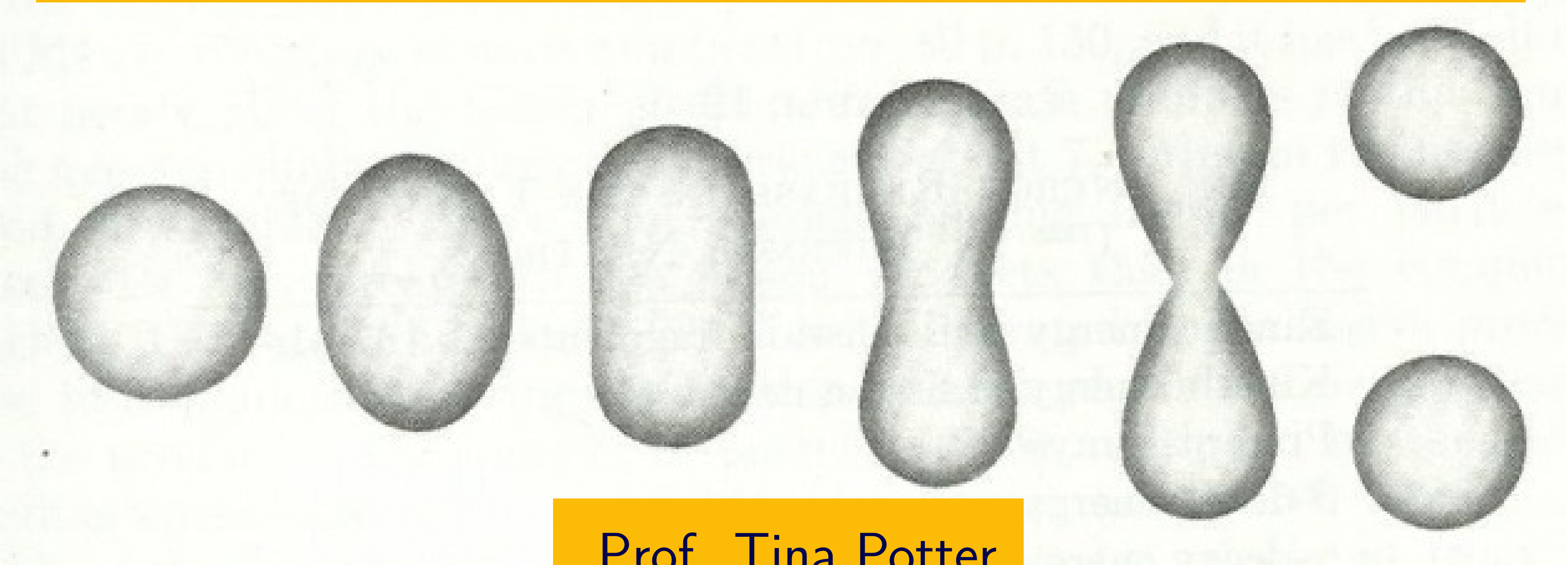
13. Basic Nuclear Properties

Particle and Nuclear Physics



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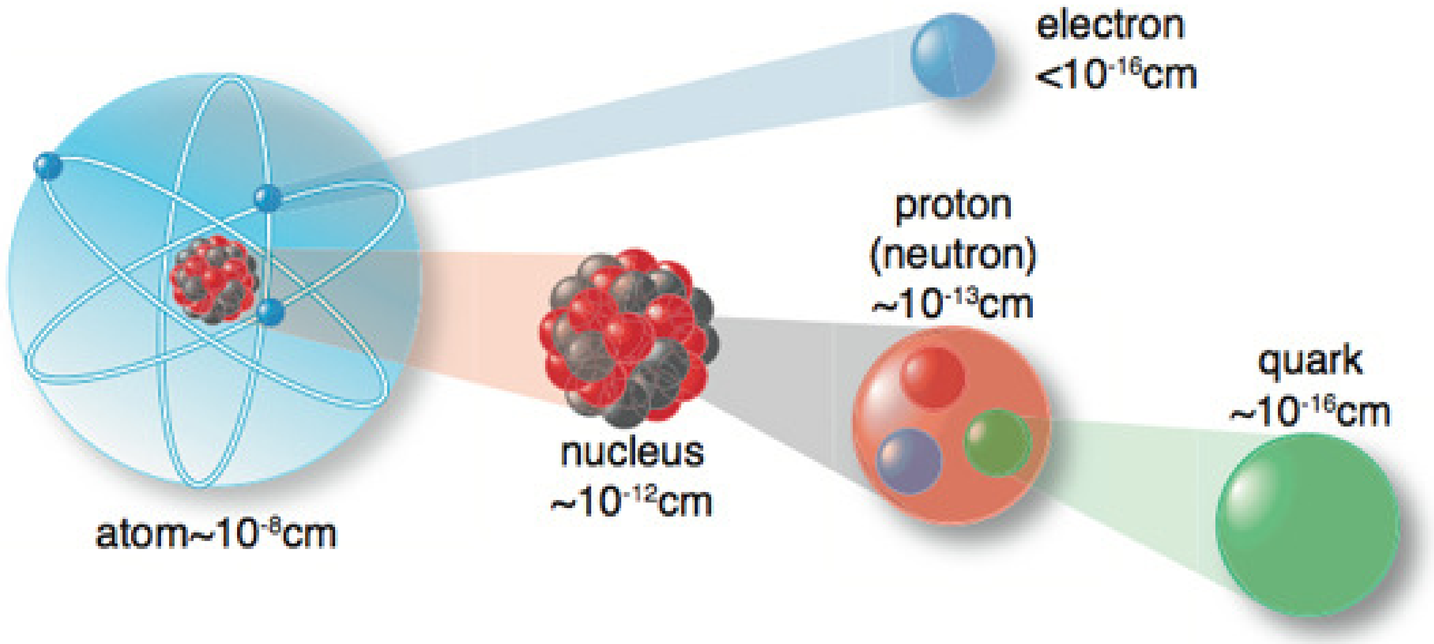
In this section...

- Motivation for study
- The strong nuclear force
- Stable nuclei
- Binding energy & nuclear mass (SEMF)
- Spin & parity
- Nuclear size (scattering, muonic atoms, mirror nuclei)
- Nuclear moments (electric, magnetic)

Introduction

Nuclear processes play a fundamental role in the physical world:

- Origin of the universe
- Creation of chemical elements
- Energy of stars
- Constituents of matter; influence properties of atoms

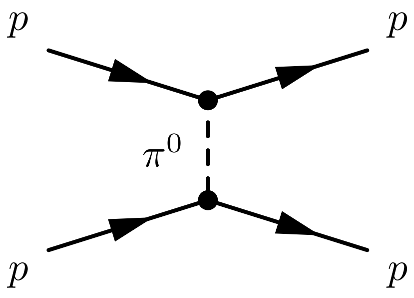


Nuclear processes also have many practical applications:

- Uses of radioactivity in research, health and industry, e.g. NMR, radioactive dating.
- Various tools for the study of materials, e.g. Mössbauer, NMR.
- Nuclear power and weapons.

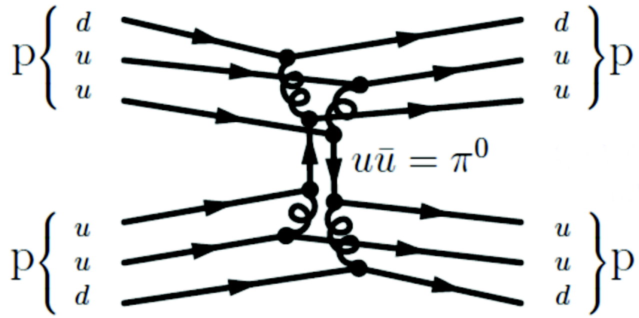
The Nuclear Force

Consider the *pp* interaction, Range $\sim \hbar / m_\pi c \sim 1\text{fm}$



Hadron level

$\equiv$



Quark-gluon level

Pion vs. gluon exchange is similar to the Coulomb potential vs. van der Waals' force in QED.

The treatment of the strong nuclear force between nucleons is a many-body problem in which

- quarks do not behave as if they were completely independent.
- nor do they behave as if they were completely bound.

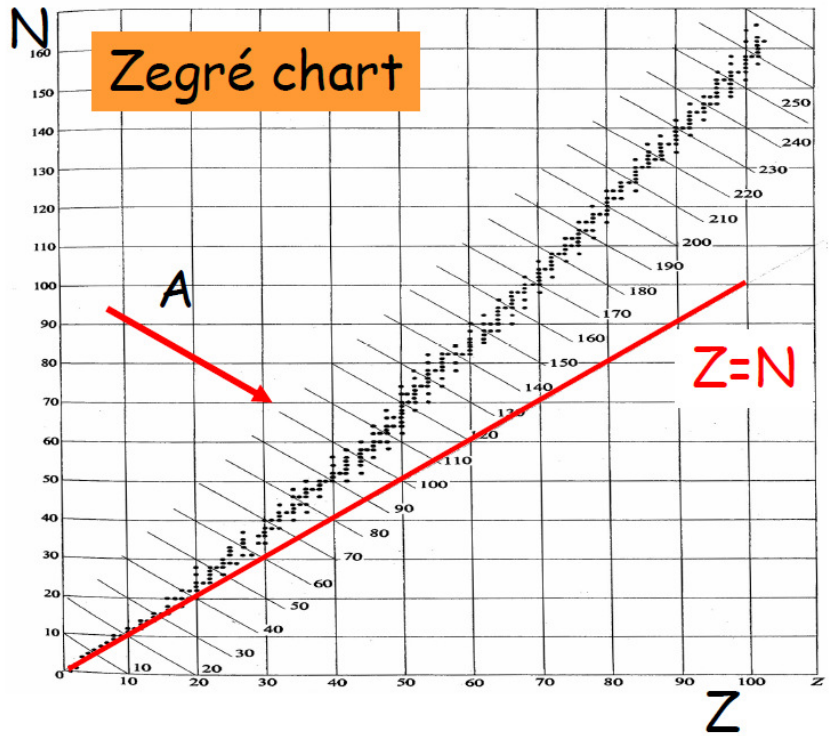
The nuclear force is not yet calculable in detail at the quark level and can only be deduced empirically from nuclear data.

Stable Nuclei

Stable nuclei do not decay by the strong interaction.
They may transform by β and α emission (weak or electromagnetic) with long lifetimes.

Characteristics

- Light nuclei tend to have $N{=}Z$.
  Heavy nuclei have more neutrons, $N > Z$.
- Most have even N and/or Z.
  Protons and neutrons tend to form pairs (only 8/284 have odd N *and* Z).
- Certain values of Z and N exhibit larger numbers of isotopes and isotones.



Binding Energy

Binding Energy is the energy required to split a nucleus into its constituents.

$$\text{Mass of nucleus } m(N,Z) = Zm_p + Nm_n - B$$

Binding energy is very important: gives information on

- forces between nucleons
- stability of nucleus
- energy released or required in nuclear decays or reactions

Relies on precise measurement of nuclear masses (mass spectrometry).

Used less in this course, but important nonetheless.

Separation Energy of a nucleon is the energy required to remove a single nucleon from a nucleus.

e.g. n: $B({}^{A}_{Z}X) - B({}^{A-1}_{Z}X) = m({}^{A-1}_{Z}X) + m(n) - m({}^{A}_{Z}X)$

p: $B({}^{A}_{Z}X) - B({}^{A-1}_{Z-1}X') = m({}^{A-1}_{Z-1}X') + m({}^{1}H) - m({}^{A}_{Z}X)$

Key Observations

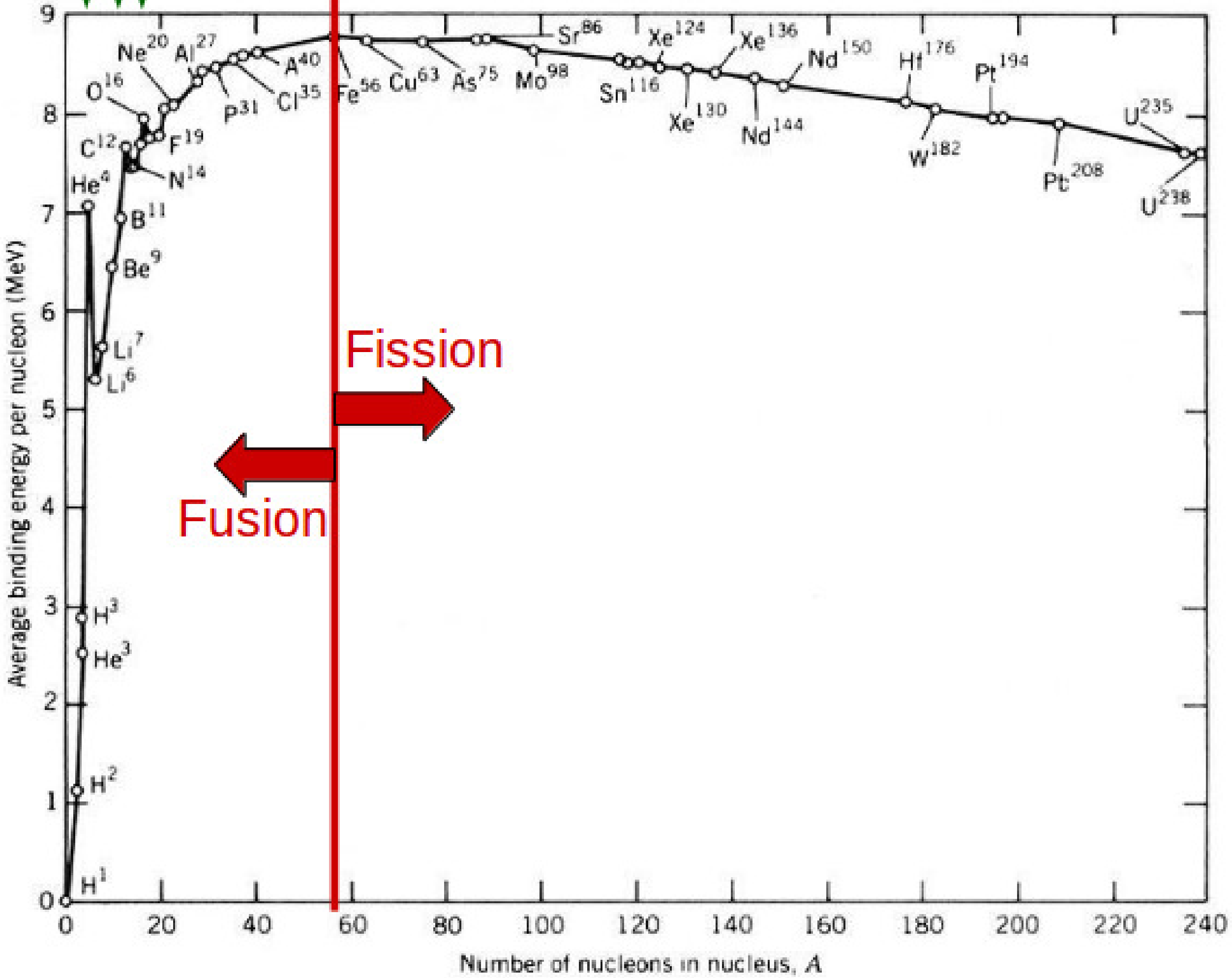
Peaks for light nuclei with A = 4n. "α stability"

Broad maximum at A~60

For A>20, B/A ~ constant
(~ 8 MeV per nucleon)
Compare to B of atomic electrons per nucleon <3 keV

Implies that nucleons are only attracted by nearby nucleons

→ Nuclear force is short range and saturated
"Saturated" means each nucleus only interacts with a limited number of neighbours; not with all nucleons.



Nuclear mass *The liquid drop model*

Atomic mass: $M(A,Z) = Z(m_p + m_e) + (A-Z)m_n \; -B$

Nuclear mass: $m(A,Z) = Zm_p + (A-Z)m_n \; -B$

Liquid drop model

Approximate the nucleus as a sphere with a uniform interior density, which drops to zero at the surface.

Liquid Drop

- Short-range intermolecular forces.
- Density independent of drop size.
- Heat required to evaporate fixed mass independent of drop size.

Nucleus

- Nuclear force short range.
- Density independent of nuclear size.
- $B/A \sim$ constant.

Nuclear mass *The liquid drop model*

Predicts the binding energy as: $B = a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}}$

$a_V A$

Volume term

Strong force between nucleons increases B and reduces mass by a constant amount per nucleon.

Nuclear volume $\sim A$

$-a_S A^{2/3}$

Surface term

Nucleons on surface are not as strongly bound $\Rightarrow$ decreases B.

Surface area $\sim R^2 \sim A^{2/3}$

$-\frac{a_c Z^2}{A^{1/3}}$

Coulomb term

Protons repel each other $\Rightarrow$ decreases B.

Electrostatic P.E. $\sim Q^2/R \sim Z^2/A^{1/3}$

But there are problems. Does not account for

- $N \sim Z$
- Nucleons tend to pair up; even N, Z favoured

Fermi gas model: assume the nucleus is a Fermi gas, in which confined nucleons can only assume certain discrete energies in accordance with the Pauli Exclusion Principle.

Addresses problems with the liquid drop model with additional terms:

$$-a_A \frac{(N-Z)^2}{A}$$

Asymmetry term Nuclei tend to have $N \sim Z$.

Kinetic energy of Z protons and N neutrons is minimised if $N{=}Z$. The greater the departure from $N{=}Z$, the smaller the binding energy. Correction scaled down by $1/A$, as levels are more closely spaced as A increases.

$$+\delta(A)$$

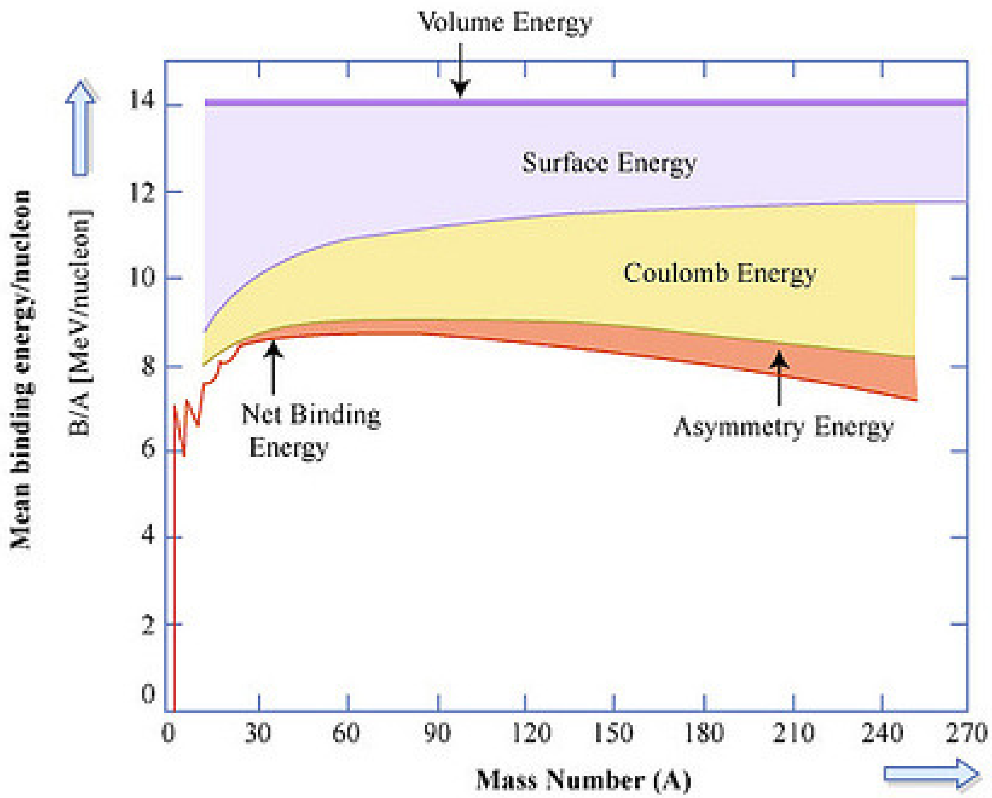
Pairing term Nuclei tend to have even Z, even N.

Pairing interaction energetically favours the formation of pairs of like nucleons (pp, nn) with spins $\uparrow\downarrow$ and symmetric spatial wavefunction.

The form is simply empirical.

$$\begin{aligned}\delta(A) &= +a_P A^{-3/4} && N, Z \text{ even-even}\\ &= -a_P A^{-3/4} && N, Z \text{ odd-odd}\\ &= 0 && N, Z \text{ even-odd}\end{aligned}$$

Putting all these terms together, we have various contributions to B/A:



Nuclear mass is well described by the **semi-empirical mass formula**

$$m(A,Z) = Zm_p + (A-Z)m_n - B$$

$$B = a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(A)$$

with the following coefficients (in MeV) obtained by fitting to data

$$a_V = 15.8,\ a_S = 18.0,\ a_C = 0.72,\ a_A = 23.5,\ a_P = 33.5$$

Nuclear Spin

The nucleus is an isolated system and so has a well defined nuclear spin

Nuclear spin quantum number J

$$|J| = \sqrt{J(J+1)} \qquad \hbar = 1$$

$$m_J = -J,\ -(J-1), \ldots,\ J-1,\ J.$$

Nuclear spin is the sum of the individual nucleons total angular momentum, j_i,

$$\vec{J} = \sum_i \vec{j}_i, \qquad \vec{j}_i = \vec{L}_i + \vec{S}_i$$

$j-j$ coupling always applies because of strong spin-orbit interaction (see later)

where the total angular momentum of a nucleon is the sum of its intrinsic spin and orbital angular momentum

- intrinsic spin of p or n is $s = 1/2$
- orbital angular momentum of nucleon is integer

A even $\rightarrow$ J must be integer

A odd $\rightarrow$ J must be $1/2$ integer

All nuclei with even N and even Z have $J = 0$.

Nuclear Parity

- All particles are eigenstates of parity $\hat{P}|\Psi\rangle = P|\Psi\rangle, \quad P = \pm 1$
- Label nuclear states with the nuclear spin and parity quantum numbers.
  Example: 0^+ ($J = 0$, parity even), 2^- ($J = 2$, parity odd)
- The parity of a nucleus is given by the product of the parities of all the neutrons and protons

$$P = \left(\prod_i P_i\right)(-1)^L$$

for ground state nucleus, $L = 0$

- The parity of a single proton or neutron is $P = (+1)(-1)^L$
  intrinsic $P = +1$ (3 quarks) nucleon L is important
- For an odd A, the parity is given by the unpaired p or n. *(Nuclear Shell Model)*
- Parity is conserved in nuclear processes (strong interaction).
- Parity of nuclear states can be extracted from experimental measurements, e.g. γ transitions.

Nuclear Size

The size of a nucleus may be determined using two sorts of interaction:

Electromagnetic Interaction gives the **charge** distribution of protons inside the nucleus, e.g.

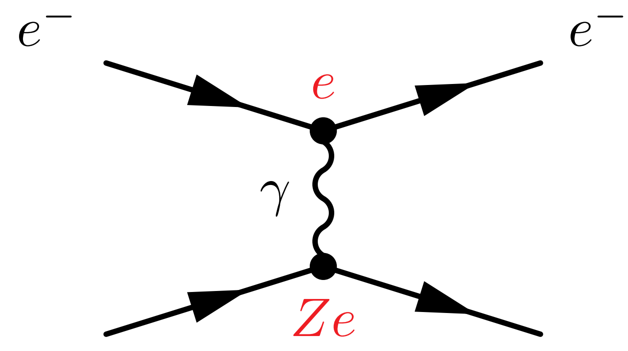
- electron scattering
- muonic atoms
- mirror nuclei

Strong Interaction gives **matter** distribution of protons and neutrons inside the nucleus. Sample nuclear and charge interactions at the same time $\Rightarrow$ more complex, e.g.

- α particle scattering (Rutherford)
- proton and neutron scattering
- Lifetime of α particle emitters (see later)
- π-mesic X-rays.

$\Rightarrow$ Find charge and matter radii EQUAL for all nuclei.

Nuclear Size *Electron scattering*



Nucleus, Z protons

Use electron as a probe to study deviations from a point-like nucleus.

Electromagnetic Interaction

Coulomb potential $$V(\vec{r}) = -\frac{Z\alpha}{r}$$

Born Approximation $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{E^2}{(2\pi)^2}\left|\int \mathrm{e}^{-i\vec{q}.\vec{r}} V(\vec{r})\,\mathrm{d}^3\vec{r}\right|^2$$

$\vec{q} = \vec{p}_i - \vec{p}_f$ is the momentum transfer

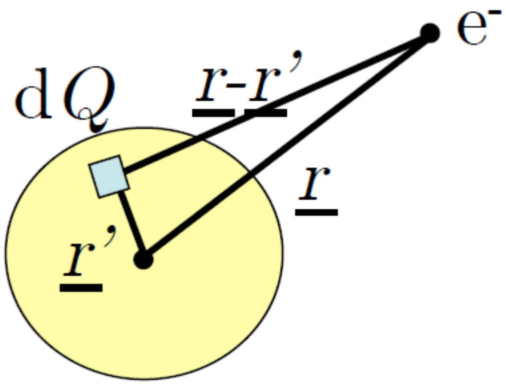
Rutherford Scattering $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2}{4E^2\sin^4\theta/2}$$

To measure a distance of $\sim$1 fm, need large energy (*ultra-relativistic*)

$$E = \frac{1}{\lambda} = 1\ \mathrm{fm}^{-1} \sim 200\ \mathrm{MeV} \qquad \hbar c = 197\ \mathrm{MeV.fm}$$

Nuclear Size *Scattering from an extended nucleus*

But the nucleus is not point-like!
$V(\vec{r})$ depends on the distribution of charge in nucleus.



Potential energy of electron due to charge $\mathrm{d}Q$

$$\mathrm{d}V = -\frac{e\,\mathrm{d}Q}{4\pi\left|\vec{r} - \vec{r'}\right|}$$

where $\mathrm{d}Q = Ze\rho(\vec{r'})\,\mathrm{d}^3\vec{r'}$

$\rho(\vec{r'})$ is the charge distribution (normalised to 1)

$$V(\vec{r}) = \int -\frac{\mathrm{e}^2 Z\rho(\vec{r'})}{4\pi\left|\vec{r} - \vec{r'}\right|} = -Z\alpha\int\frac{\rho(\vec{r'})}{\left|\vec{r} - \vec{r'}\right|}\,\mathrm{d}^3\vec{r'} \qquad \alpha = \frac{\mathrm{e}^2}{4\pi}$$

This is just a convolution of the pure Coulomb potential $Z\alpha/r$ with the normalised charge distribution $\rho(r)$.
Hence we can use the convolution theorem to help evaluate the matrix element which enters into the Born Approximation.

Nuclear Size *Scattering from an extended nucleus*

Matrix Element $\quad M_{\mathrm{if}} = \int \mathrm{e}^{\mathrm{i}\vec{q}\vec{r}} V(\vec{r})\,\mathrm{d}^3\vec{r} = -Z\alpha \int \frac{\mathrm{e}^{\mathrm{i}\vec{q}\vec{r}}}{r}\,\mathrm{d}^3\vec{r} \int \rho(\vec{r})\mathrm{e}^{\mathrm{i}\vec{q}\vec{r}}\,\mathrm{d}^3\vec{r}$

Rutherford scattering $\quad F(q^2)$

Hence,
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{point}} \left|F(q^2)\right|^2$$

where $F(q^2) = \int \rho(\vec{r})\mathrm{e}^{\mathrm{i}\vec{q}\vec{r}}\,\mathrm{d}^3\vec{r}$ is called the Form Factor and is the fourier transform of the normalised charge distribution.

Spherical symmetry, $\rho = \rho(r)$, a simple calculation (similar to our treatment of the Yukawa potential) shows that

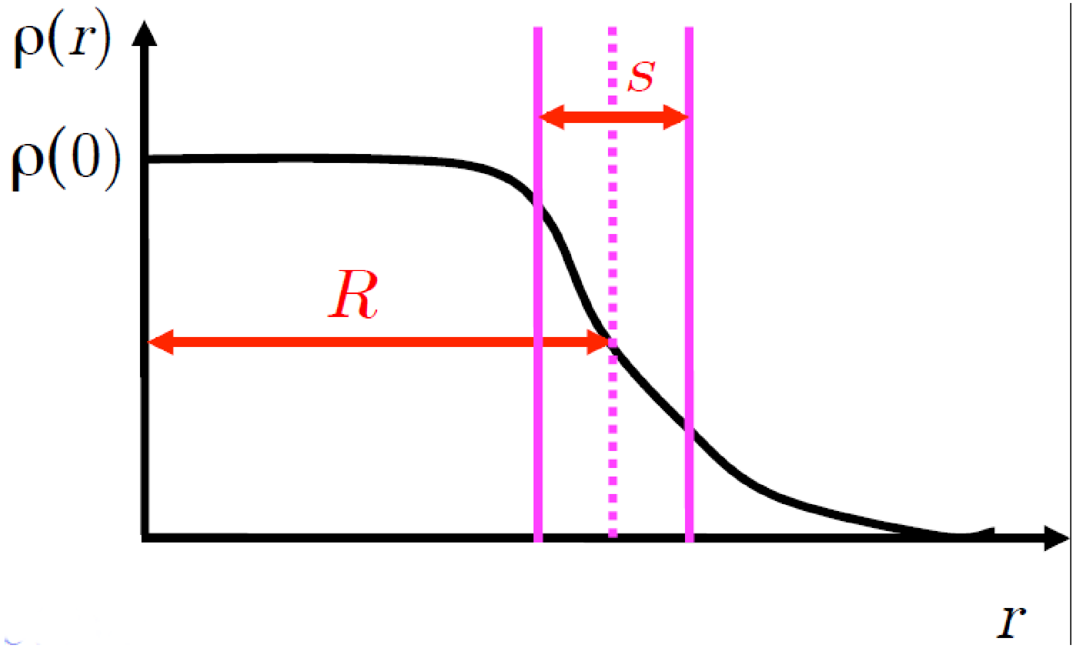
$$F(q^2) = \int_0^\infty \rho(r)\frac{\sin qr}{qr} 4\pi r^2\,\mathrm{d}r \quad ; \quad \rho(r) = \frac{1}{2\pi^2}\int_0^\infty F(q^2)\frac{\sin qr}{qr} q^2\,\mathrm{d}q$$

So if we measure cross-section, we can infer $F(q^2)$ and get the charge distribution by Fourier transformation.

Nuclear Size *Modelling charge distribution*

Use nuclear diffraction to measure scattering, and find the charge distribution inside a nucleus is well described by the Fermi parametrisation.

$$\rho(r) = \frac{\rho(0)}{1 + e^{(r-R)/s}}$$



Fit this to data to determine parameters R and s.

- R is the radius at which $\rho(r) = \rho(0)/2$

  Find R increases with A: $R = r_0 A^{1/3}$ $r_0 \sim 1.2\,\text{fm}$.

- s is the surface width or skin thickness over which $\rho(r)$ falls from 90%→10%.

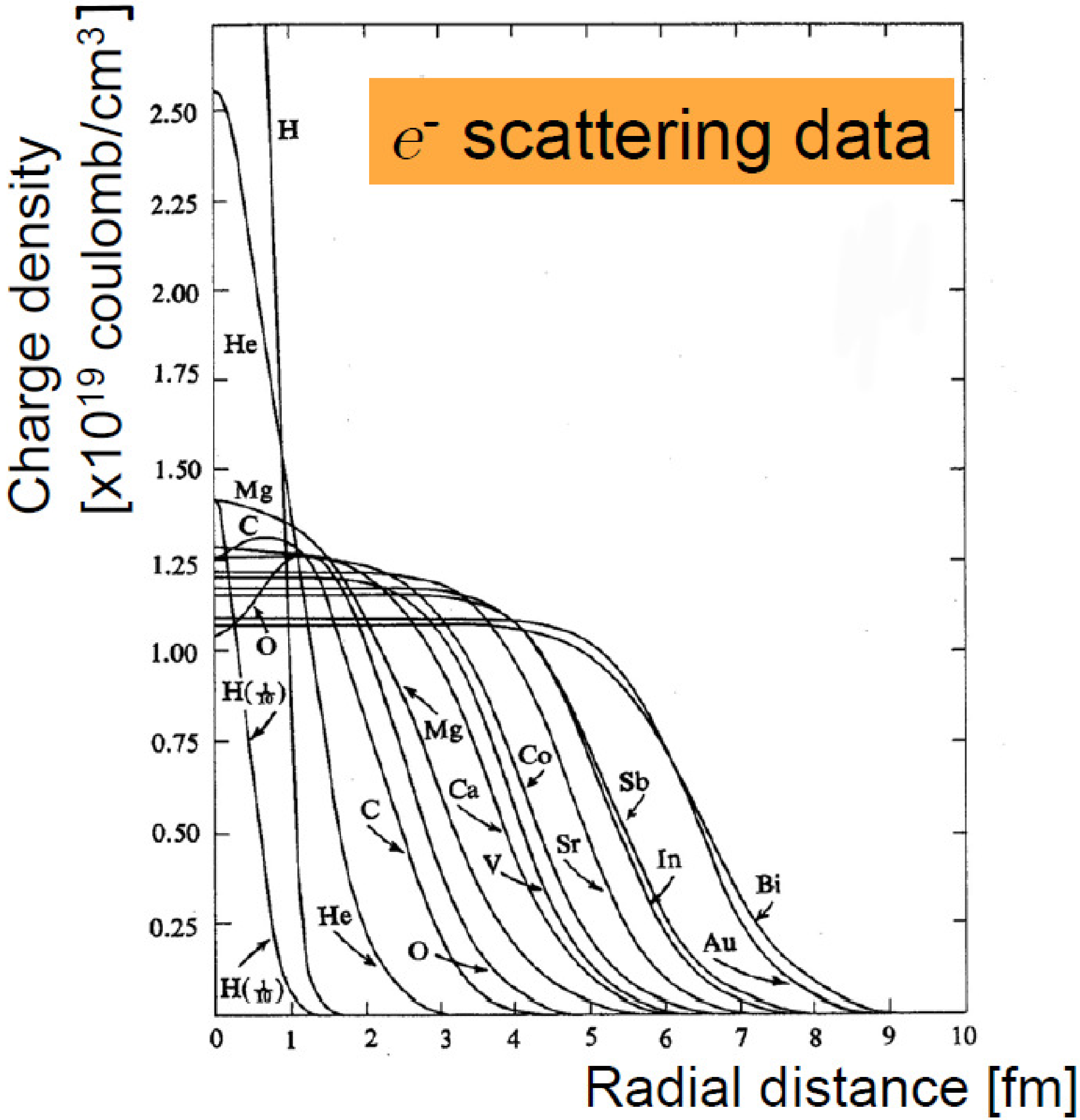
  Find s is is approximately the same for all nuclei ($s \sim 2.5$ fm); governed by the range of the strong nuclear interaction

Nuclear Size *Modelling charge distribution*

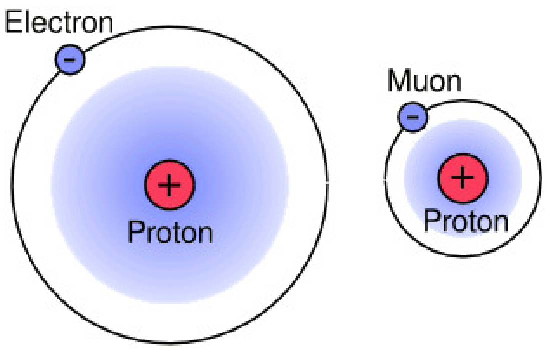
Fits to e^- scattering data show the Fermi parametrisation models nuclear charge distributions well.

Shows that all nuclei have roughly the same density in their interior.

Radius $\sim R_0 A^{1/3}$ with $R_0 \sim 1.2$ fm $\Rightarrow$ consistent with short-range saturated forces.



Nuclear Size *Muonic Atoms*



Muons can be brought to rest in matter and trapped in orbit $\rightarrow$ probe EM interactions with nucleus.
The large muon mass affects its orbit, $m_\mu \sim 207\ m_e$

Bohr radius, $r \propto 1/Zm$

Hydrogen atom with electrons: $r = a_0 \sim 53,000$ fm

with muons: $r \sim 285$ fm

Lead ($Z = 82$) with muons: $r \sim 3$ fm   Inside nucleus!

Energy levels, $E \propto Z^2 m$

Rapid transitions to lower energy levels $\sim\ 10^{-9}$s

Factor of 2 effect seen from nuclear size in muonic lead

Transition energy ($2P_{3/2} \rightarrow 1S_{1/2}$) : 16.41 MeV (Bohr theory) vs 6.02 MeV (measured)

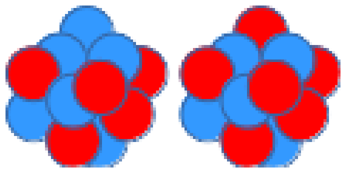
Muon lifetime, $\tau_\mu \sim 2\mu$s

Decays via $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ – Plenty of time spent in $1s$ state.

$Z_{\text{effective}}$ and E are changed relative to electrons.
Measure X-ray energies $\rightarrow$ nuclear radius.

Nuclear Size *Mirror Nuclei*

$^{11}_{5}$B $^{11}_{6}$C



Different nuclear masses from *p*-*n* difference and the different Coulomb terms.

$$m(A,Z) = Zm_p + (A-Z)m_n - \left[a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(A)\right]$$

For the *atomic* mass difference, don't forget the electrons!

$$M(A,Z+1) - M(A,Z) = \Delta E_c + m_p + m_e - m_n$$

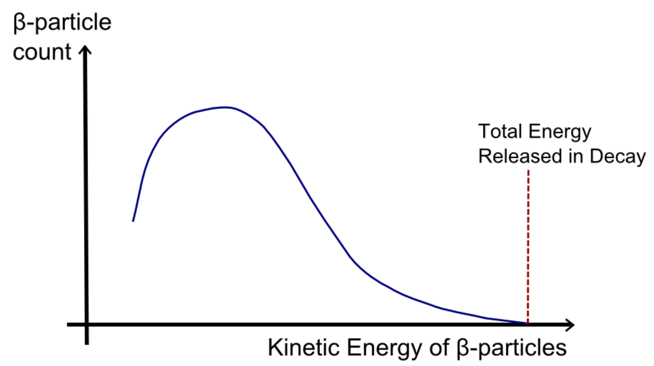
where $\Delta E_c = \frac{3}{5}\frac{A\alpha}{R}$ (see Question 33)

Probe the atomic mass difference between two mirror nuclei by observing β^+ decay spectra (3-body decay).

$$^{11}_{6}\text{C} \rightarrow {}^{11}_{5}\text{B} + e^+ + \nu_e \qquad (p \rightarrow n + e^+ + \nu_e)$$

$$M(A,Z+1) - M(A,Z) = 2m_e + E_{\max} \qquad m_\nu \sim 0$$

where $E_{\max}$ is the maximum kinetic energy of the positron.

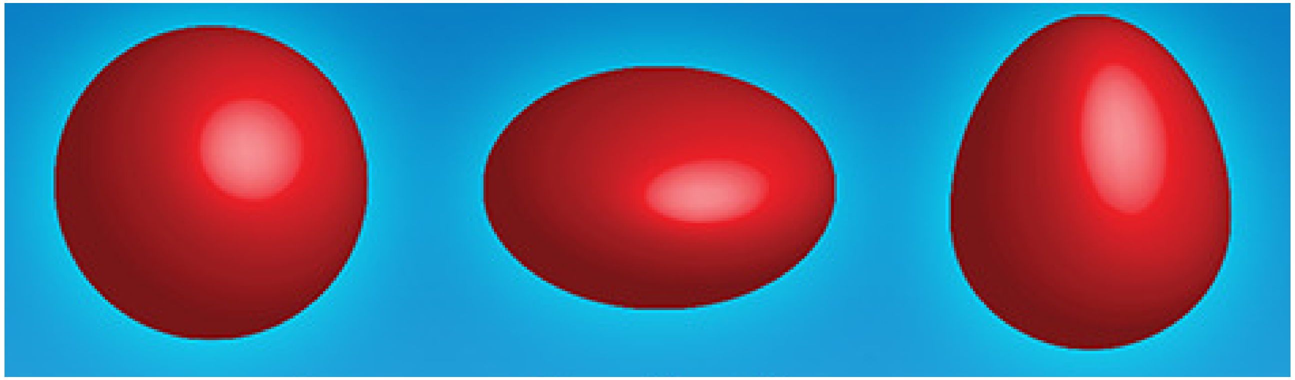


Relate mass difference to ΔE_c and extract the nuclear radius

$$R = \frac{3A\alpha}{5}\left[\frac{1}{E_{\max} - m_p + m_n + m_e}\right]$$

Nuclear Shape

The shape of nuclei can be inferred from measuring their electromagnetic moments.



Nuclear moments give information about the way magnetic moment and charge is distributed throughout the nucleus.

The two most important moments are:

Electric Quadrupole Moment Q

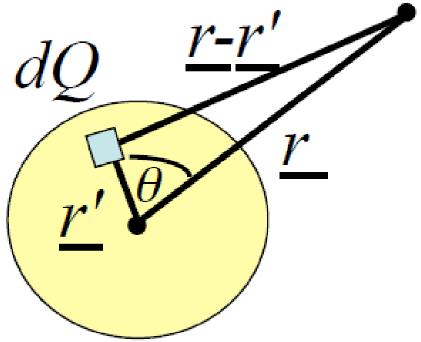
Magnetic Dipole Moment μ

Nuclear Shape *Electric Moments*

Electric moments depend on the charge distribution inside the nucleus.

Parameterise the nuclear shape using a multipole expansion of the external electric field or potential

$$V(r) = \frac{1}{4\pi}\int \frac{\rho(\vec{r'})}{\left|\vec{r} - \vec{r'}\right|}\,\mathrm{d}^3\vec{r'}$$



where $\rho(\vec{r'})\,\mathrm{d}^3\vec{r'} = Ze$ and $r(r') =$ distance to observer (charge element) from origin.

$$\left|\vec{r} - \vec{r'}\right| = \left[r^2 + r'^2 - 2rr'\cos\theta\right]^{1/2} \Rightarrow \left|\vec{r} - \vec{r'}\right|^{-1} = r^{-1}\left[1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right]^{-1/2}$$

$$\left|\vec{r} - \vec{r'}\right|^{-1} = r^{-1}\left[1 - \frac{1}{2}\left(\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right)^2 + \ldots\right]$$

$$\sim r^{-1}\left[1 + \frac{r'}{r}\cos\theta + \frac{1}{2}\frac{r'^2}{r^2}\left(3\cos^2\theta - 1\right) + \ldots\right]$$

$r' \ll r \Rightarrow$ expansion in powers of $r'r$; or equivalently Legendre polynomials

$$V(r) = \frac{1}{4\pi r}\left[Ze + \frac{1}{r}\int r'\cos\theta\rho(r')\,\mathrm{d}^3\vec{r'} + \frac{1}{2r^2}\int r'^2(3\cos\theta - 1)\rho(r')\,\mathrm{d}^3\vec{r'} + \ldots\right]$$

Nuclear Shape *Electric Moments*

Let r define z-axis, $z = r' \cos\theta$

$$V(r) = \frac{1}{4\pi r}\left[Ze + \frac{1}{r}\int z\rho(r')\,\mathrm{d}^3\vec{r'} + \frac{1}{2r^2}\int (3z^2 - r'^2)\rho(r')\,\mathrm{d}^3\vec{r'} + \ldots\right]$$

Quantum limit: $\rho(r') = Ze.\left|\psi(\vec{r'})\right|^2$

The electric moments are the coefficients of each successive power of $1/r$

E0 moment $\int Ze.\psi^*\psi\,\mathrm{d}^3\vec{r'} = Ze$ *charge*

No shape information

E1 moment $\int \psi^* z\psi\,\mathrm{d}^3\vec{r'}$ *electric dipole*

Always zero since ψ have definite parity

$$|\psi(\vec{r})|^2 = |\psi(-\vec{r})|^2$$

E2 moment $\int \frac{1}{e}\psi^*(3z^2 - r'^2)\psi\,\mathrm{d}^3\vec{r'}$ *electric quadrupole*

First interesting moment!

Nuclear Shape *Electric Moments*

Electric Quadrupole Moment $$Q = \frac{1}{e}\int (3z^2 - r^2)\rho(\vec{r})\,\mathrm{d}^3\vec{r}$$

Units: m^2 or barns (though sometimes the factor of e is left in)

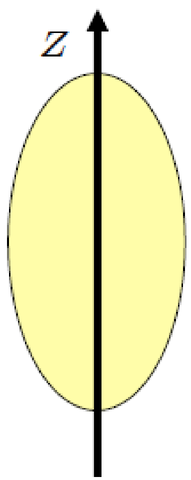
If spherical symmetry, $\overline{z^2} = \frac{1}{3}\overline{r^2} \Rightarrow Q = 0$

- $Q = 0$ Spherical nucleus. All $J = 0$ nuclei have $Q = 0$.
- Large Q Highly deformed nucleus. e.g. Na

  Two cases:

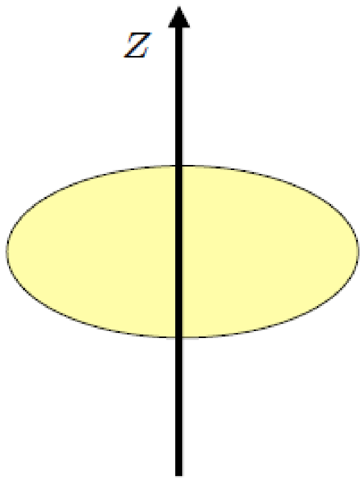
Prolate spheroid

$Q > 0$



Oblate spheroid

$Q < 0$

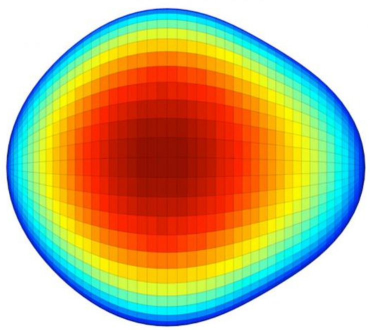


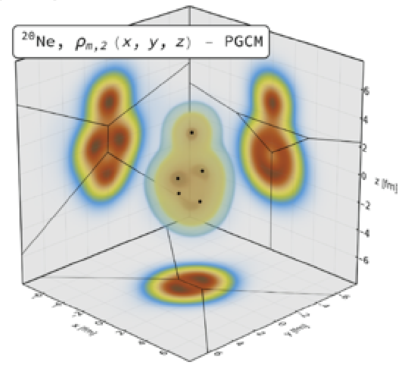
Aside.

Radium-224 is pear-shaped, Ne-20 is a snowman!

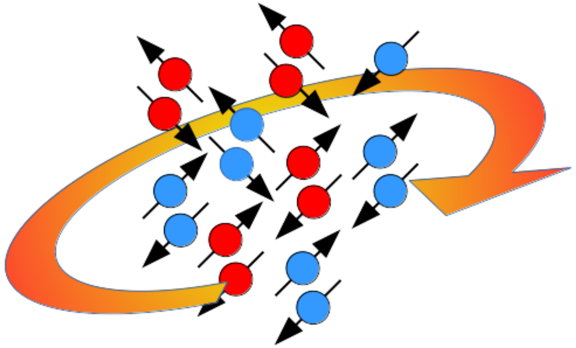
Non-zero quadrupole and octupole moments.

(ISOLDE, CERN, 2013), (LHCb, CERN, 2025)





Nuclear Shape *Magnetic Moments*



Nuclear magnetic dipole moments arise from

- intrinsic spin magnetic dipole moments of the protons and neutrons
- circulating currents (motion of the protons)

The nuclear magnetic dipole moment can be written as

$$\vec{\mu} = \frac{\mu_N}{\hbar} \sum_i \left[g_L \vec{L} + g_s \vec{s} \right]$$

summed over all p, n

where $\mu_N = e\hbar/2m_p$ is the Nuclear Magneton.

or $\mu = g_J \mu_N J$ where J total nuclear spin quantum number

g_J nuclear g-factor (analogous to Landé g-factor in atoms)

g_J may be predicted using the Nuclear Shell Model (see later), and measured using magnetic resonance (see Advanced Quantum course).

All even-even nuclei have $\mu = 0$ since $J = 0$

Summary

- Nuclear binding energy – short range saturated forces
- Semi-empirical Mass Formula – based on liquid drop model + simple inclusion of quantum effects

$$m(A,Z) = Zm_p + (A-Z)m_n - B$$

$$B = a_V A - a_S A^{2/3} - \frac{a_c Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(A)$$

- Nuclear size from electron scattering, muonic atoms, and mirror nuclei. Constant density; radius $\propto A^{1/3}$
- Nuclear spin, parity, electric and magnetic moments.

Problem Sheet: q.31-33

Up next...
Section 14: The Structure of Nuclei