

Integration-by-parts identities and multi-loop QCD amplitudes

Herschel A. Chawdhry

Cavendish Laboratory
University of Cambridge

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Based on `hep-ph/1805.09182` with M. A. Lim and A. Mitov

Introduction

- Precise calculations of scattering amplitudes in gauge theories (such as QCD) require the evaluation of multi-loop diagrams
- Multi-loop integrals are often evaluated using integration-by-parts (IBP) identities
- We introduce a new strategy for solving IBP identities
- As an example, we use it to reduce all of the planar integrals contributing to any massless 2-loop 5-point QCD scattering amplitude with quarks and/or gluons (e.g. $q\bar{q} \rightarrow q'\bar{q}'g$)

Outline

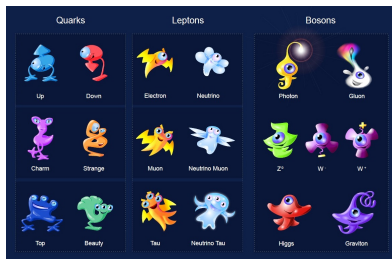
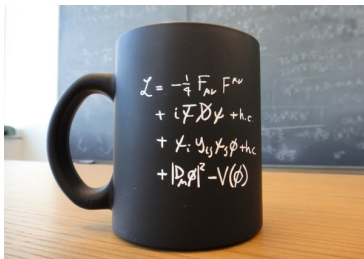
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 - Why scattering amplitudes?
 - Why 5-point QCD amplitudes?
- 2 Integration-by-parts (IBP) identities
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Why scattering amplitudes?

● Theory:

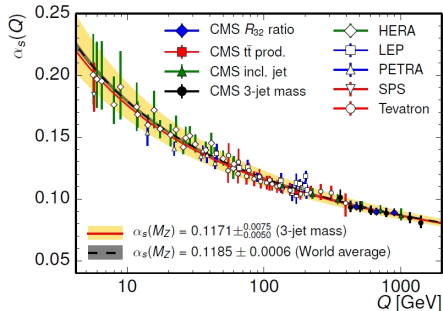
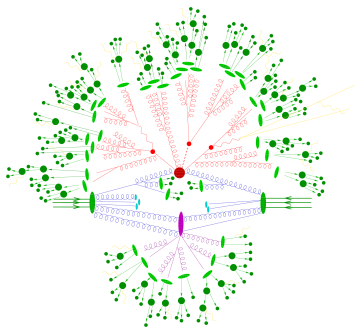


● Experiment:

- masses
- decay rates
- cross-sections

Images: www.quantumdiaries.org & Andre-Pierre Olivier

Scattering amplitudes at particle colliders



$$\sigma_{\text{hard}} = \sigma_{\text{LO}}\alpha_s^n + \sigma_{\text{NLO}}\alpha_s^{n+1} + \sigma_{\text{NNLO}}\alpha_s^{n+2} + \dots$$

Images: Stefan Höche [arXiv:1411.4085](https://arxiv.org/abs/1411.4085) & CMS [arXiv:1412.1633](https://arxiv.org/abs/1412.1633)

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Why 5-point QCD amplitudes?

- NNLO phenomenology at the LHC has cleared $2 \rightarrow 2$ processes
- However, $2 \rightarrow 3$ processes remain an open problem.
 - e.g. 3-jet production; $H + 2$ jets; $\gamma\gamma + 1$ jet
- 2-loop amplitudes are the biggest missing ingredient
- Many developments in recent years:

<ul style="list-style-type: none"> ● Badger, Frellesvig, Zhang (2013) ● Ita (2015) ● Badger, Mogull, Ochirov, O'Connell (2015) ● Gehrmann, Henn, Presti (2015) ● Dunbar, Perkins (2016) ● Dunbar, Godwin, Jehu, Perkins (2017) ● Badger, Brønnum-Hansen, Hartanto, Peraro (2017) ● Abreu, Cordero, Ita, Page, Zeng (2017) ● Böhmer, Georgoudis, Larsen, Schönemann, Zhang (2018) 	<ul style="list-style-type: none"> ● Kosower (2018) ● HAC, Lim, Mitov (2018) ● Badger, Brønnum-Hansen, Gehrmann, Hartanto, Henn, Lo Presti, Peraro (2018) ● Abreu, Cordero, Ita, Page, Zeng (2018) ● Gehrmann, Henn, Lo Presti (2018) ● Abreu, Page, Zeng (2018) ● Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser (2018) ● Abreu, Cordero, Ita, Page, Sotnikov (2018)
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Why IBPs?

- The IBPs are linear relations between Feynman integrals
- They contain an incredible amount of information about the problem
- They have been widely successful in the computation of multi-loop QCD amplitudes
- Some problems have so far remained beyond the reach of current IBP-solving methods
 - e.g. $2 \rightarrow 3$ at 2 loops; $2 \rightarrow 2$ at 3 loops; massive $2 \rightarrow 2$ at 2 loops
- Highly desirable to improve methods of solving IBPs
- Note: KIRA v1.1 makes steps in a similar direction

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Integration-by-parts (IBP) identities

- Generic bare amplitude is a sum of many Feynman integrals

$$M = \sum_{i=1}^N f_i I_i$$

- I_i are (wlog) scalar Feynman integrals

$$I_i = \int d^d k_1 \dots d^d k_L \left[\frac{1}{\Pi_1^{a_1} \Pi_2^{b_2} \dots} \right], \text{ where } \Pi \sim (q^2(p, k) - m^2)$$

- N is large: in our 2-loop 5-point case, we have $N \sim \mathcal{O}(10^4 - 10^5)$
- f_i are rational functions, whereas I_i are dilogs/polylogs/etc.
- Integration-by-parts (IBP) identities:

$$\int d^d k_1 \dots d^d k_L \left[\frac{\partial}{\partial k_j^\mu} \left(\frac{v^\mu}{\Pi_1^a \Pi_2^b \dots} \right) \right] = 0$$

Hence, many linear relations between integrals.

Integration-by-parts (IBP) identities (continued)

- *The challenge:* Solve this system of equations to express all integrals in terms of a small basis of **master integrals**.

$$I_i = \sum_{m=1}^{\hat{N}} c_{i,m} \hat{I}_m$$

- in our 2-loop 5-point case, we have $\hat{N} \sim \mathcal{O}(100)$
- Can hence write original amplitude in terms of master integrals

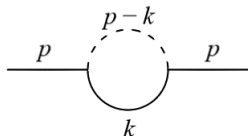
$$M = \sum_{m=1}^{\hat{N}} \hat{c}_m \hat{I}_m, \text{ with } \hat{c}_m = \sum_{i=1}^N c_{i,m} f_i$$

- In our work we focus on solving the IBPs. Evaluating the masters is a separate problem (but the IBP solutions help here too).
 - In our 5-point case, all planar masters are already known, as well as some non-planars.

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Example: 1-loop bubble

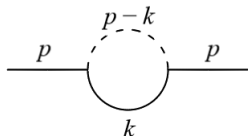


$$\mathcal{I}_{a,b}(p^2, m^2) = \int d^d k \frac{1}{[k^2 - m^2]^a [(p-k)^2]^b}$$

- Numerators absorbed: take $\int d^d k \left(\frac{N}{[k^2 - m^2]^a [(p-k)^2]^b} \right)$, where **N** is:
 - m^2, p^2
 - $k^2 = (k^2 - m^2) + m^2$
 - $k \cdot p = -\frac{1}{2}(p-k)^2 + \frac{1}{2}p^2 + \frac{1}{2}k^2$
 - k^μ :
 - let $\int d^d k \left(\frac{k^\mu}{[k^2 - m^2]^a [(p-k)^2]^b} \right) = A p^\mu$
 - then $A = \frac{1}{p^2} \int d^d k \left(\frac{k \cdot p}{[k^2 - m^2]^a [(p-k)^2]^b} \right)$
- More propagators upstairs \rightarrow more complicated integral
- In general, might need extra propagators

Example: 1-loop bubble (2)

further details: see V. A. Smirnov, "Evaluating Feynman Integrals" (2004)



$$\mathcal{I}_{a,b}(p^2, m^2) = \int d^d k \frac{1}{[k^2 - m^2]^a [(p-k)^2]^b}$$

- Two IBP identities:

1

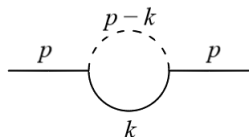
$$\int d^d k \frac{\partial}{\partial k^\mu} \left(\frac{p^\mu}{[k^2 - m^2]^a [(p-k)^2]^b} \right) = 0$$

2

$$\int d^d k \frac{\partial}{\partial k^\mu} \left(\frac{k^\mu}{[k^2 - m^2]^a [(p-k)^2]^b} \right) = 0$$

Example: 1-loop bubble (3)

further details: see V. A. Smirnov, "Evaluating Feynman Integrals" (2004)



$$\mathcal{I}_{a,b}(p^2, m^2) = \int d^d k \frac{1}{[k^2 - m^2]^a [(p-k)^2]^b}$$

- Two IBP identities:

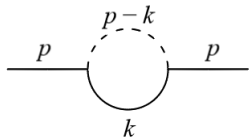
- $$(b-a)\mathcal{I}_{a,b} - (m^2 a + p^2 a)\mathcal{I}_{a+1,b} + a\mathcal{I}_{a+1,b-1} + (m^2 b - p^2 b)\mathcal{I}_{a,b+1} - b\mathcal{I}_{a-1,b+1} = 0$$

- $$(d-2a-b)\mathcal{I}_{a,b} - 2m^2 a\mathcal{I}_{a+1,b} + (m^2 b - p^2 b)\mathcal{I}_{a,b+1} - b\mathcal{I}_{a-1,b+1} = 0$$

- Note: the identities are **linear** and **homogeneous**.

Example: 1-loop bubble (4)

further details: see V. A. Smirnov, "Evaluating Feynman Integrals" (2004)



$$\mathcal{I}_{a,b}(p^2, m^2) = \int d^d k \frac{1}{[k^2 - m^2]^a [(p-k)^2]^b}$$

General solution:

- $\mathcal{I}_{a,b} = f_{a,b}(p^2, m^2)\mathcal{I}_{1,0} + g_{a,b}(p^2, m^2)\mathcal{I}_{1,1}$

Special case: $b = 0$

- IBPs lead to recurrence relation: $\mathcal{I}_{a,0} = \frac{d-2a+2}{2(a-1)m^2} \mathcal{I}_{a-1,0}$

- Hence, closed-form solution: $\mathcal{I}_{a,0} = \frac{(-1)^a(1-d/2)}{\Gamma(a)(m^2)^{a-1}} \mathcal{I}_{1,0}$

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Our strategy

- Recall:
 - We want to write each integral in terms of master integrals

$$I_i = \sum_{m=1}^{\hat{N}} c_{i,m} \hat{f}_m$$

- We'll use the IBP equations. They are **linear** and **homogeneous**.

Our strategy:

- Generate the IBP equations
- Identify the master integrals (straight-forward e.g. using Reduze)
- Pick one master integral. Set all other masters to zero.
- Solve the (greatly simplified!) IBP equations
- This will give the projection of each integral onto the chosen master integral.
- Now repeat for each of the other masters; this way, we build up the full solution (i.e. we obtain each $c_{i,m}$).

Note: we use Laporta algorithm but this is not essential.

Cartoon: Integration-by-parts identities (IBPs)

- IBP equations (coefficients not shown)

$$\text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} + \dots$$

$$\text{Diagram 4} = \text{Diagram 5} + \text{Diagram 6} + \dots$$

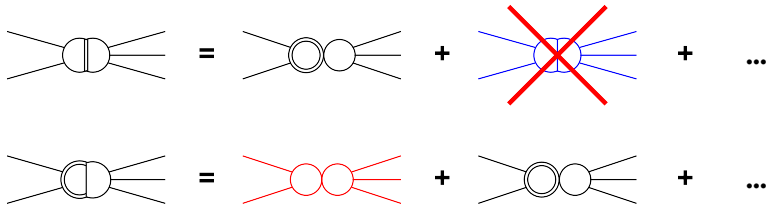
- Goal: Solve this system of equations to express all integrals in terms of a *small* basis of **master integrals**.

$$\text{Diagram 7} = \text{Diagram 8} + \text{Diagram 9} + \dots$$

$$\text{Diagram 10} = \text{Diagram 11} + \text{Diagram 12} + \dots$$

Cartoon: Our strategy

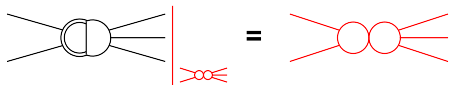
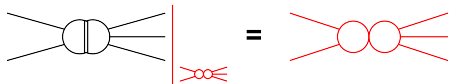
- Derive projections of the IBP equations onto a single master integral by setting all other master integrals to be zero



- This simplifies the IBP equations because many non-master integrals only project onto a subset of the master integrals

Cartoon: Our strategy (continued)

- By solving these simplified equations, one obtains the projections of all integrals onto a single master integral (coefficients still not shown)



- The full solution to the original IBP equations is obtained by repeating for each of the other master integrals and summing the solutions

Benefits of this strategy

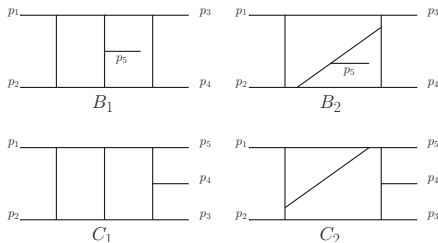
- Simplifies the problem
 - Many integrals only have projections onto a subset of all the possible masters. (A given integral, I , only projects onto a master, M , if the propagators of M are a subset of the propagators of I)
- Parallelisation
 - The IBP equations are “solved” many times – once for each master integral. These runs are independent of one another, so they can run in parallel.
 - Run times for different masters vary by several orders of magnitude. The overall running time is limited by a handful of ‘difficult’ masters.
- Reduced memory requirements
 - RAM usage is reduced, since far fewer coefficients need to be kept in memory at one time.

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Application to 2 \rightarrow 3: Setup

- We have implemented our strategy in a private C++ code and applied it to the 2-loop 5-point massless QCD amplitudes
- The most complicated 2-loop 5-point topologies have 8 propagators:



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Application to 2 \rightarrow 3: Results

- Identified the masters:
 - 113 masters in B_1
 - 75 masters in B_2
 - 62 masters in C_1
 - 28 masters in C_2
- Full reduction for all planar (C_1 and C_2) integrals that contribute to 2-loop 5-point massless QCD processes.
- Some results for non-planar (B_1 and B_2) topologies:
 - Coefficients of the highest-weight masters, for all integrals with up to 6 numerator powers and up to 1 squared denominator
- Results available to download from:
www.precision.hep.phy.cam.ac.uk/results/amplitudes/
 - Rational expressions are fully expanded
 - Files compressed (22GB)

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Application to 2 \rightarrow 3: Checks

- Computed $q\bar{q} \rightarrow Q\bar{Q}$ and cross-checked against `Reduze`
- Checked our results in the B_2 topology against those from `hep-th/1805.01873` (Böhm et al.)
- C_1 integrals with 5 numerator powers can be related to integrals with fewer numerator powers using `hep-th/1009.0472` (Gluza et al.) and `hep-th/1804.00131` (Kosower). We carried this out as a check and find full consistency with our solutions.

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Conclusions and future work

- Proposed a new strategy for solving the IBP identities. Particularly useful for multi-scale problems.
- Derived analytic expressions for all integral coefficients needed to construct any planar 2-loop 5-point massless QCD amplitude with quarks and/or gluons. Results are now publicly available.
- Further reading – see our paper:

`hep-ph/1805.09182`

Outlook:

- We are working on the non-planar topologies
- Eventually intend to make our IBP-solving program public
- Ultimately, one would like a closed-form solution to the IBPs. This remains an open problem, although we hope our strategy could be of some help in this direction.