathrm{s}}^2$ | $P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} \;:\; \alpha_{\mathrm{s}}^2$ | $P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} \neq P^{\mathrm{s}}_{\mathrm{qq}} \;:\; \alpha_{\mathrm{s}}^3$

$\Rightarrow$ **three types of independent difference (non-singlet, ns) combinations**

$2(n_f - 1)$ **flavour asymmetries of** $q_i \pm \bar{q}_i$ **+ one total valence distribution**

with

$$q^{\pm}_{\mathrm{ns},ik} = q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) \,, \quad q^{\mathrm{v}}_{\mathrm{ns}} = \sum_{r=1}^{n_f} (q_r - \bar{q}_r)$$

$$P^{\pm}_{\mathrm{ns}} = P^{\mathrm{v}}_{\mathrm{qq}} \pm P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} \,, \quad P^{\mathrm{v}}_{\mathrm{ns}} = P^{-}_{\mathrm{ns}} + P^{\mathrm{s}}_{\mathrm{ns}} \,.$$

Large n_c: $P^{+}_{\mathrm{ns}}(x) = P^{-}_{\mathrm{ns}}(x)$

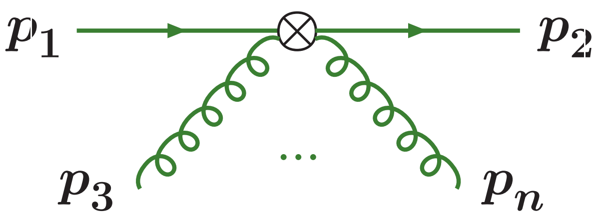
N**-space calculations (OPE, 'NIKHEF method'): even (+) or odd (−, v)** N

Splitting functions via the OPE (non-singlet)

Spin-N twist-two flavour non-singlet quark operators (symmetric, traceless)

$$O^{\rm ns}_{\{\mu_1,\ldots,\mu_N\}} = \overline{\psi}\,\lambda^\alpha\,\gamma_{\{\mu_1} D_{\mu_2} \ldots D_{\mu_N\}}\,\psi$$

ψ: quark field, $D_\mu = \partial_\mu - igA_\mu$: covariant derivative, A_μ: gluon field,
λ^α, $\alpha = 3, 8, \ldots, (n_f^2 - 1)$: diagonal generators of flavour group SU(n_f)



L loops: up to L gluons at $N > L$

Here: operator matrix elements with off-shell quarks, $p_1 = p_2 = p$, $p^2 < 0$

$$A^{\rm ns}(N) \;=\; \Delta^{\mu_1} \ldots \Delta^{\mu_N} \langle p | O^{\rm ns}_{\{\mu_1,\ldots,\mu_N\}} | p \rangle \;, \quad \Delta^2 = 0$$

Renormalization constants $Z_{\rm ns}(N)$ $\rightarrow$ **anomalous dimensions** $\gamma_{\rm ns}(N)$

2 and 3 loops: Floratos, Ross, Sachrajda ('77); Blümlein et al. [heavy quarks, $p^2 = 0$] ('09), ...

MRUVV ('17): 4 loops to $N = 16$ in general; to $N = 20$ for large-n_c parts

$N = 2, 3, 4$: Velizhanin ('12, '14); Baikov, Chetyrkin [, Kühn] (..., '15)

Towards all-N expressions (I)

If analogous to lower orders: harmonic sums S_w and simple denominators

$$\gamma_{\rm ns}^{(n)}(N) \;=\; \sum_{w=0}^{2n+1} c_{00w}\, S_w(N) \,+\, \sum_{a} \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{akw}\, D_a^{\,k}\, S_w(N) \;,$$

$D_a^k \equiv (N+a)^{-k}$. Denominators at calculated N values: $a = 0, 1$ for $\gamma_{\rm ns}^{\pm}$

Coefficients c_{00w}, c_{akw} : integer modulo low powers of $1/2$ and $1/3$

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Coefficients c_{00w}, c_{akw}: integer modulo low powers of $1/2$ and $1/3$

$\gamma_{\rm ns}(N)$: constrained by 'self-tuning' (conjecture, conformal symmetry)

$$\gamma_{\rm ns}(N) = \gamma_{\rm u}\,(N + \sigma\,\gamma_{\rm ns}(N) - \beta(a_{\sf s})/a_{\sf s}))$$

Space-like/time-like anom. dimensions: $\sigma = -1/+1$; universal kernel $\gamma_{\rm u}$: reciprocity-respecting (RR), i.e., invariant under replacement $N \to (1\!-\!N)$

Dokshitzer, Marchesini ('06); Basso, Korchemsky ('06); . . .

Non-RR parts, spacelike/timelike difference: 'inherited' from lower orders, need to find 'only' $\gamma_{\rm u}$; weight w: 2^{w-1} RR (combinations of) harmonic sums

Towards all-N expressions (II)

Limit of a large number of colours n_c: $\gamma_{\mathbf{ns}}^{+} = \gamma_{\mathbf{ns}}^{-}$, no alternating sums

$\Rightarrow$ remaining sums at $w = 1, \ldots, 7$: 1, 1, 2, 3, 5, 8, 13 $=$ Fibonacci(w)

List to $w = 9$: Velizhanin [website] ('10)

Together with powers of $\eta = [N(N+1)]^{-1}$ (RR): 87 basis funct's at $w = 7$

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List to $w = 9$: Velizhanin [website] ('10)

Together with powers of $\eta = [N(N+1)]^{-1}$ (RR): 87 basis funct's at $w = 7$

Large-N and small-x limits: more than 40 constraints. Large-N:

$$\gamma_{\rm ns}^{(n-1)}(N) = A_n \ln \widetilde{N} - B_n + N^{-1}\{C_n \ln \widetilde{N} - \widetilde{D}_n + \tfrac{1}{2} A_n\} + O(N^{-2})$$

C_n, $\widetilde{D}_n$: fixed by lower-order information   Dokshitzer, Marchesini, Salam ('05), ...

Small-x resummation: 4-loop coeff's of $x^a \ln^b x$ known for all a, $4 \leq b \leq 6$

$a = 0$ large-n_c single logs: $\gamma_{\rm ns}(N) \cdot (\gamma_{\rm ns}(N) + N - \beta(a_{\sf s})/a_{\sf s}) = O(1)$

Kirschner, Lipatov ('83), Blümlein, A,V. (95); [Davies,] Kom, A.V. ('12, '16); Velizhanin ('14)

$N \leq 18$ Diophantine eqs. $\Rightarrow$ remaining large-n_c coeff's. Check: $N = 19, 20$

LLL algorithm via www.numbertheory.org/php/axb.html (Matthews); Velizhanin ('12); MVV ('14)

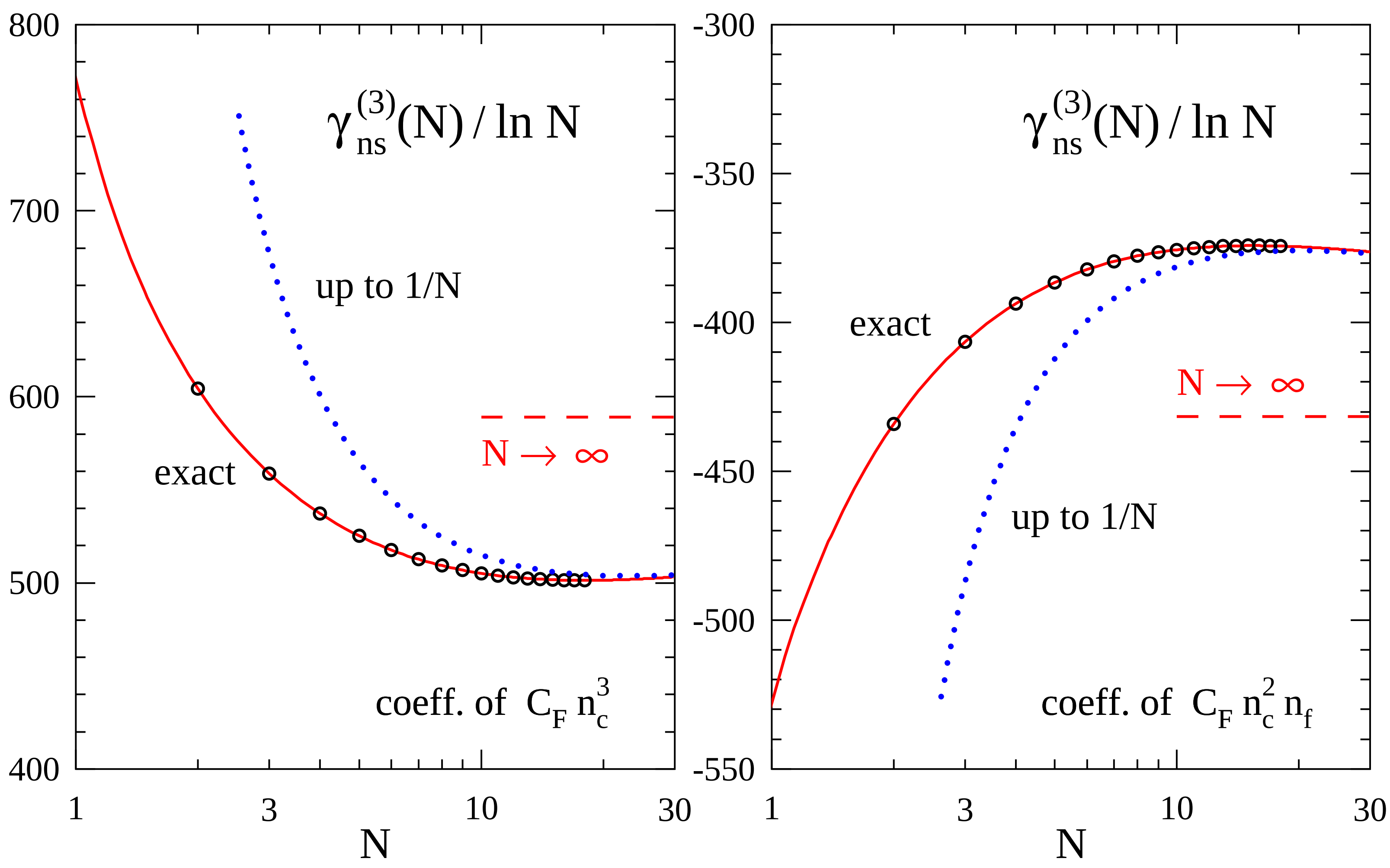
All-N anomalous dimension in the large-n_c limit (I)

$$\gamma_{\rm ns}^{(3)\pm}(N) \;=$$

$$16 C_F n_c^3 \,(\ldots - 21 S_{3,1}\eta^2 - 4 S_{3,1}\eta^3 - 4 S_{3,1}\zeta_3 - 5581/72\, S_{3,2} + 22 S_{3,2} D_1^2 - 53/3\, S_{3,2}\eta$$
$$-16 S_{3,2}\eta^2 + 143/6\, S_{3,3} - 11 S_{3,3}\eta - 14 S_{3,4} - 6899/72\, S_{4,1} + 24 S_{4,1} D_1^2 - 74/3\, S_{4,1}\eta$$
$$-11 S_{4,1}\eta^2 + 57/2\, S_{4,2} - 25 S_{4,2}\eta - 26 S_{4,3} + 63/2\, S_{5,1} - 23 S_{5,1}\eta - 36 S_{5,2} - 28 S_{6,1}$$
$$-12 S_{1,1,1} D_1^4 + 12 S_{1,1,1}\eta^2 + 24 S_{1,1,1}\eta^3 + 6 S_{1,1,1}\eta^4 + 18 S_{1,1,2}\eta^2 + 6 S_{1,1,2}\eta^3 - 20 S_{1,1,3}\eta$$
$$+8 S_{1,1,3}\eta^2 + 20/3\, S_{1,1,4} - 20 S_{1,1,4}\eta + 8 S_{1,1,5} + 18 S_{1,2,1}\eta^2 + 6 S_{1,2,1}\eta^3 + 134/3\, S_{1,2,2}$$
$$-12 S_{1,2,2} D_1^2 + 12 S_{1,2,2}\eta + 6 S_{1,2,2}\eta^2 - 22/3\, S_{1,2,3} - 6 S_{1,2,4} + 1447/18\, S_{1,3,1} - 16 S_{1,3,1} D_1^2$$
$$+104/3\, S_{1,3,1}\eta - 6 S_{1,3,1}\eta^2 - 38/3\, S_{1,3,2} + 16 S_{1,3,2}\eta + 22 S_{1,3,3} - 56/3\, S_{1,4,1} + 12 S_{1,4,1}\eta$$
$$+50 S_{1,4,2} + 46 S_{1,5,1} + 18 S_{2,1,1}\eta^2 + 6 S_{2,1,1}\eta^3 + 134/3\, S_{2,1,2} - 12 S_{2,1,2} D_1^2 + 12 S_{2,1,2}\eta$$
$$+6 S_{2,1,2}\eta^2 - 22/3\, S_{2,1,3} - 6 S_{2,1,4} + 134/3\, S_{2,2,1} - 12 S_{2,2,1} D_1^2 + 12 S_{2,2,1}\eta + 6 S_{2,2,1}\eta^2$$
$$-13 S_{2,2,2} + 6 S_{2,2,2}\eta + 12 S_{2,2,3} - 44/3\, S_{2,3,1} + 38 S_{2,3,2} + 36 S_{2,4,1} + 307/6\, S_{3,1,1}$$
$$-20 S_{3,1,1} D_1^2 + 86/3\, S_{3,1,1}\eta + 16 S_{3,1,1}\eta^2 - 43/3\, S_{3,1,2} + 10 S_{3,1,2}\eta + 14 S_{3,1,3} - 43/3\, S_{3,2,1}$$
$$+10 S_{3,2,1}\eta + 24 S_{3,2,2} + 22 S_{3,3,1} - 37/3\, S_{4,1,1} + 26 S_{4,1,1}\eta + 28 S_{4,1,2} + 28 S_{4,2,1} + 44 S_{5,1,1}$$
$$+40 S_{1,1,1,4} - 16/3\, S_{1,1,3,1} + 16 S_{1,1,3,1}\eta - 32 S_{1,1,3,2} - 24 S_{1,1,4,1} - 12 S_{1,2,2,2} - 28/3\, S_{1,3,1,1}$$
$$-16 S_{1,3,1,1}\eta - 20 S_{1,3,1,2} - 20 S_{1,3,2,1} - 52 S_{1,4,1,1} - 12 S_{2,1,2,2} - 12 S_{2,2,1,2} - 12 S_{2,2,2,1}$$
$$-36 S_{2,3,1,1} - 12 S_{3,1,1,2} - 12 S_{3,1,2,1} - 12 S_{3,2,1,1} - 12 S_{4,1,1,1} - 32 S_{1,1,1,3,1} + 32 S_{1,1,3,1,1})$$

$+$ **large-n_c terms with n_f (known) $+$ terms suppressed at large n_c (see below)**

All-N anomalous dimension in the large-n_c limit (II)



Limit $N \to \infty$: large-n_c cusp anomalous dimension A_4 **Korchemsky ('89), ...**

$\Leftrightarrow$ UV anom. dimension of vacuum average of Wilson loop with cusp, Polyakov ('80); HQET, ...

Large-N coefficients in the large-n_c limit

Cusp anomalous dimension, expansion in $a_s = \alpha_s/(4\pi)$

$$
\begin{aligned}
A_{L,4} \;=\; & C_F n_c^3 \Big(\frac{84278}{81} - \frac{88832}{81}\,\zeta_2 + \frac{20992}{27}\,\zeta_3 + 1804\,\zeta_4 - \frac{352}{3}\,\zeta_2\zeta_3 - 352\,\zeta_5 \\
& \quad -32\,\zeta_3^2 - 876\,\zeta_6 \Big) \\
& - C_F n_c^2 n_f \Big(\frac{39883}{81} - \frac{26692}{81}\,\zeta_2 + \frac{16252}{27}\,\zeta_3 + \frac{440}{3}\,\zeta_4 - \frac{256}{3}\,\zeta_2\zeta_3 - 224\,\zeta_5 \Big) \\
& + C_F n_c n_f^2 \Big(\frac{2119}{81} - \frac{608}{81}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{64}{3}\,\zeta_4 \Big) - C_F n_f^3 \Big(\frac{32}{81} - \frac{64}{27}\,\zeta_3 \Big)
\end{aligned}
$$

B. Ruijl [Seminar, Zurich, 6 Dec '16]

Agreement with result obtained from the large-n_c photon-quark form factor

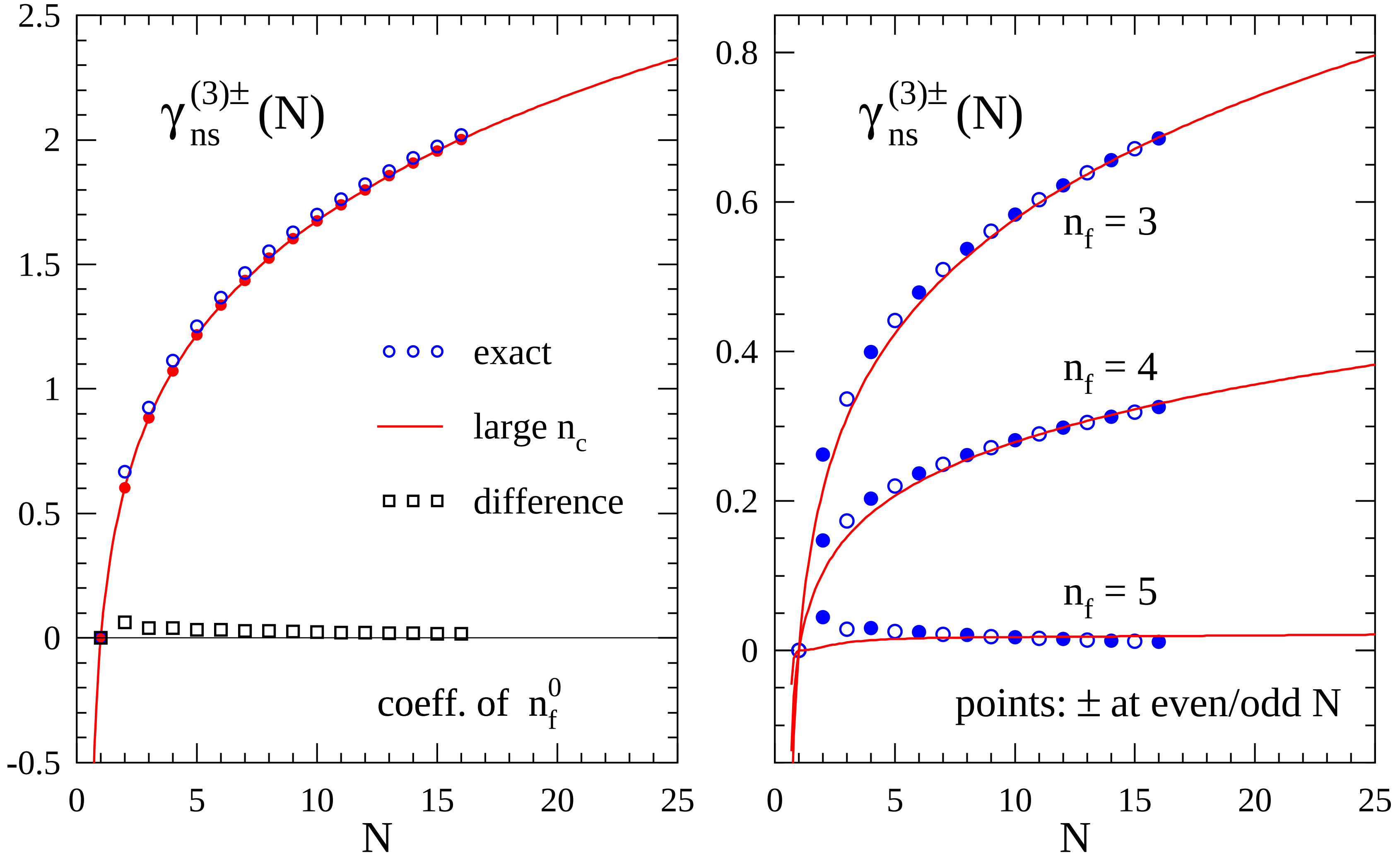
Henn, Smirnov, Smirnov, Steinhauser [, Lee] (n_f: Apr '16, n_f^0: 13 Dec '16)

ζ_3^2, ζ_6 ($\mathcal{N} = 4$ SYM): Bern, Czakon, Dixon, Kosower, Smirnov ('06); ...

Further non-trivial check of the determination of the all-N form of $\gamma_{\rm ns}(N)$

Also relevant beyond parton evolution: $\delta(1-x)$ coeff. $B_{L,4}$ ..., Dixon ('17)

$\gamma_{\rm ns}^{(3)\pm}(N)$: large-$n_c$ limit vs QCD



Cancellations between powers of n_f: non large-n_c terms relevant at low N

Approximations of large-n_c suppressed parts

Analogous, but more accurate than those used before 2004 at 3 loops

van Neerven, A.V. ('99, '00); MVV [photon structure] ('01), ...

n_f^0 and n_f^1 parts $P^{(3)+}_{\mathrm{N},0/1}$ of $P^{(3)+}_{\mathrm{ns}}(x)$: ansatz consisting of

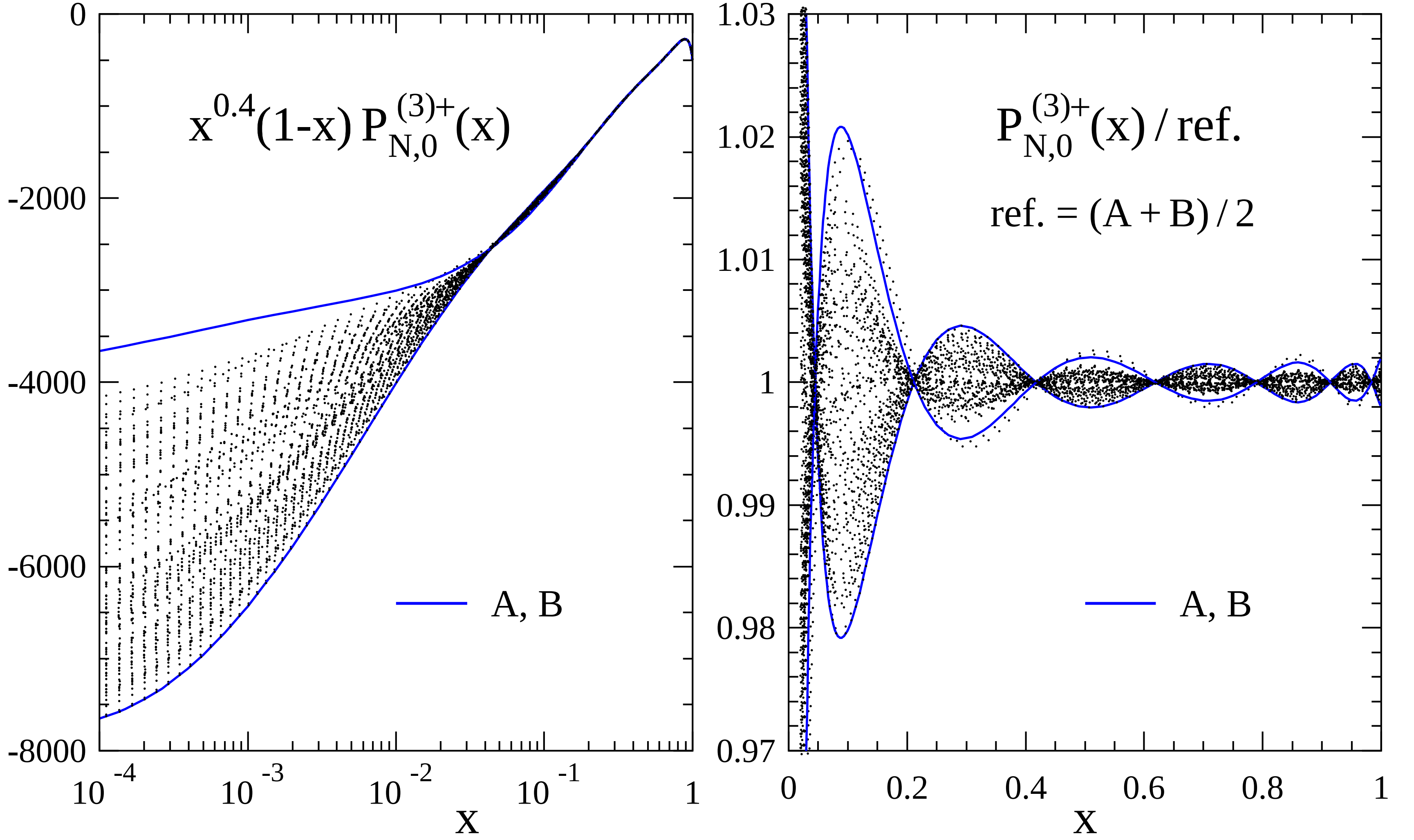
- the 2 large-x parameters A_4 and B_4
- 2 of 3 suppressed large-x logs $(1-x)\ln^k(1-x)$, $k = 1, 2, 3$
- one of ten 2-parameter polynomials in x vanishing for $x \to 1$
- 2 of the 3 unknown small-x logarithms $\ln^k x$, $k = 1, 2, 3$

90 resulting trial functions, parameters fixed from the 8 available moments, two representatives chosen that indicate the remaining uncertainty

Checks: compare same treatment for the large-n_c parts to exact results; compare $N = 18$ prediction for n_f^1 part due exact result, ...

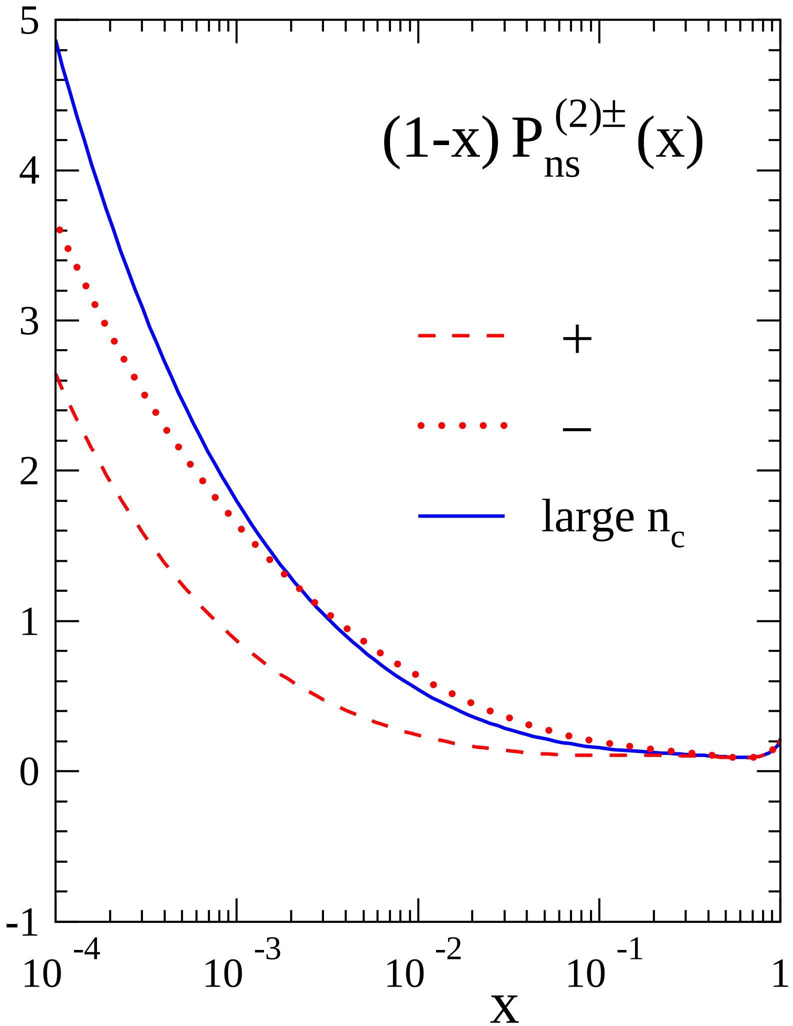
$P^{(3)-}_{\mathrm{ns}}(x)$ and $P^{(3)\mathrm{s}}_{\mathrm{ns}}(x)$: similar, but with less small-x information

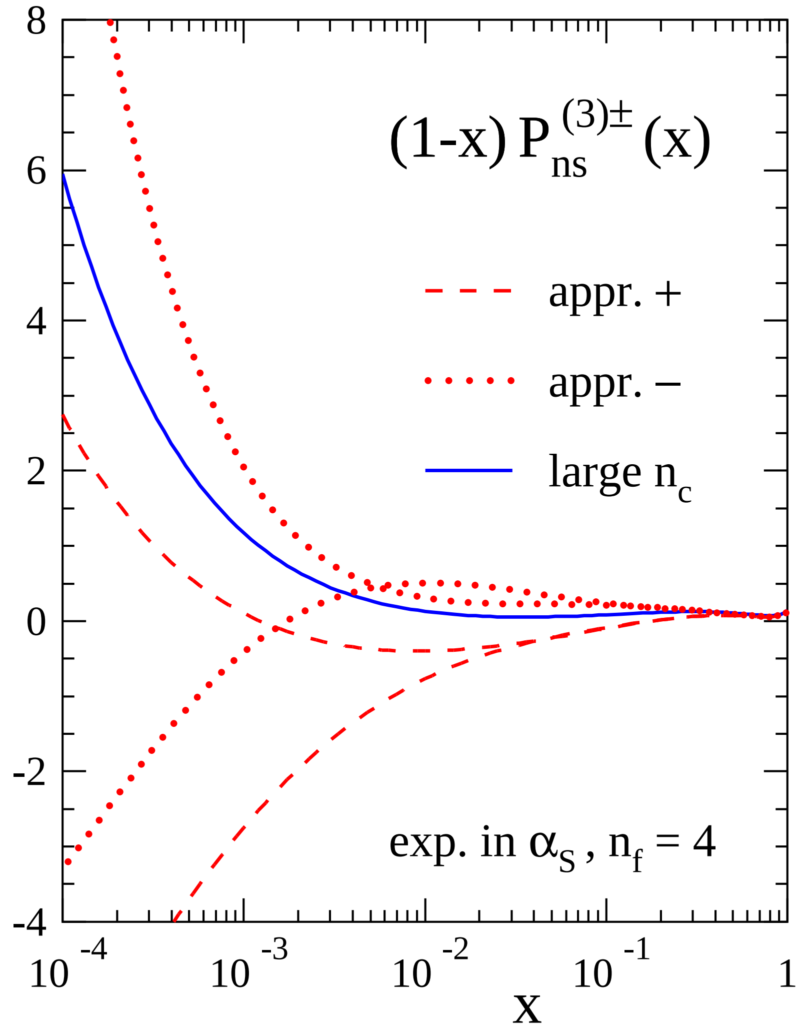
Example: approximations of the n_f^0 component



Factor $(1-x)$ $\rightarrow$ value at $x=1$: contribution to cusp anom. dimension A_4

3- and 4-loop large-n_c limits vs $P^{\pm}_{\mathbf{ns}}(x)$ in QCD





Weight-6 harmonic polylogarithms, numerically: Gehrmann, Remiddi (ext. by T.G.); Ablinger, Blümlein, Round, Schneider $\rightarrow$ compact high-accuracy parametrizations

Numerical applications: soft-gluon coefficients

4-loop cusp anom. dim. for QCD with n_f quark flavours, expansion in $\alpha_S/4\pi$

$$A_4 \;=\; 20702(2) - 5171.9(2)\, n_f + 195.5772\, n_f^2 + 3.272344\, n_f^3$$

In brackets: uncertainty of the preceeding digit, conservative estimate

$$
\begin{aligned}
A_q(\alpha_S, n_f\!=\!3) \;&=\; 0.42441\, \alpha_S\, (1 + 0.72657\, \alpha_S + 0.73405\, \alpha_S^2 + 0.6647(2)\, \alpha_S^3 + \ldots) \\
A_q(\alpha_S, n_f\!=\!4) \;&=\; 0.42441\, \alpha_S\, (1 + 0.63815\, \alpha_S + 0.50998\, \alpha_S^2 + 0.3168(2)\, \alpha_S^3 + \ldots) \\
A_q(\alpha_S, n_f\!=\!5) \;&=\; 0.42441\, \alpha_S\, (1 + 0.54973\, \alpha_S + 0.28403\, \alpha_S^2 + 0.0133(2)\, \alpha_S^3 + \ldots)
\end{aligned}
$$

$n_f = 5$: much smaller than prev. Padé estimate ($\Leftarrow$ quartic Casimirs, cf. β_3)

Corresponding coefficient of $\delta(1\!-\!x)$: similarly benign expansion

$$B_4 \;=\; 23393(10) - 5551(1)\, n_f + 193.8554\, n_f^2 + 3.014982\, n_f^3$$

$$
\begin{aligned}
B_q(\alpha_S, n_f\!=\!3) \;&=\; 0.31831\, \alpha_S\, (1 + 0.99712\, \alpha_S + 1.24116\, \alpha_S^2 + 1.0791(13)\, \alpha_S^3 + \ldots) \\
B_q(\alpha_S, n_f\!=\!4) \;&=\; 0.31831\, \alpha_S\, (1 + 0.87192\, \alpha_S + 0.97833\, \alpha_S^2 + 0.5649(13)\, \alpha_S^3 + \ldots) \\
B_q(\alpha_S, n_f\!=\!5) \;&=\; 0.31831\, \alpha_S\, (1 + 0.74672\, \alpha_S + 0.71907\, \alpha_S^2 + 0.1085(13)\, \alpha_S^3 + \ldots)
\end{aligned}
$$

A_4, B_4: individual colour factors

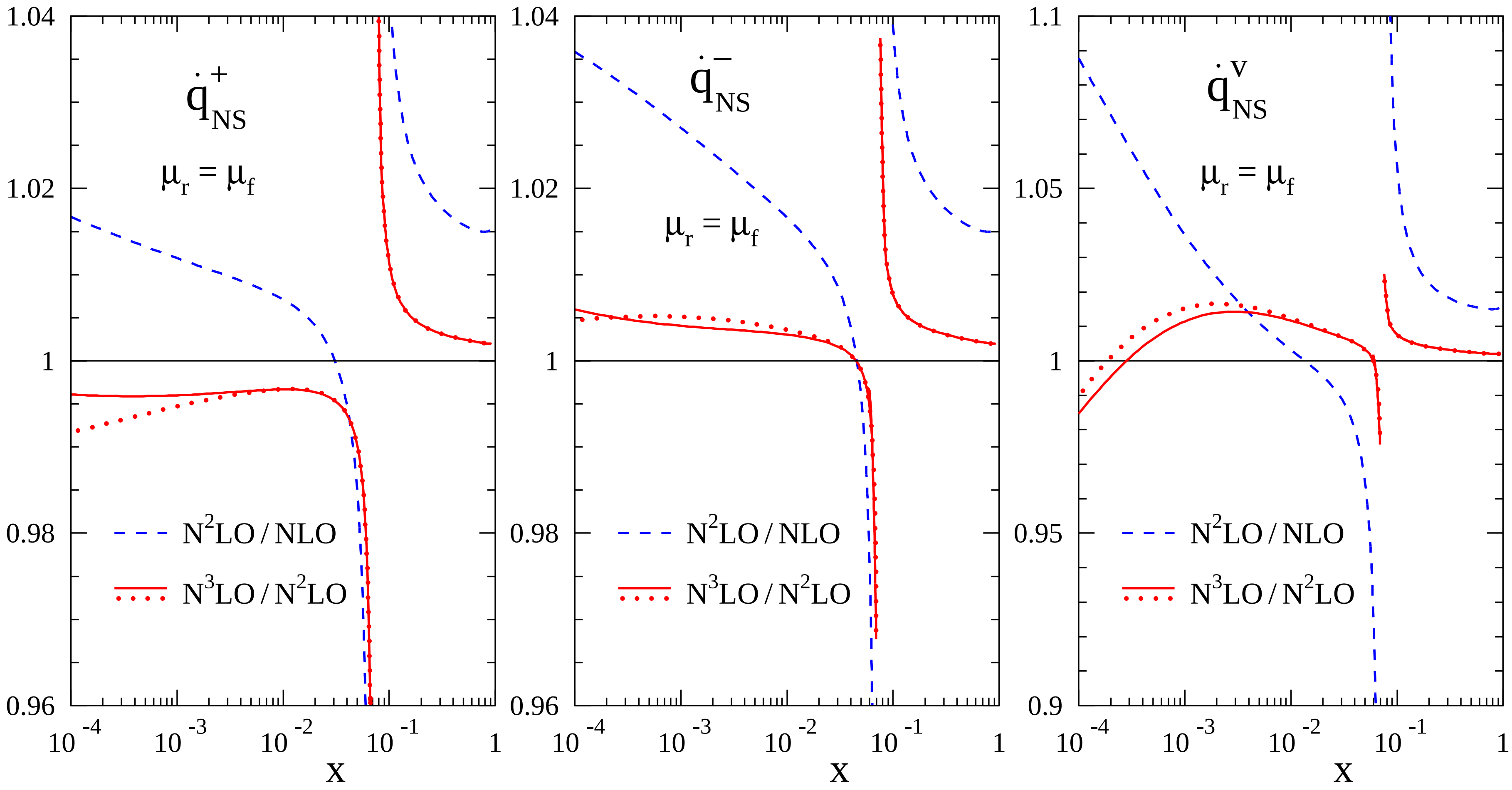
	A_4	B_4
C_F^4	0	$197. \pm 3.$
$C_F^3\, C_A$	0	$-687. \pm 10.$
$C_F^2 C_A^2$	0	$1219. \pm 12.$
$C_F C_A^3$	610.3 ± 0.3	295.6 ± 2.4
$d_R^{abcd} d_A^{abcd}/N_R$	-507.5 ± 6.0	$-996. \pm 45.$
$n_f\, C_F^3$	-31.00 ± 0.4	81.4 ± 2.2
$n_f\, C_F^2 C_A$	38.75 ± 0.2	-455.7 ± 1.1
$n_f\, C_F C_A^2$	-440.65 ± 0.2	-274.4 ± 1.1
$n_f\, d_R^{abcd} d_R^{abcd}/N_R$	-123.90 ± 0.2	-143.5 ± 1.2
$n_f^2\, C_F^2$	-21.31439	-5.775288
$n_f^2\, C_F C_A$	58.36737	51.03056
$n_f^3\, C_F$	2.454258	2.261237

Exact large-n_c limit: errors highly correlated. Quartic Casimirs: definitely $A_4 \neq 0$

see Gardi, Magnea ('09); Becher, Neubert ('09), . . . , Grozin, Henn, Korchemsky, Marquard ('15); Boels, Huber, Yang ('17); Grozin, Henn, Stahlhofen ('17)

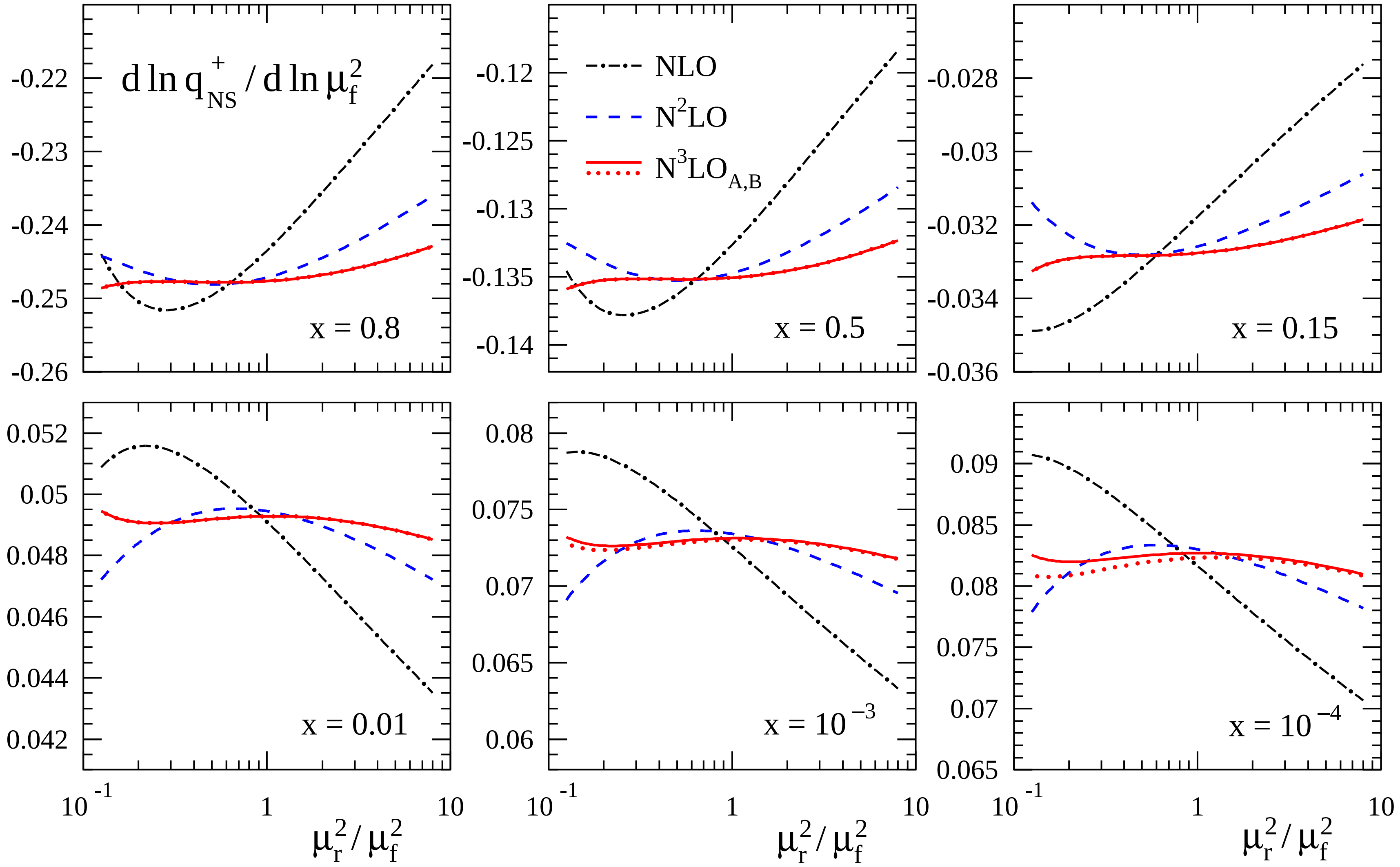
Higher-order corr's to the non-singlet evolutions

Logarithmic derivatives w.r.t. the factorization scale, $\dot{q}^{\,i}_{\rm ns} \equiv d \ln q^{\,i}_{\rm ns} / d \ln \mu_f^2$



Reference point: $x q_{\rm ns}^{\pm,v}(x,\mu_0^2) = x^{0.5}(1-x)^3$ with $\alpha_s(\mu_0^2) = 0.2$

NS$^+$ evolution: renormalization scale dependence



$\frac{1}{2}\mu_f \leq \mu_r \leq 2\mu_f$: **scale uncertainty below 1% except close to sign change**

Summary and Outlook

5-loop beta fct., $H \to gg$ (+ check of $H \to b\bar{b}$ & R-ratio in e^+e^- annihil.)

$\Rightarrow$ **uncertainty < 1% at the relevant scales from truncating the pert. series**

4-loop splitting functions/anomalous dimensions via DIS/OPE

$N = 6$ / $N = 16$ **reached for complete singlet/non-singlet results**

$\Rightarrow$ **approximate results for $P_{\rm ns}^{(3)a}(x)$ incl. cusp anomalous dimension**

Large-n_c : $P_{\rm ns}^{+} = P_{\rm ns}^{-}$, no alternating sums in $\gamma_{\rm ns}^{\pm}$, RR, endpoints

$1 \leq N \leq 20$, **LLL** $\Rightarrow$ **all-N/all-x results (construction and checks)**

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4-loop splitting functions/anomalous dimensions via DIS/OPE

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$\Rightarrow$ **approximate results for $P_{\rm ns}^{(3)a}(x)$ incl. cusp anomalous dimension**

Large-n_c : $P_{\rm ns}^{+} = P_{\rm ns}^{-}$, no alternating sums in $\gamma_{\rm ns}^{\pm}$, RR, endpoints

$1 \leq N \leq 20$, **LLL $\Rightarrow$ all-N/all-x results (construction and checks)**

4-loop singlet: high-N sufficient for approximations only via OPE:

theoretically and computationally much more challenging

2-loop ($p^2 < 0$): Floratos, Ross, Sachradja ('78), . . . ; Matiounine, Smith, van Neerven ('98)

5-loop non-singlet: (very - for now) low N possible via local R*

Check of the setup, with ξ^1: $\gamma_{\rm ns}^{(4)-}(N = 1)$. First result: $\gamma_{\rm ns}^{(4)+}(N = 2)$

5-loop non-singlet anomalous dimension at $N = 2$

$$\gamma_{\rm ns}^{(4)+}(N=2) = C_F^5\left[\frac{9306376}{19683} - \frac{802784}{729}\zeta_3 - \frac{557440}{81}\zeta_5 + \frac{12544}{9}\zeta_3^2 + 8512\,\zeta_7\right]$$
$$- C_A C_F^4\left[\frac{81862744}{19683} - \frac{1600592}{243}\zeta_3 + \frac{59840}{81}\zeta_4 - \frac{142240}{27}\zeta_5 + 3072\,\zeta_3^2 - \frac{35200}{9}\zeta_6 + 19936\,\zeta_7\right]$$
$$+ C_A^2 C_F^3\left[\frac{63340406}{6561} - \frac{1003192}{243}\zeta_3 - \frac{229472}{81}\zeta_4 + \frac{61696}{27}\zeta_5 + \frac{30976}{9}\zeta_3^2 - \frac{35200}{9}\zeta_6 + 15680\,\zeta_7\right]$$
$$- C_A^3 C_F^2\left[\frac{220224724}{19683} + \frac{4115536}{729}\zeta_3 - \frac{170968}{27}\zeta_4 - \frac{3640624}{243}\zeta_5 + \frac{70400}{27}\zeta_3^2 + \frac{123200}{27}\zeta_6 + \frac{331856}{27}\zeta_7\right]$$
$$+ C_A^4 C_F\left[\frac{266532611}{39366} + \frac{2588144}{729}\zeta_3 - \frac{221920}{81}\zeta_4 - \frac{3102208}{243}\zeta_5 + \frac{74912}{81}\zeta_3^2 + \frac{334400}{81}\zeta_6 + \frac{178976}{27}\zeta_7\right]$$
$$- \frac{d_A^{abcd}d_A^{abcd}}{N_A} C_F\left[\frac{15344}{81} - \frac{12064}{27}\zeta_3 - \frac{704}{3}\zeta_4 + \frac{58400}{27}\zeta_5 - \frac{6016}{3}\zeta_3^2 - \frac{19040}{9}\zeta_7\right]$$
$$+ \frac{d_R^{abcd}d_A^{abcd}}{N_R} C_F\left[\frac{23968}{81} - \frac{733504}{81}\zeta_3 + \frac{176320}{81}\zeta_5 + \frac{6400}{3}\zeta_3^2 + \frac{77056}{9}\zeta_7\right]$$
$$- \frac{d_R^{abcd}d_A^{abcd}}{N_R} C_A\left[\frac{82768}{81} - \frac{555520}{81}\zeta_3 + \frac{10912}{9}\zeta_4 - \frac{1292960}{81}\zeta_5 + \frac{84352}{27}\zeta_3^2 + \frac{140800}{27}\zeta_6 + 12768\,\zeta_7\right]$$

+ **terms with n_f.** **Numerical expansion for $n_f = 4$:**

$$\gamma_{\rm ns}^{+}(N=2) \cong 0.2829\alpha_{\rm S}(1+0.7987\alpha_{\rm S}+0.5451\alpha_{\rm S}^2+0.5215\alpha_{\rm S}^3+1.2229\alpha_{\rm S}^4+\ldots)$$

π^2 terms (ζ_4, ζ_6) can be predicted by/support the conjecture of Jamin, Miravitllas ('17)