# The HERWIG Event Generator

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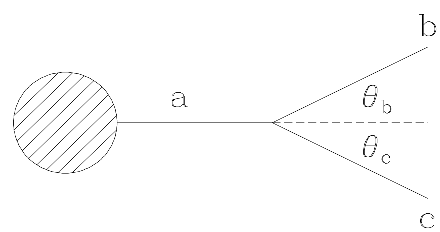
## Lecture 1: Parton branching & showering

- Parton branching
  - Kinematics
  - Splitting functions
  - Phase space
- Parton showering
  - Evolution of parton distributions
  - Sudakov form factor
  - Monte Carlo method
  - Soft gluon coherence
  - Angular ordering

# Parton branching

- Leading soft and collinear enhanced terms in QCD matrix elements (and corresponding virtual corrections) can be identified and summed to all orders. Consider splitting of outgoing parton $a$ into $b + c$.




- Can assume $p_b^2, p_c^2 \ll p_a^2 \equiv t$. Opening angle is $\theta = \theta_a + \theta_b$, energy fraction is

$$z = E_b/E_a = 1 - E_c/E_a \ .$$

- For small angles

$$\begin{aligned} t &= 2E_bE_c(1-\cos\theta) = z(1-z)E_a^2\theta^2 \ , \\ \theta &= \frac{1}{E_a}\sqrt{\frac{t}{z(1-z)}} = \frac{\theta_b}{1-z} = \frac{\theta_c}{z} \ . \end{aligned}$$

- Consider first $g \to gg$ branching:
  - Amplitude has triple-gluon vertex factor

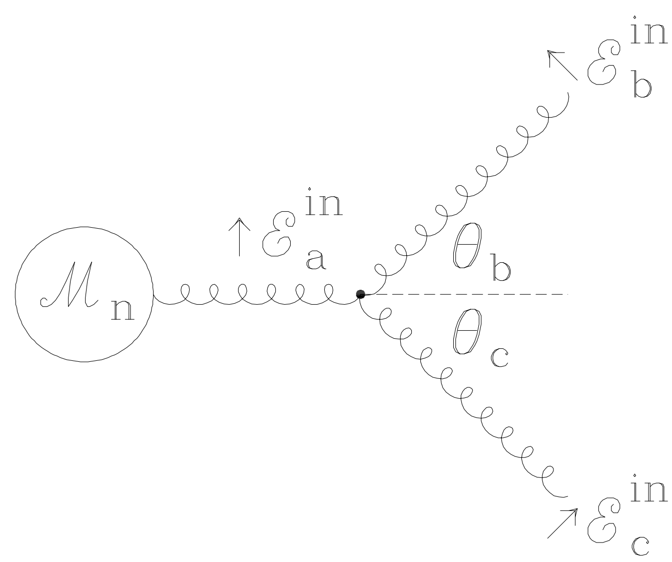
$$g f^{ABC} \epsilon_a^\alpha \epsilon_b^\beta \epsilon_c^\gamma [g_{\alpha\beta}(p_a - p_b)_\gamma + g_{\beta\gamma}(p_b - p_c)_\alpha + g_{\gamma\alpha}(p_c - p_a)_\beta]$$

    $\epsilon_i^\mu$ is polarization vector for gluon $i$. All momenta defined as outgoing here, so $p_a = -p_b - p_c$. Using this and $\epsilon_i \cdot p_i = 0$, vertex factor becomes

$$-2g f^{ABC}[(\epsilon_a \cdot \epsilon_b)(\epsilon_c \cdot p_b) - (\epsilon_b \cdot \epsilon_c)(\epsilon_a \cdot p_b) - (\epsilon_c \cdot \epsilon_a)(\epsilon_b \cdot p_c)] \,.$$

  - Resolve polarization vectors into $\epsilon_i^{\text{in}}$ in plane of branching and $\epsilon_i^{\text{out}}$ normal to plane, so that

$$\begin{aligned} \epsilon_i^{\text{in}} \cdot \epsilon_j^{\text{in}} &= \epsilon_i^{\text{out}} \cdot \epsilon_j^{\text{out}} = -1 \\ \epsilon_i^{\text{in}} \cdot \epsilon_j^{\text{out}} &= \epsilon_i^{\text{out}} \cdot p_j \;=\; 0 \,. \end{aligned}$$




- For small $\theta$, neglecting terms of order $\theta^2$, we have

$$\begin{aligned}
\epsilon_a^{\rm in}\cdot p_b &= -E_b\theta_b = -z(1-z)E_a\theta \\
\epsilon_b^{\rm in}\cdot p_c &= +E_c\theta = (1-z)E_a\theta \\
\epsilon_c^{\rm in}\cdot p_b &= -E_b\theta = -zE_a\theta\,.
\end{aligned}$$

- Vertex factor proportional to $\theta$, together with propagator factor of $1/t \propto 1/\theta^2$, gives $1/\theta$ collinear singularity in amplitude.
- $(n+1)$-parton matrix element squared (in small-angle region) is given in

terms of that for $n$ partons:

$$|\mathcal{M}_{n+1}|^2 \sim \frac{4g^2}{t} C_A F(z;\epsilon_a,\epsilon_b,\epsilon_c)|\mathcal{M}_n|^2$$

where colour factor $C_A = 3$ comes from $f^{ABC}f^{ABC}$ and functions $F$ are given below

| $\epsilon_a$ | $\epsilon_b$ | $\epsilon_c$ | $F(z;\epsilon_a,\epsilon_b,\epsilon_c)$ |
|---|---|---|---|
| in | in | in | $(1-z)/z + z/(1-z) + z(1-z)$ |
| in | out | out | $z(1-z)$ |
| out | in | out | $(1-z)/z$ |
| out | out | in | $z/(1-z)$ |

- Sum/averaging over polarizations gives

$$C_A \langle F \rangle \equiv \hat{P}_{gg}(z) = C_A \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] .$$

This is (unregularized) gluon splitting function.

- Enhancements at $z \to 0$ ($b$ soft) and $z \to 1$ ($c$ soft) due to soft gluon polarized in plane of branching.

- Correlation between polarization and plane of branching (angle $\phi$):

$$\begin{aligned} F_\phi &\propto \sum_{\epsilon_{b,c}} |\cos\phi\,\mathcal{M}(\epsilon_a^{\rm in},\epsilon_b,\epsilon_c) + \sin\phi\,\mathcal{M}(\epsilon_a^{\rm out},\epsilon_b,\epsilon_c)|^2 \\ &\propto \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) + z(1-z)\cos 2\phi\,. \end{aligned}$$

  Hence branching in plane of gluon polarization preferred.

- Consider next $g \to q\bar{q}$ branching:
  - Vertex factor is

$$-ig\bar{u}^b\gamma_\mu\epsilon_a^\mu v^c$$

    where $u^b$ and $v^c$ are quark and antiquark spinors.
  - Spin-averaged splitting function is

$$T_R\,\langle F\rangle \equiv \hat{P}_{qg}(z) = T_R\left[z^2 + (1-z)^2\right].$$

    No soft ($z \to 0$ or 1) singularities since these are associated only with gluon emission.
  - Vector quark-gluon coupling implies (for $m_q \simeq 0$) $q$ and $\bar{q}$ helicities always opposite (helicity conservation).

- Correlation between gluon polarization and plane of branching:

$$F_\phi = z^2 + (1-z)^2 - 2z(1-z)\cos 2\phi$$

  i.e. strong preference for splitting perpendicular to polarization.

- Branching $q \to qg$:
  - Spin-averaged splitting function is

$$C_F \langle F \rangle \equiv \hat{P}_{qq}(z) = C_F \frac{1+z^2}{1-z} \; .$$

  - Helicity conservation ensures that quark does not change helicity in branching.
  - Gluon polarized in plane of branching preferred, polarization angular correlation being

$$F_\phi = \frac{1+z^2}{1-z} + \frac{2z}{1-z}\cos 2\phi \; .$$

# Phase space

- Phase space factors before and after branching are related by

$$d\Phi_{n+1} = d\Phi_n \frac{1}{4(2\pi)^3} dt\, dz\, d\phi \; .$$

- Hence cross sections before and after branching are related by

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_{\mathrm{S}}}{2\pi} C F$$

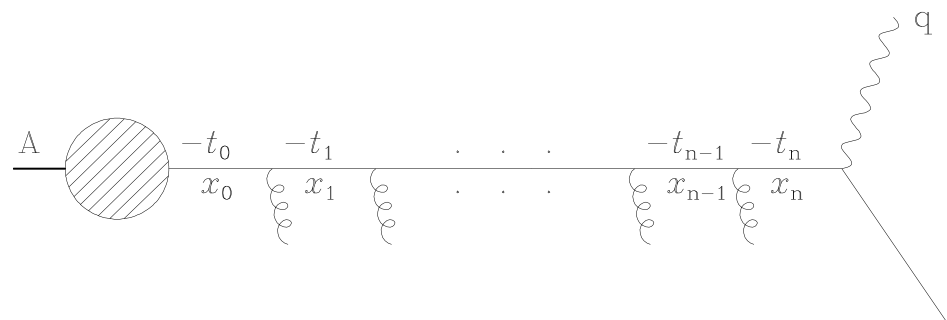
where $C$ and $F$ are colour factor and polarization-dependent $z$-distribution introduced earlier. Integrating over azimuthal angle gives

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_{\mathrm{S}}}{2\pi} \hat{P}_{ba}(z) \; .$$

where $\hat{P}_{ba}(z)$ is $a \to b$ splitting function.

# Evolution of quark distribution

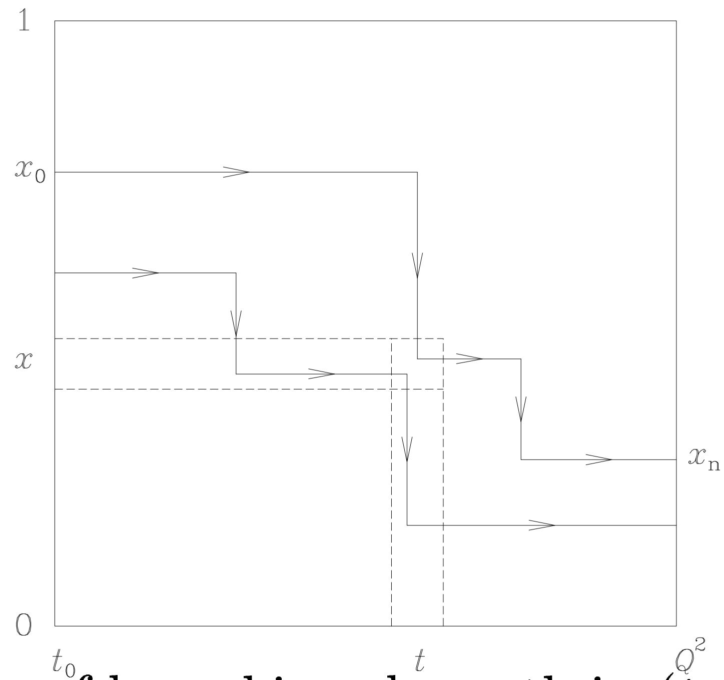
- Consider enhancement of higher-order contributions due to multiple small-angle gluon emission, for example in deep inelastic scattering (DIS)




- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction $x_0$ of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.

- Cross section will depend on $Q^2$ and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

- To derive evolution equation for $Q^2$-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution, also useful later for Monte Carlo simulation.




- Represent sequence of branchings by path in $(t, x)$-space. Each branching is a step downwards in $x$, at a value of $t$ equal to (minus) the virtual mass-squared after the branching.

- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale $t$ is just the $x$-distribution of paths at that scale.

- Consider change in the parton distribution $D(x, t)$ when $t$ is increased to $t + \delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by $\delta x$.

- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\begin{aligned} \delta D_{\rm in}(x,t) &= \frac{\delta t}{t}\int_x^1 dx'\, dz \frac{\alpha_{\rm S}}{2\pi}\hat{P}(z) D(x',t)\,\delta(x - zx') \\ &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_{\rm S}}{2\pi}\hat{P}(z) D(x/z,t) \end{aligned}$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\begin{aligned} \delta D_{\rm out}(x,t) &= \frac{\delta t}{t} D(x,t)\int_0^x dx'\, dz \frac{\alpha_{\rm S}}{2\pi}\hat{P}(z)\,\delta(x' - zx) \\ &= \frac{\delta t}{t} D(x,t)\int_0^1 dz \frac{\alpha_{\rm S}}{2\pi}\hat{P}(z) \end{aligned}$$

- Change in population of element is

$$\begin{aligned} \delta D(x,t) &= \delta D_{\rm in} - \delta D_{\rm out} \\ &= \frac{\delta t}{t}\int_0^1 dz\, \frac{\alpha_{\rm S}}{2\pi}\hat{P}(z)\left[\frac{1}{z} D(x/z,t) - D(x,t)\right]\,. \end{aligned}$$

- Introduce plus-prescription with definition

$$\int_0^1 dz\, f(z)\, g(z)_+ \;=\; \int_0^1 dz\, [f(z) - f(1)]\, g(z)\,.$$

Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+\,,$$

and obtain Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P(z)D(x/z,t)\,.$$

$$\text{[N.B. } \textstyle\int_x^1 dz\, f(z)g(z)_+ = \int_0^1 dz\, \Theta(z-x)f(z)g(z)_+$$
$$= \textstyle\int_x^1 dz\, [f(z)-f(1)]g(z) - f(1)\int_0^x dz\, g(z).\text{]}$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale $t$. In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

# Quark and gluon distributions

- For several different types of partons, must take into account different processes by which parton of type $i$ can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P_{ij}(z)D_j(x/z,t)\,.$$

- Quark $(i=q)$ can enter element via either $q \to qg$ or $g \to q\bar{q}$, but can only leave via $q \to qg$. Thus plus-prescription applies only to $q \to qg$ part, giving

$$\begin{aligned} P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\ P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\left[z^2+(1-z)^2\right] \end{aligned}$$

- Gluon can arrive either from $g \to gg$ (2 contributions) or from $q \to qg$ (or $\bar{q} \to \bar{q}g$). Thus number arriving is

$$\begin{aligned}\delta D_{g,\mathrm{in}} &= \frac{\delta t}{t}\int_0^1 dz\frac{\alpha_S}{2\pi}\left\{\hat{P}_{gg}(z)\left[\frac{D_g(x/z,t)}{z}+\frac{D_g(x/(1-z),t)}{1-z}\right]\right.\\&\left.+\frac{\hat{P}_{qg}(z)}{1-z}\left[D_q\left(\frac{x}{1-z},t\right)+D_{\bar{q}}\left(\frac{x}{1-z},t\right)\right]\right\}\\&= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\left\{2\hat{P}_{gg}(z)D_g\left(\frac{x}{z},t\right)+\hat{P}_{qg}(1-z)\left[D_q\left(\frac{x}{z},t\right)+D_{\bar{q}}\left(\frac{x}{z},t\right)\right]\right\},\end{aligned}$$

- Gluon can leave by splitting into either $gg$ or $q\bar{q}$, so that

$$\delta D_{g,\mathrm{out}} = \frac{\delta t}{t}D_g(x,t)\int_0^1 dz\frac{\alpha_S}{2\pi}\left[\hat{P}_{gg}(z)+N_f\hat{P}_{qg}(z)\,dz\right] .$$

- After some manipulation we find

$$\begin{aligned}P_{gg}(z) &= 2C_A\left[\left(\frac{z}{1-z}+\tfrac{1}{2}z(1-z)\right)_+ +\frac{1-z}{z}+\tfrac{1}{2}z(1-z)\right]-\frac{2}{3}N_fT_R\,\delta(1-z),\\P_{gq}(z) &= P_{g\bar{q}}(z)=\hat{P}_{qq}(1-z)=C_F\frac{1+(1-z)^2}{z} .\end{aligned}$$

- Using definition of the plus-prescription, can check that

$$\begin{aligned}\left(\frac{z}{1-z}+\tfrac{1}{2}z(1-z)\right)_+ &= \frac{z}{(1-z)_+}+\tfrac{1}{2}z(1-z)+\frac{11}{12}\delta(1-z)\ ,\\ \left(\frac{1+z^2}{1-z}\right)_+ &= \frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\ ,\end{aligned}$$

so $P_{qq}$ and $P_{gg}$ can be written in more common forms

$$\begin{aligned}P_{qq}(z) &= C_F\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right]\\ P_{gg}(z) &= 2C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right]+\frac{1}{6}(11C_A-4N_fT_R)\,\delta(1-z)\ .\end{aligned}$$

# Sudakov form factor

- DGLAP equations convenient for evolution of parton distributions. To study structure of final states, slightly different form is useful. Consider again simplified treatment with only one type of branching. Introduce Sudakov form factor:

$$\Delta(t) \equiv \exp\left[-\int_{t_0}^{t} \frac{dt'}{t'} \int dz \, \frac{\alpha_S}{2\pi} \hat{P}(z)\right] ,$$

Then

$$\begin{aligned} t\frac{\partial}{\partial t} D(x,t) &= \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t) + \frac{D(x,t)}{\Delta(t)} t \frac{\partial}{\partial t} \Delta(t) , \\ t\frac{\partial}{\partial t}\left(\frac{D}{\Delta}\right) &= \frac{1}{\Delta} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t) . \end{aligned}$$

- This is similar to DGLAP, except $D$ replaced by $D/\Delta$ and regularized splitting function $P$ replaced by unregularized $\hat{P}$. Integrating,

$$\begin{aligned} D(x,t) &= \Delta(t) D(x,t_0) \\ &\quad + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t') . \end{aligned}$$

- This has simple interpretation. First term is contribution from paths that do not branch between scales $t_0$ and $t$. Thus Sudakov form factor $\Delta(t)$ is probability of evolving from $t_0$ to $t$ without branching. Second term is contribution from paths which have their last branching at scale $t'$. Factor of $\Delta(t)/\Delta(t')$ is probability of evolving from $t'$ to $t$ without branching.

- Generalization to several species of partons straightforward. Species $i$ has Sudakov form factor

$$\Delta_i(t) \equiv \exp\left[-\sum_j \int_{t_0}^{t} \frac{dt'}{t'} \int dz\, \frac{\alpha_S}{2\pi} \hat{P}_{ji}(z)\right] ,$$

which is probability of it evolving from $t_0$ to $t$ without branching. Then

$$t\frac{\partial}{\partial t}\left(\frac{D_i}{\Delta_i}\right) = \frac{1}{\Delta_i}\sum_j \int \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}_{ij}(z) D_j(x/z,t) \; .$$

# Infrared cutoff

- In DGLAP equation, infrared singularities of splitting functions at $z = 1$ are regularized by plus-prescription. However, in above form we must introduce an explicit infrared cutoff, $z < 1 - \epsilon(t)$. Branchings with $z$ above this range are unresolvable: emitted parton is too soft to detect. Sudakov form factor with this cutoff is probability of evolving from $t_0$ to $t$ without any resolvable branching.
- Sudakov form factor sums enhanced virtual (parton loop) as well as real (parton emission) contributions. No-branching probability is the sum of virtual and unresolvable real contributions: both are divergent but their sum is finite.
- Infrared cutoff $\epsilon(t)$ depends on what we classify as resolvable emission. For timelike branching, natural resolution limit is given by cutoff on parton virtual mass-squared, $t > t_0$. When parton energies are much larger than virtual masses, transverse momentum in $a \to bc$ is

$$p_T^2 = z(1-z)p_a^2 - (1-z)p_b^2 - zp_c^2 > 0 \,.$$

Hence for $p_a^2 = t$ and $p_b^2,\, p_c^2 > t_0$ we require $z(1-z) > t_0/t$, that is,

$$z,\ 1-z > \epsilon(t) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4t_0/t} \simeq t_0/t \,.$$

- Quark Sudakov form factor is then

$$\Delta_q(t) \simeq \exp\left[-\int_{2t_0}^{t} \frac{dt'}{t'} \int_{t_0/t'}^{1-t_0/t'} dz \frac{\alpha_S}{2\pi} \hat{P}_{qq}(z)\right] .$$

- Careful treatment of running coupling suggests its argument should be $p_T^2 \sim z(1-z)t'$. Then at large $t$

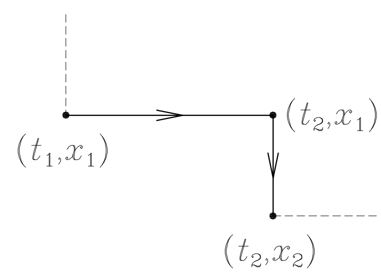
$$\Delta_q(t) \sim \left(\frac{\alpha_S(t)}{\alpha_S(t_0)}\right)^{p \ln t} ,$$

($p =$ a constant), which tends to zero faster than any negative power of $t$.

- Infrared cutoff discussed here follows from kinematics. We shall see later that QCD dynamics effectively reduces phase space for parton branching, leading to a more restrictive effective cutoff.

# Monte Carlo method

- Formulation in terms of Sudakov form factor is well suited to computer implementation, and is basis of “parton shower” Monte Carlo programs.
- Monte Carlo branching algorithm operates as follows: given virtual mass scale and momentum fraction $(t_1, x_1)$ after some step of the evolution, or as initial conditions, it generates values $(t_2, x_2)$ after the next step.




- Since probability of evolving from $t_1$ to $t_2$ without branching is $\Delta(t_2)/\Delta(t_1)$, $t_2$ can be generated with the correct distribution by solving

$$\frac{\Delta(t_2)}{\Delta(t_1)} = \mathcal{R}$$

where $\mathcal{R}$ is random number (uniform on $[0, 1]$).

❖ If $t_2$ is higher than hard process scale $Q^2$, this means branching has finished.

- Otherwise, generate $z = x_2/x_1$ with distribution proportional to $(\alpha_S/2\pi)P(z)$, where $P(z)$ is appropriate splitting function, by solving

$$\int_{\epsilon}^{x_2/x_1} dz \frac{\alpha_S}{2\pi} P(z) = \mathcal{R}' \int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_S}{2\pi} P(z)$$

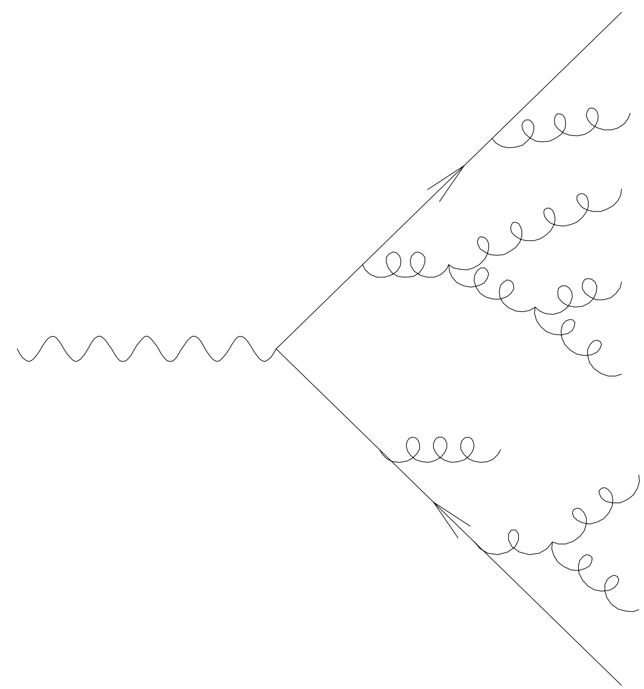
  where $\mathcal{R}'$ is another random number and $\epsilon$ is cutoff for resolvable branching.

- In DIS, $(t_i, x_i)$ values generated define virtual masses and momentum fractions of exchanged quark, from which momenta of emitted gluons can be computed. Azimuthal emission angles are then generated uniformly in the range $[0, 2\pi]$. More generally, e.g. when exchanged parton is a gluon, azimuths must be generated with polarization angular correlations discussed earlier.

- Each emitted (timelike) parton can itself branch. In that case $t$ evolves downwards towards cutoff value $t_0$, rather than upwards towards hard process scale $Q^2$. Probability of evolving downwards without branching between $t_1$ and $t_2$ is now given by

$$\frac{\Delta(t_1)}{\Delta(t_2)} = \mathcal{R}\,.$$

  Thus branching stops when $\mathcal{R} < \Delta(t_1)$.



- Due to successive branching, parton cascade or shower develops. Each outgoing line is source of new cascade, until all outgoing lines have stopped branching. At this stage, which depends on cutoff scale $t_0$, outgoing partons have to be converted into hadrons via a hadronization model.

# Soft gluon emission

- Parton branching formalism discussed so far takes account of collinear enhancements to all orders in PT. There are also soft enhancements: When external line with momentum $p$ and mass $m$ (not necessarily small) emits gluon with momentum $q$, propagator factor is

$$\frac{1}{(p \pm q)^2 - m^2} = \frac{\pm 1}{2p \cdot q} = \frac{\pm 1}{2\omega E(1 - v\cos\theta)}$$

  where $\omega$ is emitted gluon energy, $E$ and $v$ are energy and velocity of parton emitting it, and $\theta$ is angle of emission. This diverges as $\omega \to 0$, for any velocity and emission angle.

- Including numerator, soft gluon emission gives a colour factor times a universal, spin-independent factor in amplitude

$$F_{\rm soft} = \frac{p \cdot \epsilon}{p \cdot q}$$

  where $\epsilon$ is polarization of emitted gluon.

- For example, emission from quark gives numerator factor $N \cdot \epsilon$, where

$$\begin{aligned} N^\mu &= (\not{p} + \not{q} + m)\gamma^\mu u(p) \underset{\omega \to 0}{\to} (\gamma^\nu \gamma^\mu p_\nu + \gamma^\mu m) u(p) \\ &= (2p^\mu - \gamma^\mu \not{p} + \gamma^\mu m) u(p) = 2p^\mu u(p) \, . \end{aligned}$$

(using Dirac equation for on-mass-shell spinor $u(p)$).

- Universal factor $F_{\text{soft}}$ coincides with classical eikonal formula for radiation from current $p^\mu$, valid in long-wavelength limit.
  - No soft enhancement of radiation from off-mass-shell internal lines, since associated denominator factor $(p+q)^2 - m^2 \to p^2 - m^2 \neq 0$ as $\omega \to 0$.
- Enhancement factor in amplitude for each external line implies cross section enhancement is sum over all pairs of external lines $\{i,j\}$:

$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega} \frac{d\Omega}{2\pi} \frac{\alpha_S}{2\pi} \sum_{i,j} C_{ij} W_{ij}$$

where $d\Omega$ is element of solid angle for emitted gluon, $C_{ij}$ is a colour factor, and radiation function $W_{ij}$ is given by

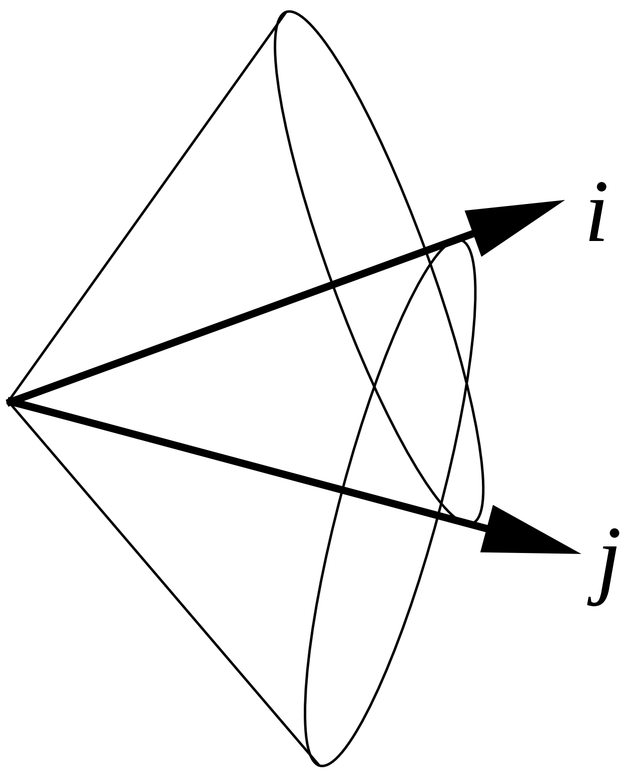
$$W_{ij} = \frac{\omega^2 p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} = \frac{1 - v_i v_j \cos\theta_{ij}}{(1 - v_i \cos\theta_{iq})(1 - v_j \cos\theta_{jq})} \, .$$

- Colour-weighted sum of radiation functions $C_{ij}W_{ij}$ is antenna pattern of hard process.
- Radiation function can be separated into two parts containing collinear singularities along lines $i$ and $j$. Consider for simplicity massless particles, $v_{i,j} = 1$. Then $W_{ij} = W_{ij}^i + W_{ij}^j$ where

$$W_{ij}^i = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos\theta_{iq}} - \frac{1}{1-\cos\theta_{jq}}\right) .$$

- This function has remarkable property of angular ordering. Write angular integration in polar coordinates w.r.t. direction of $i$, $d\Omega = d\cos\theta_{iq}\, d\phi_{iq}$. Performing azimuthal integration, we find

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W_{ij}^i = \frac{1}{1-\cos\theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}, \text{ otherwise } 0.$$




Thus, after azimuthal averaging, contribution from $W_{ij}^{i}$ is confined to cone, centred on direction of $i$, extending in angle to direction of $j$. Similarly, $W_{ij}^{j}$, averaged over $\phi_{jq}$, is confined to cone centred on line $j$ extending to direction of $i$.

# Angular ordering

- To prove angular ordering property, write $1-\cos\theta_{jq} = a - b\cos\phi_{iq}$ where $a = 1-\cos\theta_{ij}\cos\theta_{iq}$ and $b = \sin\theta_{ij}\sin\theta_{iq}$.

  Defining $z = \exp(i\phi_{iq})$, we have

$$I^i_{ij} \equiv \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{1-\cos\theta_{jq}} = \frac{1}{i\pi b} \oint \frac{dz}{(z-z_+)(z-z_-)}$$

  where $z$-integration contour the unit circle and

$$z_\pm = \frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}\,.$$

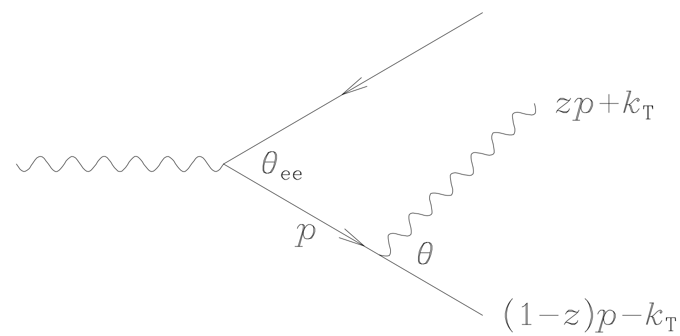
  Now only pole at $z = z_-$ can lie inside unit circle, so

$$I^i_{ij} = \sqrt{\frac{1}{a^2-b^2}} = \frac{1}{|\cos\theta_{iq} - \cos\theta_{ij}|}\,.$$

  Hence

$$\begin{aligned} \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W^i_{ij} &= \frac{1}{2(1-\cos\theta_{iq})}[1 + (\cos\theta_{iq} - \cos\theta_{ij}) I^i_{ij}] \\ &= \frac{1}{1-\cos\theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}, \text{ otherwise } 0. \end{aligned}$$

- Angular ordering is coherence effect common to all gauge theories. In QED it causes Chudakov effect – suppression of soft bremsstrahlung from $e^+e^-$ pairs, which has simple explanation in old-fashioned (time-ordered) perturbation theory.




- Consider emission of soft photon at angle $\theta$ from electron in pair with opening angle $\theta_{ee} < \theta$. For simplicity assume $\theta_{ee}, \theta \ll 1$.
- Transverse momentum of photon is $k_T \sim zp\theta$ and energy imbalance at $e \to e\gamma$ vertex is

$$\Delta E \sim k_T^2/zp \sim zp\theta^2 \ .$$

- Time available for emission is $\Delta t \sim 1/\Delta E$. In this time transverse separation of pair will be $\Delta b \sim \theta_{ee}\Delta t$.

- For non-negligible probability of emission, photon must resolve this transverse separation of pair, so

$$\Delta b > \lambda/\theta \sim (zp\theta)^{-1}$$

  where $\lambda$ is photon wavelength.

- This implies that

$$\theta_{ee}(zp\theta^2)^{-1} > (zp\theta)^{-1} \;,$$

  and hence $\theta_{ee} > \theta$. Thus soft photon emission is suppressed at angles larger than opening angle of pair, which is angular ordering.

- Photons at larger angles cannot resolve electron and positron charges separately – they see only total charge of pair, which is zero, implying no emission.

- More generally, if $i$ and $j$ come from branching of parton $k$, with (colour) charge $\boldsymbol{Q}_k = \boldsymbol{Q}_i + \boldsymbol{Q}_k$, then radiation outside angular-ordered cones is emitted coherently by $i$ and $j$ and can be treated as coming directly from (colour) charge of $k$.

# Coherent branching

- Angular ordering provides basis for coherent parton branching formalism, which includes leading soft gluon enhancements to all orders.

- In place of virtual mass-squared variable $t$ in earlier treatment, use angular variable

$$\zeta = \frac{p_b \cdot p_c}{E_b\, E_c} \simeq 1 - \cos\theta$$

as evolution variable for branching $a \to bc$, and impose angular ordering $\zeta' < \zeta$ for successive branchings. Iterative formula for $n$-parton emission becomes

$$d\sigma_{n+1} = d\sigma_n \frac{d\zeta}{\zeta} dz \frac{\alpha_S}{2\pi} \hat{P}_{ba}(z)\ .$$

- In place of virtual mass-squared cutoff $t_0$, must use angular cutoff $\zeta_0$ for coherent branching. This is to some extent arbitrary, depending on how we classify emission as unresolvable. Simplest choice is

$$\zeta_0 = t_0/E^2$$

for parton of energy $E$.

- For radiation from particle $i$ with mass-squared $t_0 > 0$, radiation function is

$$\omega^2 \left( \frac{p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} - \frac{p_i^2}{(p_i \cdot q)^2} \right) \simeq \frac{1}{\zeta} \left( 1 - \frac{t_0}{E^2 \zeta} \right) ,$$

so angular distribution of radiation is cut off at $\zeta = t_0/E^2$. Thus $t_0$ can still be interpreted as minimum virtual mass-squared.

- With this cutoff, most convenient definition of evolution variable is not $\zeta$ itself but rather

$$\tilde{t} = E^2 \zeta \geq t_0 \, .$$

Angular ordering condition $\zeta_b, \zeta_c < \zeta_a$ for timelike branching $a \to bc$ ($a$ outgoing) becomes

$$\tilde{t}_b < z^2 \tilde{t} \, , \quad \tilde{t}_c < (1-z)^2 \tilde{t}$$

where $\tilde{t} = \tilde{t}_a$ and $z = E_b/E_a$. Thus cutoff on $z$ becomes

$$\sqrt{t_0/\tilde{t}} < z < 1 - \sqrt{t_0/\tilde{t}} \, .$$

- Neglecting masses of $b$ and $c$, virtual mass-squared of $a$ and transverse momentum of branching are

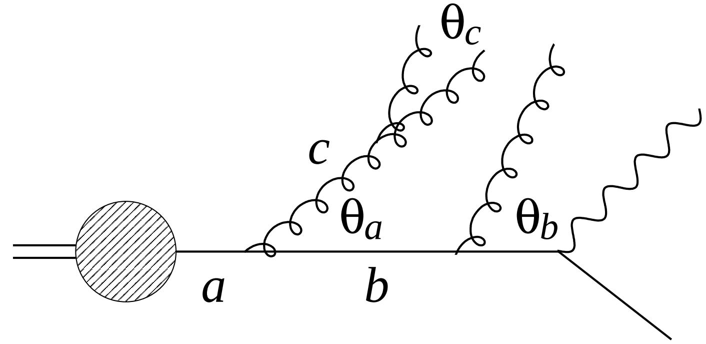
$$t = z(1-z)\tilde{t}\,, \quad p_t^2 = z^2(1-z)^2\tilde{t}\,.$$

- Thus for coherent branching Sudakov form factor of quark becomes

$$\tilde{\Delta}_q(\tilde{t}) = \exp\left[-\int_{4t_0}^{\tilde{t}} \frac{dt'}{t'} \int_{\sqrt{t_0/t'}}^{1-\sqrt{t_0/t'}} \frac{dz}{2\pi} \alpha_{\mathrm{S}}(z^2(1-z)^2 t') \hat{P}_{qq}(z)\right]$$

At large $\tilde{t}$ this falls more slowly than form factor without coherence, due to the suppression of soft gluon emission by angular ordering.

- Note that for spacelike branching $a \to bc$ ($a$ incoming, $b$ spacelike), angular ordering condition is




$$\theta_b > \theta_a > \theta_c \ ,$$

and so for $z = E_b/E_a$ we now have

$$\tilde{t}_b > z^2 \tilde{t}_a \ , \quad \tilde{t}_c < (1-z)^2 \tilde{t}_a \ .$$

- Thus we can have either $\tilde{t}_b > \tilde{t}_a$ or $\tilde{t}_b < \tilde{t}_a$, especially at small $z$ — spacelike branching becomes disordered at small $x$.

# Summary of Lecture 1

- Parton branching approximation describes collinear-enhanced contribution to multi-parton cross sections in terms of splitting functions $P_{ij}(z)$.
- Evolution of parton distributions (or parton fragmentation) is controlled by DGLAP equation.
- Sudakov form factor is probability of evolution without resolvable branching – useful for Monte Carlo implementation.
- Successive branching leads to parton showering, terminated by infrared cutoff, followed by hadronization.
- Soft gluon coherence implies angular ordered parton showers.