Integration-by-parts reductions via algebraic geometry

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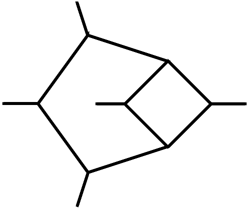

University of Cambridge, Cavendish Laboratory
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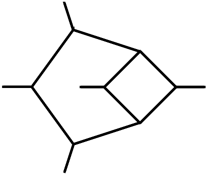
Based on PRD **93**(2016)041701, PRD **98**(2018)025023, JHEP **09**(2018)024
with J. Böhm, A. Georgoudis, H. Schönemann, M. Schulze, Y. Zhang

Overview

1. Motivation
2. IBP identities on unitarity cuts
3. Syzygy equations and their solution
4. Main example: 

Integration-by-parts reductions

IBP identities arise from the vanishing integration of total derivatives,

[Chertyrkin, Tchakov, Nucl. Phys. B **192**, 159 (1981)]

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu \, P}{D_1^{a_1} \cdots D_k^{a_k}} \; = \; 0$$

where P and v_j^μ are polynomials in ℓ_i, p_j, and $a_i \in \mathbb{N}$.

Role in perturbative QFT calculations:

- **Reduction.** Reduce number of contributing loop integrals by factor of $\mathcal{O}(10^2) - \mathcal{O}(10^6)$ to basis.
- **Computing master integrals.** Enable setting up differential equations for basis integrals $\mathcal{I}_j$:

  [Gehrmann and Remiddi, Nucl. Phys. B **580**, 485 (2000)]
  [Henn, PRL **110** (2013) 251601]

$$\frac{\partial}{\partial x_m} \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon) = A_m(\boldsymbol{x}, \epsilon) \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon)$$

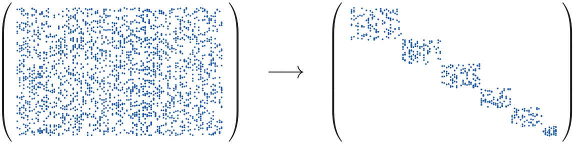
where x_m denotes a kinematical invariant.

IBP reductions on unitarity cuts

Standard approach: enumerate all linear relations and apply Gauss-Jordan elimination to *large* linear systems

[Laporta, Int.J.Mod.Phys. A **15** (2000) 5087-5159]

Idea here: use *unitarity cuts* to block-diagonalize system



We use the Baikov representation $\left(k = \frac{L(L+1)}{2} + LE\right)$,

$$I(N;a) \equiv \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{i\pi^{D/2}} \frac{N}{D_1^{a_1} \cdots D_k^{a_k}} = \int \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_k}{z_1^{a_1} \cdots z_k^{a_k}} \underset{(\widehat{\boldsymbol{p}},\boldsymbol{\ell})}{\mathrm{Gram}}(z)^{\frac{D-L-E-1}{2}} N$$

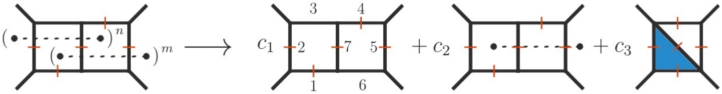
[Baikov, Phys.Lett. B **385** (1996) 404-410]

in which cuts are straightforward to apply,

$$\int \frac{\mathrm{d}z_i}{z_i^{a_i}} \xrightarrow{\text{cut}} \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}z_i}{z_i^{a_i}} \qquad i \in \mathcal{S}_{\text{cut}}$$

Example: Zurich-flag cut

Let us construct IBP identities on the Zurich-flag cut



Define $S_{\text{cut}} = \{1,2,4,5,7\}$ and $G = \text{Gram}_{(\widehat{\boldsymbol{p}},\boldsymbol{\ell})}$.

On S_{cut}, the double-box integral takes the form

$$I^{\text{DB}}_{\text{cut}}[P] \; = \; \prod_{i\in S_{\text{cut}}} \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}\widetilde{z}_i}{\widetilde{z}_i} \; \int \prod_{j\notin S_{\text{cut}}} \mathrm{d}\widetilde{z}_j \, \frac{G(\widetilde{\boldsymbol{z}})^{\frac{D-6}{2}}}{\widetilde{z}_3\, \widetilde{z}_6} \, P(\widetilde{\boldsymbol{z}})$$

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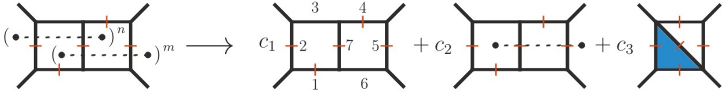

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On S_{cut}, the double-box integral takes the form

$$I^{\text{DB}}_{\text{cut}}[P] = \prod_{i \in S_{\text{cut}}} \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}\tilde{z}_i}{\tilde{z}_i} \int \prod_{j \notin S_{\text{cut}}} \mathrm{d}\tilde{z}_j \frac{G(\tilde{\boldsymbol{z}})^{\frac{D-6}{2}}}{\tilde{z}_3\,\tilde{z}_6} P(\tilde{\boldsymbol{z}})$$

Relabeling $z_{\{1,2,3,4\}} = \tilde{z}_{\{3,6,8,9\}}$, this becomes

$$I^{\text{DB}}_{\text{cut}}[P] = \int \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_1 z_2} G(\boldsymbol{z})^{\frac{D-6}{2}} P(\boldsymbol{z})$$

Generic total derivative on cut

Need to find IBP identities which involve

$$I_{\text{cut}}^{\text{DB}}[P] = \int \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_1 z_2} G(\boldsymbol{z})^{\frac{D-6}{2}} P(\boldsymbol{z})$$

Total derivatives ⟶ IBP identities. Generic total derivative on cut:

$$\begin{aligned} 0 &= \int \left[\sum_{i=1}^{4} \frac{\partial}{\partial z_i} \left(\frac{a_i(\boldsymbol{z}) G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \right) \right] \mathrm{d}z_1 \cdots \mathrm{d}z_4 \\ &= \int \left[\sum_{i=1}^{4} \left(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2G} a_i \frac{\partial G}{\partial z_i} \right) - \sum_{j=1,2} \frac{a_j}{z_j} \right] \frac{G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \cdots \mathrm{d}z_4 \end{aligned}$$

The red term corresponds to an integral in $(D-2)$ dimensions, and the purple term in general produces doubled propagators.

IBP identities from syzygies

To avoid dimension shifts and doubled propagators in

$$0 = \int \Big[\sum_{i=1}^{4} \Big(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2G} a_i \frac{\partial G}{\partial z_i} \Big) - \sum_{j=1,2} \frac{a_j}{z_j} \Big] \frac{G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \cdots \mathrm{d}z_4$$

we demand that each term is *polynomial,*

$$\sum_{i=1}^{4} a_i \frac{\partial G}{\partial z_i} + bG = 0$$

$$a_j + b_j z_j = 0$$

with a_i, b_i, b polynomials in z. Such eqs. are known as *syzygy equations*.
[Gluza, Kajda, Kosower, PRD**83**(2011)045012], [Schabinger, JHEP**01**(2012)077], [Ita, PRD**94**(2016)116015]

Obtain IBPs by plugging (a_i, b) into the top equation.
Note: (qa_i, qb) is also a solution, for polynomial q.

Strategy to solve syzygy equations

Solve syzygy equations with c cuts

$$a_j + b_j z_j = 0\,, \quad j = 1, \ldots, k-c \tag{1}$$

$$\sum_{j=1}^{m-c} a_j \frac{\partial G}{\partial z_k} + bG = 0 \tag{2}$$

as follows.

1) The generators of (1) are trivial:

$$\mathcal{M}_1 = \langle z_1 \mathbf{e}_1, \ldots, z_k \mathbf{e}_k, \mathbf{e}_{k+1}, \ldots, \mathbf{e}_m \rangle$$

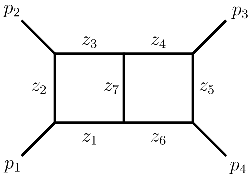
2) Generators $\mathcal{M}_2 = \Big\langle (a_1, \ldots, a_m, b),\ \ldots \Big\rangle$ of (2) for the *off-shell case* $c = 0$ **can be explicitly found:**

$$(a_\alpha, b) = \left(\sum_{k=1}^{E+L} (1+\delta_{ik}) x_{jk} \frac{\partial z_\alpha}{\partial x_{ik}},\ 2\delta_{ij} \right)$$

where $\quad x_{ij} = v_i \cdot v_j \quad$ with $\quad v_{i,j} \in \{p_1, \ldots, p_E, \ell_1, \ldots, \ell_L\}$.

[Böhm, Georgoudis, KJL, Schulze, Zhang, PRD **98**(2018)025023]

3) Take module intersection $\mathcal{M}_1\big|_{\text{cut}} \cap \mathcal{M}_2\big|_{\text{cut}}$



Set $P_{12} = p_1 + p_2$ and

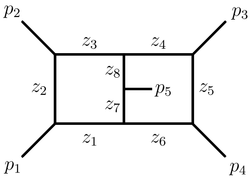
$$
\begin{aligned}
z_1 &= \ell_1^2\,, & z_2 &= (\ell_1 - p_1)^2\,, & z_3 &= (\ell_1 - P_{12})^2 \\
z_4 &= (\ell_2 + P_{12})^2\,, & z_5 &= (\ell_2 - p_4)^2\,, & z_6 &= \ell_2^2 \\
z_7 &= (\ell_1 + \ell_2)^2\,, & z_8 &= (\ell_1 + p_4)^2\,, & z_9 &= (\ell_2 + p_1)^2
\end{aligned}
$$

Only need to find explicit relation $z = Ax + B$. Here $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ -2 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Set $t_{i,j} = (a_\alpha, b)$. The syzygy generators are *linear* in the z_k

$$
\begin{aligned}
t_{4,1} &= (z_1 - z_2,\ z_1 - z_2,\ -s + z_1 - z_2,\ 0,\ 0,\ 0,\ z_1 - z_2 - z_6 + z_9,\ t + z_1 - z_2,\ 0,\ 0) \\
t_{4,2} &= (s + z_2 - z_3,\ z_2 - z_3,\ z_2 - z_3,\ 0,\ 0,\ 0,\ z_2 - z_3 + z_4 - z_9,\ -t + z_2 - z_3,\ 0,\ 0) \\
t_{4,3} &= (-s + z_3 - z_8,\ t + z_3 - z_8,\ z_3 - z_8,\ 0,\ 0,\ 0,\ z_3 - z_4 + z_5 - z_8,\ z_3 - z_8,\ 0,\ 0) \\
t_{4,4} &= (2z_1,\ z_1 + z_2,\ -s + z_1 + z_3,\ 0,\ 0,\ 0,\ z_1 - z_6 + z_7,\ z_1 + z_8,\ 0,\ -2) \\
t_{4,5} &= (-z_1 - z_6 + z_7,\ -z_1 + z_7 - z_9,\ s - z_1 - z_4 + z_7,\ 0,\ 0,\ 0,\ -z_1 + z_6 + z_7,\ -z_1 - z_5 + z_7,\ 0,\ 0) \\
t_{5,1} &= (0,\ 0,\ 0,\ s - z_6 + z_9,\ -t - z_6 + z_9,\ z_9 - z_6,\ z_1 - z_2 - z_6 + z_9,\ 0,\ z_9 - z_6,\ 0) \\
t_{5,2} &= (0,\ 0,\ 0,\ z_4 - z_9,\ t + z_4 - z_9,\ -s + z_4 - z_9,\ z_2 - z_3 + z_4 - z_9,\ 0,\ z_4 - z_9,\ 0) \\
t_{5,3} &= (0,\ 0,\ 0,\ z_5 - z_4,\ z_5 - z_4,\ s - z_4 + z_5,\ z_3 - z_4 + z_5 - z_8,\ 0,\ -t - z_4 + z_5,\ 0) \\
t_{5,4} &= (0,\ 0,\ 0,\ s - z_3 - z_6 + z_7,\ -z_6 + z_7 - z_8,\ -z_1 - z_6 + z_7,\ z_1 - z_6 + z_7,\ 0,\ -z_2 - z_6 + z_7,\ 0) \\
t_{5,5} &= (0,\ 0,\ 0,\ -s + z_4 + z_6,\ z_5 + z_6,\ 2z_6,\ -z_1 + z_6 + z_7,\ 0,\ z_6 + z_9,\ -2)
\end{aligned}
$$

Example 2: syzygies of non-planar double pentagon



Set $P_{i,j} \equiv p_i + p_j$ and

$$z_1 = \ell_1^2\,, \qquad z_2 = (\ell_1 - p_1)^2\,, \qquad z_3 = (\ell_1 - P_{1,2})^2\,,$$
$$z_4 = (\ell_2 - P_{3,4})^2\,, \qquad z_5 = (\ell_2 - p_4)^2\,, \qquad z_6 = \ell_2^2\,,$$
$$z_7 = (\ell_1 + \ell_2)^2\,, \qquad z_8 = (\ell_1 + \ell_2 + p_5)^2\,, \qquad z_9 = (\ell_1 + p_3)^2\,,$$
$$z_{10} = (\ell_1 + p_4)^2\,, \qquad z_{11} = (\ell_2 + p_1)^2$$

Here $z = Ax + B$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the syzygy generators are again compact:

$$\begin{aligned}
t_{5,1} = (&z_1-z_2,\, z_1-z_2,\, -s_{1,2}+z_1-z_2,\, 0,\, 0,\, 0,\\
&z_1-z_2-z_6+z_{11},\, -s_{1,2}-s_{1,3}-s_{1,4}+z_1-z_2-z_6+z_{11},\\
&s_{1,3}+z_1-z_2,\, s_{1,4}+z_1-z_2,\, 0,\, 0)\,,\\
t_{5,2} = (&s_{1,2}+z_2-z_3,\, z_2-z_3,\, z_2-z_3,\, 0,\, 0,\, 0,\\
&-s_{3,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&s_{1,3}+s_{1,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&s_{1,2}+s_{2,3}+z_2-z_3,\, -s_{1,3}-s_{1,4}-s_{2,3}-s_{3,4}+z_2-z_3,\, 0,\, 0)\,,\\
t_{5,3} = (&z_9-z_1,\, -s_{1,3}-z_1+z_9,\, -s_{1,3}-s_{2,3}-z_1+z_9,\, 0,\, 0,\, 0,\\
&s_{3,4}-z_1-z_4+z_5+z_9,\, -s_{1,3}-s_{2,3}-z_1-z_4+z_5+z_9,\\
&z_9-z_1,\, s_{3,4}-z_1+z_9,\, 0,\, 0)\,,\\
t_{5,4} = (&z_{10}-z_1,\, -s_{1,4}-z_1+z_{10},\, -s_{1,4}-s_{2,4}-z_1+z_{10},\\
&0,\, 0,\, 0,\, -z_1-z_5+z_6+z_{10},\\
&s_{1,2}+s_{1,3}+s_{2,3}-z_1-z_5+z_6+z_{10},\\
&s_{3,4}-z_1+z_{10},\, z_{10}-z_1,\, 0,\, 0)\,,\\
t_{5,5} = (&2z_1,\, z_1+z_2,\, -s_{1,2}+z_1+z_3,\, 0,\, 0,\, 0,\\
&z_1-z_6+z_7,\, -s_{1,2}+2z_1+z_3-z_6+z_7-z_9-z_{10},\\
&z_1+z_9,\, z_1+z_{10},\, 0,\, -2)\,,\\
t_{5,6} = (&-z_1-z_6+z_7,\, -z_1+z_7-z_{11},\\
&s_{1,2}+s_{3,4}-2z_1-z_3-z_4+z_8+z_9+z_{10},\, 0,\, 0,\, 0,\\
&-z_1+z_6+z_7,\, s_{1,2}-2z_1-z_3+z_6+z_8+z_9+z_{10},\\
&s_{3,4}-z_1-z_4+z_5-z_6+z_7,\, -z_1-z_5+z_7,\, 0,\, 0)\,, \qquad (6.11)\\
t_{6,1} = (&0,\, 0,\, 0,\, -s_{1,3}-s_{1,4}-z_6+z_{11},\, -s_{1,4}-z_6+z_{11},\\
&z_{11}-z_6,\, z_1-z_2-z_6+z_{11},\\
&-s_{1,2}-s_{1,3}-s_{1,4}+z_1-z_2-z_6+z_{11},\, 0,\, 0,\, z_{11}-z_6,\, 0)\,,\\
t_{6,2} = (&0,0,0,\, s_{1,3}+s_{1,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&s_{1,3}+s_{1,4}+s_{2,3}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&-s_{1,2}-s_{3,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&-s_{3,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\\
&s_{1,3}+s_{1,4}+z_1+z_2+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, 0,\, 0,\\
&-s_{3,4}+z_1+z_3+z_4+z_7-z_8-z_9-z_{10}-z_{11},\, 0)\,,\\
t_{6,3} = (&0,\, 0,\, 0,\, z_5-z_4,\, z_5-z_4,\, s_{3,4}-z_4+z_5,\\
&s_{3,4}-z_1-z_4+z_5+z_9,\, -s_{1,3}-s_{2,3}-z_1-z_4+z_5+z_9,\, 0,\, 0,\\
&s_{1,3}+s_{3,4}-z_4+z_5,\, 0)\,,\\
t_{6,4} = (&0,\, 0,\, 0,\, -s_{3,4}-z_5+z_6,\, z_6-z_5,\, z_6-z_5,\\
&-z_1-z_5+z_6+z_{10},\, s_{1,2}+s_{1,3}+s_{2,3}-z_1-z_5+z_6+z_{10},\, 0,\, 0,\\
&s_{1,4}-z_5+z_6,\, 0)\,,\\
t_{6,5} = (&0,\, 0,\, 0,\, z_1-z_6+z_7-z_9-z_{10},\, -z_6+z_7-z_{10},\\
&-z_1-z_6+z_7,\, z_1-z_6+z_7,\, -s_{1,2}+2z_1+z_3-z_6+z_7-z_9-z_{10},\\
&0,\, 0,\, -z_2-z_6+z_7,\, 0)\,,\\
t_{6,6} = (&0,\, 0,\, 0,\, -s_{3,4}+z_4+z_6,\, z_5+z_6,\, 2z_6,\, -z_1+z_6+z_7,\\
&s_{1,2}-2z_1-z_3+z_6+z_8+z_9+z_{10},\, 0,\, 0,\, z_6+z_{11},\, -2)\,.
\end{aligned}$$

Computing module intersections

Given $\mathcal{M}_1 = \langle v_1, \ldots, v_p \rangle$ and $\mathcal{M}_2 = \langle w_1, \ldots, w_q \rangle$ with v_i, w_j m-tuples of polynomials. Let Q denote the $m \times (p+q)$ matrix

$$Q = \begin{pmatrix} \vdots & & \vdots & \vdots & & \vdots \\ v_1 & \cdots & v_p & w_1 & \cdots & w_q \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] \succ [s_{ij}]$

$$\langle h_1, \ldots, h_t \rangle \equiv \text{Gröbner basis of column space of } \left(\begin{array}{ccc} & Q & \\ \hline 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array}\right)$$

Selecting $h_i = (\overbrace{0, \ldots, 0}^{m}, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^{p} x_j v_j + \sum_{k=1}^{q} y_k w_k$$

Computing module intersections

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$$Q = \begin{pmatrix} \vdots & & \vdots & \vdots & & \vdots \\ v_1 & \cdots & v_p & w_1 & \cdots & w_q \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] \succ [s_{ij}]$

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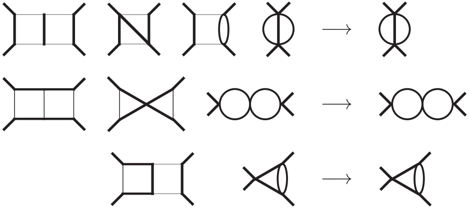
Selecting $h_i = (\overbrace{0, \ldots, 0}^{m}, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^{p} x_j v_j + \sum_{k=1}^{q} y_k w_k \quad \Longrightarrow \quad \sum_{j=1}^{p} x_j v_j = - \sum_{k=1}^{q} y_k w_k \in \mathcal{M}_1 \cap \mathcal{M}_2$$

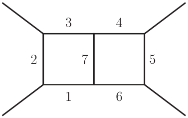
Hence $\sum_{j=1}^{p} x_j v_j$ generate $\mathcal{M}_1 \cap \mathcal{M}_2$, taking $(x_1, \ldots, x_p)$ from each h_i.

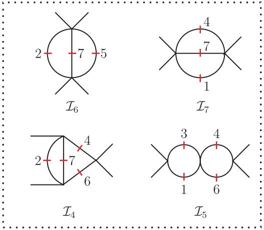
Spanning set of cuts for IBPs

To find the complete IBP reduction, we must consider the cuts associated with "uncollapsible" masters:



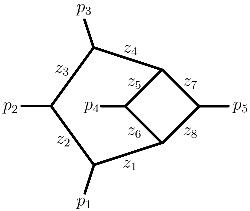
A bit more explicitly, the cuts we need to consider are



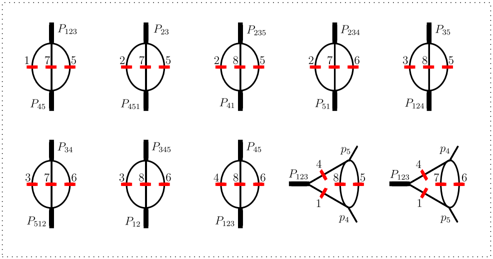


Task: IBP reduce non-planar hexagon box with numerator insertions of degree four in the z_i

[Chicherin, Henn, Mitev JHEP **05**(2018)164]
[Badger, Brønnum-Hansen, Hartanto, Peraro, PRL **120**(2018)092001]
[Abreu, Cordero, Ita, Page, Zeng, PRD **97**(2018)116014]
[Chawdhry, Lim, Mitov, 1805.09182]
[S. Abreu, B. Page, M. Zeng, 1807.11522]
[D. Chicherin, T. Gehrmann, J. Henn, N.A. Lo Presti, V. Mitev, P. Wasser, 1809.06240]

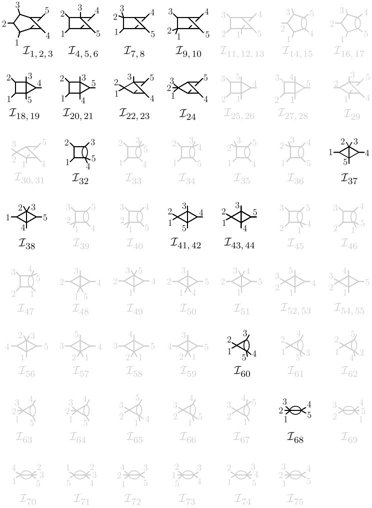


There are 10 cuts to consider:



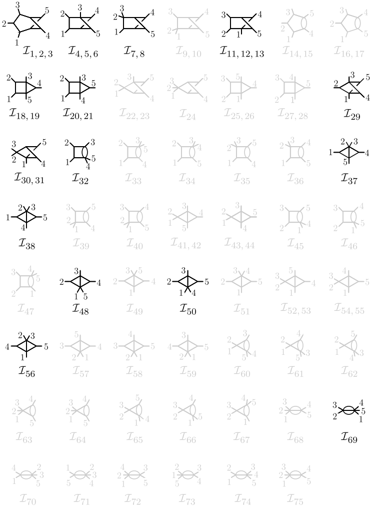
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.



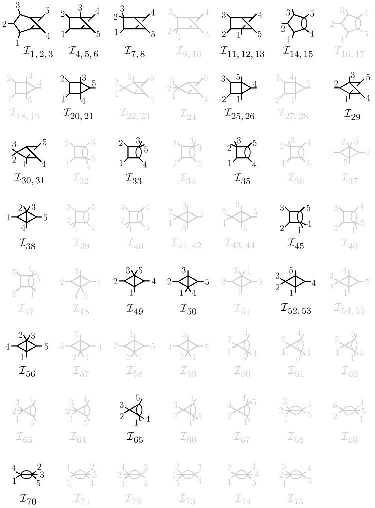
Cut $\{1, 5, 7\}$

Construct and solve IBP identities on a spanning set of cuts.



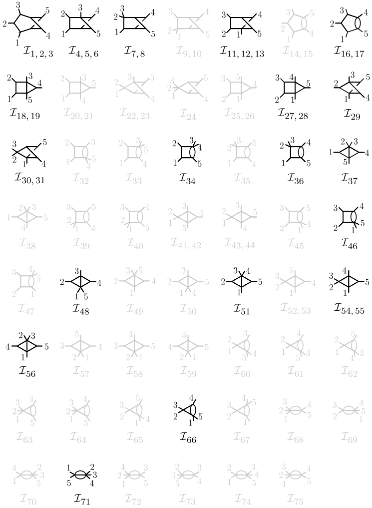
Cut $\{2, 5, 7\}$

Construct and solve IBP identities on a spanning set of cuts.

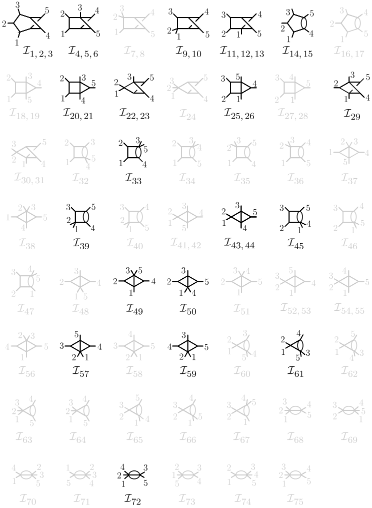


Cut $\{2, 5, 8\}$

Construct and solve IBP identities on a spanning set of cuts.



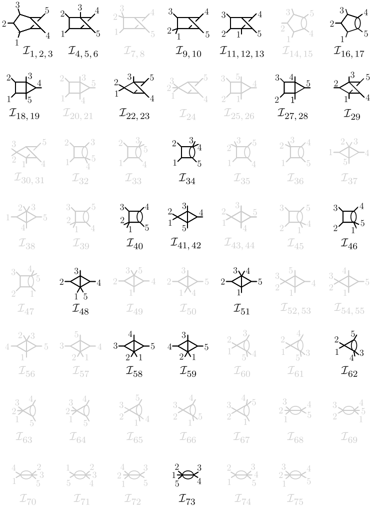
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{3, 5, 8\}$

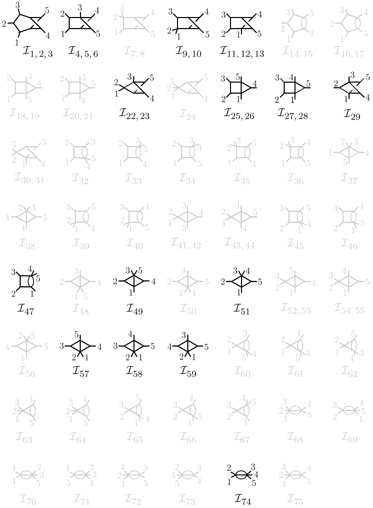
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.



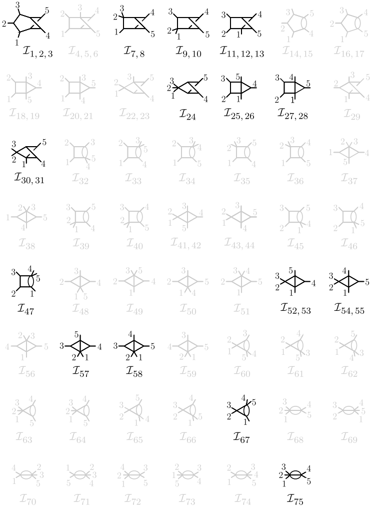
Cut $\{3, 6, 7\}$

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{3, 6, 8\}$

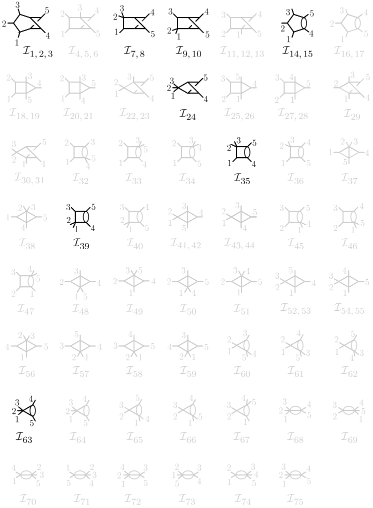
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{4, 6, 8\}$

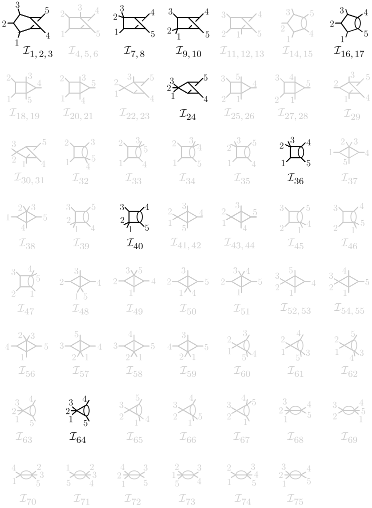
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{1, 4, 5, 8\}$

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{1,4,6,7\}$

Syzygies for the non-planar hexagon box

Syzygies for ensuring *D*-dimensionality:

$$
\begin{aligned}
M_1 = \langle &(z_1 - z_2, z_1 - z_2, -s_{12} + z_1 - z_2, -s_{12} - s_{13} + z_1 - z_2, s_{14} + z_1 - z_2 - z_8 + z_{10}, z_1 - z_2 - z_8 + z_{10}, 0, 0, -s_{12} - s_{13} - s_{14} + z_1 - z_2, 0, 0) \\
&(0, 0, 0, 0, s_{14} + z_1 - z_2 - z_8 + z_{10}, z_1 - z_2 - z_8 + z_{10}, s_{12} + s_{13} + s_{14} - z_8 + z_{10}, z_{10} - z_8, 0, z_{10} - z_8, s_{12} - z_8 + z_{10}) \\
&(s_{12} + z_2 - z_3, z_2 - z_3, z_2 - z_3, -s_{23} + z_2 - z_3, s_{12} + s_{24} + z_2 - z_3 - z_8 + z_{11}, s_{12} + z_2 - z_3 - z_8 + z_{11}, 0, 0, -s_{23} - s_{24} + z_2 - z_3, 0, 0) \\
&(0, 0, 0, 0, s_{12} + s_{24} + z_2 - z_3 - z_8 + z_{11}, s_{12} + z_2 - z_3 - z_8 + z_{11}, s_{12} + s_{23} + s_{24} - z_8 + z_{11}, z_{11} - z_8, 0, s_{12} - z_8 + z_{11}, z_{11} - z_8) \\
&(s_{13} + s_{23} + z_3 - z_4, s_{23} + z_3 - z_4, z_3 - z_4, z_3 - z_4, -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad -s_{12} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, 0, 0, s_{12} + s_{13} + s_{14} + s_{23} + s_{24} + z_3 - z_4, 0, 0) \\
&(0, 0, 0, 0, -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} - s_{13} - s_{23} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad 0, -s_{12} - s_{23} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} - s_{13} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}) \\
&(-s_{12} - s_{13} - s_{23} + z_4 - z_9, -s_{12} - s_{13} - s_{14} - s_{23} + z_4 - z_9, -s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_9, z_4 - z_9, z_5 - z_6, z_5 - z_6, 0, 0, z_4 - z_9, 0, 0) \\
&(0, 0, 0, 0, z_5 - z_6, z_5 - z_6, -z_4 + z_5 - z_6 + z_9, s_{12} + s_{13} + s_{23} - z_4 + z_5 - z_6 + z_9, \\
&\quad 0, s_{12} + s_{13} + s_{14} + s_{23} - z_4 + z_5 - z_6 + z_9, s_{12} + s_{13} + s_{23} + s_{24} - z_4 + z_5 - z_6 + z_9) \\
&(2z_1, z_1 + z_2, -s_{12} + z_1 + z_3, -s_{12} - s_{13} - s_{23} + z_1 + z_4, -s_{12} - s_{13} - s_{23} + z_1 + z_4 + z_6 - z_8 - z_9, z_1 + z_6 - z_8, 0, 0, z_1 + z_9, 0, 0) \\
&(0, 0, 0, 0, -s_{12} - s_{13} - s_{23} + z_1 + z_4 + z_6 - z_8 - z_9, z_1 + z_6 - z_8, z_6 - z_8 - z_9, -z_1 + z_6 - z_8, 0, -z_2 + z_6 - z_8, s_{12} - z_1 + z_2 - z_3 + z_6 - z_8) \\
&(-z_1 + z_6 - z_8, -z_1 + z_6 - z_{10}, -z_1 + z_6 + z_8 - z_{10} - z_{11}, s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 - z_7 + z_9, \\
&\quad s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9, -z_1 + z_6 + z_8, 0, 0, -z_1 + z_6 - z_7, 0, 0) \\
&(0, 0, 0, 0, s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9, -z_1 + z_6 + z_8, z_7 + z_8, 2z_8, 0, z_8 + z_{10}, z_8 + z_{11})\rangle
\end{aligned}
\tag{5.9}
$$

Syzygies for ensuring no doubled propagators:

$$
\begin{aligned}
M_2 = \langle &(z_1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, z_2, 0, 0, 0, 0, 0, 0, 0, 0, 0) \\
&(0, 0, z_3, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, z_4, 0, 0, 0, 0, 0, 0, 0) \\
&(0, 0, 0, 0, z_5, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, z_6, 0, 0, 0, 0, 0) \\
&(0, 0, 0, 0, 0, 0, z_7, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, z_8, 0, 0, 0) \\
&(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0) \\
&(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)\rangle
\end{aligned}
$$

Compute intersection of $M_1\big|_{\text{cut}} \cap M_2\big|_{\text{cut}}$ on each of the 10 cuts.

Complexity of IBP systems

- Resources to compute $M_1\big|_{\text{cut}} \cap M_2\big|_{\text{cut}}$: **25-800 s** and **1-14 GB RAM**

  (on 24 cores, 3.40 GHz)

- Size of generating systems after trimming: **1.5-10 MB**

  Plug resulting generators into ansatz for total derivative:

$$0 = \int \left[\sum_{i=1}^{m-c} \left(\frac{\partial a_{r_i}}{\partial z_{r_i}} + \frac{D-L-E-1}{2G(z)} a_{r_i} \frac{\partial G}{\partial z_{r_i}} \right) - \sum_{i=1}^{k-c} \frac{a_{r_i}}{z_{r_i}} \right] \frac{G(z)^{\frac{D-L-E-1}{2}}}{z_{r_1} \cdots z_{r_{k-c}}} \mathrm{d}z_{r_1} \cdots \mathrm{d}z_{r_{m-c}}$$

- Resulting linear systems to solve:

  700-1200 equations, size **1 MB**, density **1.5%**

Gauss-Jordan elimination of IBP systems

To find the IBP reductions, Gauss-Jordan eliminate IBP systems.

Some remarks:

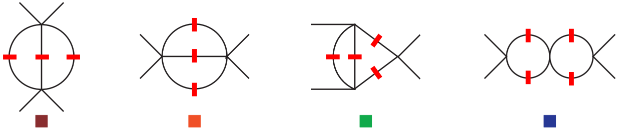
- To preserve sparsity, use a *total pivoting* strategy (i.e., allow column swaps)
- For cut $\{1,4,6,7\}$, the RREF can be performed fully analytically, requiring 31 minutes on one core and 1.5 GB RAM.

  ⋮

- For $\{3,6,7\}$, assigned numerical values to two s_{ij}.
  Ran 440 points on cluster (2.5 h and 1.8 GB RAM per job).
  Used interpolation code to get analytical results (23 min and 15 GB RAM on one core).

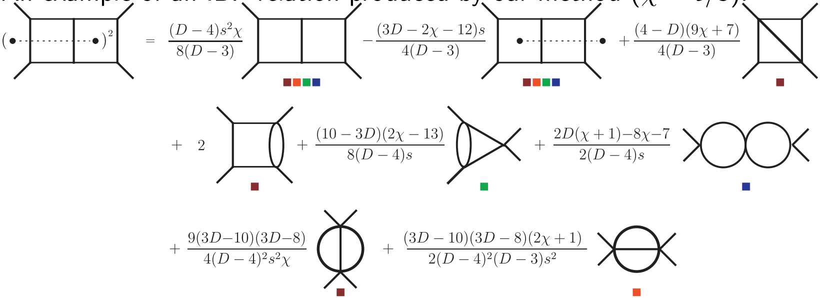
[von Manteuffel and Schabinger, PLB **744**(2015)101]
[Peraro, JHEP**12**(2016)030]

By solving the IBP identities on the following cuts



we reconstruct the *complete IBP reductions* by merging the partial results.

An example of an IBP relation produced by our method ($\chi \equiv t/s$):



Results for IBP reductions

- Fully analytic IBP reductions of the 32 hexagon boxes

$$
\begin{array}{lll}
\{ I(1,1,1,1,1,1,1,1,0,0,-4), & I(1,1,1,1,1,1,1,1,0,-1,-3), & I(1,1,1,1,1,1,1,1,0,-2,-2) \\
I(1,1,1,1,1,1,1,1,0,-3,-1), & I(1,1,1,1,1,1,1,1,0,-4,0), & I(1,1,1,1,1,1,1,1,-1,0,-3) \\
I(1,1,1,1,1,1,1,1,-1,-1,-2), & I(1,1,1,1,1,1,1,1,-1,-2,-1), & I(1,1,1,1,1,1,1,1,-1,-3,0) \\
I(1,1,1,1,1,1,1,1,-2,0,-2), & I(1,1,1,1,1,1,1,1,-2,-1,-1), & I(1,1,1,1,1,1,1,1,-2,-2,0) \\
I(1,1,1,1,1,1,1,1,-3,0,-1), & I(1,1,1,1,1,1,1,1,-3,-1,0), & I(1,1,1,1,1,1,1,1,-4,0,0) \\
I(1,1,1,1,1,1,1,1,0,0,-3), & I(1,1,1,1,1,1,1,1,0,-1,-2), & I(1,1,1,1,1,1,1,1,0,-2,-1) \\
I(1,1,1,1,1,1,1,1,0,-3,0), & I(1,1,1,1,1,1,1,1,-1,0,-2), & I(1,1,1,1,1,1,1,1,-1,-1,-1) \\
I(1,1,1,1,1,1,1,1,-1,-2,0), & I(1,1,1,1,1,1,1,1,-2,0,-1), & I(1,1,1,1,1,1,1,1,-2,-1,0) \\
I(1,1,1,1,1,1,1,1,-3,0,0), & I(1,1,1,1,1,1,1,1,0,0,-2), & I(1,1,1,1,1,1,1,1,0,-1,-1) \\
I(1,1,1,1,1,1,1,1,0,-2,0), & I(1,1,1,1,1,1,1,1,-1,0,-1), & I(1,1,1,1,1,1,1,1,-1,-1,0) \\
I(1,1,1,1,1,1,1,1,0,0,-1), & I(1,1,1,1,1,1,1,1,0,-1,0)\} &
\end{array}
$$

  can be downloaded from (268 MB compressed / 790 MB uncompressed)

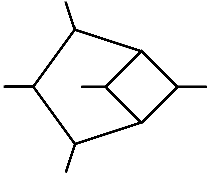
  https://github.com/yzhphy/hexagonbox_reduction/releases/download/1.0.0/hexagon_box_degree_4_Final.zip

- Our results agree with fully numerical results from FIRE5 C++ (6 hours per point).

  [A. Smirnov, CPC **189**(2015)182]

Conclusions

- New formalism for IBP reductions. Main ideas: cuts, IBP identities from syzygies, total pivoting, rational reconstruction
- Obtained the **fully analytic** IBP reductions of



  with numerator insertions up to degree 4 in the z_i.

- Powerful framework. IBP reductions for further $2 \to 3$ two-loop processes **seem well within reach**.