QCD Phenomenology at High Energy

Bryan Webber

CERN Academic Training Lectures 2008

Lecture 1: Basics of QCD

- QCD Lagrangian
  - Gauge invariance
  - Feynman rules
- Running Coupling
  - Beta function
  - Charge screening
  - Lambda parameter
- Renormalization Schemes
- History of Asymptotic Freedom
- Non-perturbative QCD
  - Infrared divergences

Lagrangian of QCD

- Feynman rules for perturbative QCD follow from Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^A_{\alpha\beta} F_A^{\alpha\beta} + \sum_{\text{flavours}} \bar{q}_a (i \not{D} - m)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$$

$F^A_{\alpha\beta}$ is field strength tensor for spin-1 gluon field $\mathcal{A}^A_\alpha$,

$$F^A_{\alpha\beta} = \partial_\alpha \mathcal{A}^A_\beta - \partial_\beta \mathcal{A}^A_\alpha - g f^{ABC} \mathcal{A}^B_\alpha \mathcal{A}^C_\beta$$

Capital indices A, B, C run over 8 colour degrees of freedom of the gluon field. Third 'non-Abelian' term distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interactions and ultimately to asymptotic freedom.

- QCD coupling strength is $\alpha_S \equiv g^2/4\pi$. Numbers f^{ABC} $(A, B, C = 1, \ldots, 8)$ are structure constants of the SU(3) colour group. Quark fields q_a $(a = 1, 2, 3)$ are in triplet colour representation. D is covariant derivative:

$$\begin{aligned} (D_\alpha)_{ab} &= \partial_\alpha \delta_{ab} + ig \left(t^C \mathcal{A}^C_\alpha \right)_{ab} \\ (D_\alpha)_{AB} &= \partial_\alpha \delta_{AB} + ig (T^C \mathcal{A}^C_\alpha)_{AB} \end{aligned}$$

- t and T are matrices in the fundamental and adjoint representations of SU(3), respectively:

$$[t^A, t^B] = i f^{ABC} t^C, \quad [T^A, T^B] = i f^{ABC} T^C$$

where $(T^A)_{BC} = -if^{ABC}$. We use the metric $g^{\alpha\beta} =$ diag(1,–1,–1,–1) and set $\hbar = c = 1$. $\not{D}$ is symbolic notation for $\gamma^\alpha D_\alpha$. Normalisation of the t matrices is

$$\text{Tr } t^A t^B = T_R \, \delta^{AB}, \quad T_R = \frac{1}{2}.$$

- Colour matrices obey the relations:

$$\sum_A t^A_{ab} t^A_{bc} = C_F \, \delta_{ac} \, , \quad C_F = \frac{N^2 - 1}{2N}$$

$$\text{Tr } T^C T^D = \sum_{A,B} f^{ABC} f^{ABD} = C_A \, \delta^{CD} \, , \quad C_A = N$$

Thus $C_F = \frac{4}{3}$ and $C_A = 3$ for SU(3).

Gauge Invariance

- QCD Lagrangian is invariant under local gauge transformations. That is, one can redefine quark fields independently at every point in space-time,

$$q_a(x) \to q'_a(x) = \exp(it \cdot \theta(x))_{ab} q_b(x) \equiv \Omega(x)_{ab} q_b(x)$$

without changing physical content.

- Covariant derivative is so called because it transforms in same way as field itself:

$$D_\alpha q(x) \to D'_\alpha q'(x) \equiv \Omega(x) D_\alpha q(x) \; .$$

(omitting the colour labels of quark fields from now on). Use this to derive transformation property of gluon field $\mathcal{A}$

$$D'_\alpha q'(x) = \big(\partial_\alpha + igt \cdot \mathcal{A}'_\alpha\big)\Omega(x) q(x)$$
$$\equiv (\partial_\alpha \Omega(x)) q(x) + \Omega(x)\partial_\alpha q(x) + igt \cdot \mathcal{A}'_\alpha \Omega(x) q(x)$$

where $t \cdot \mathcal{A}_\alpha \equiv \sum_A t^A \mathcal{A}^A_\alpha$. Hence

$$t \cdot \mathcal{A}'_\alpha = \Omega(x) t \cdot \mathcal{A}_\alpha \Omega^{-1}(x) + \frac{i}{g}\big(\partial_\alpha \Omega(x)\big)\Omega^{-1}(x) \; .$$

- Transformation property of gluon field strength $F_{\alpha\beta}$ is

$$t \cdot F_{\alpha\beta}(x) \to t \cdot F'_{\alpha\beta}(x) = \Omega(x) F_{\alpha\beta}(x) \Omega^{-1}(x) \ .$$

  Contrast this with gauge-invariance of QED field strength. QCD field strength is not gauge invariant because of self-interaction of gluons. Carriers of the colour force are themselves coloured, unlike the electrically neutral photon.

- Note there is no gauge-invariant way of including a gluon mass. A term such as

$$m^2 \mathcal{A}^\alpha \mathcal{A}_\alpha$$

  is not gauge invariant. This is similar to QED result for mass of the photon. On the other hand quark mass term is gauge invariant.

Feynman Rules

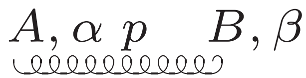
- Use free piece of QCD Lagrangian to obtain inverse quark and gluon propagators.
  - Quark propagator in momentum space obtained by setting $\partial^\alpha = -ip^\alpha$ for an incoming field. Result is in Table 1. The $i\varepsilon$ prescription for pole of propagator is determined by causality, as in QED.
  - Gluon propagator impossible to define without a choice of gauge. The choice

$$\mathcal{L}_{\mathrm{gauge-fixing}} = -\frac{1}{2\,\lambda}\left(\partial^\alpha \mathcal{A}_\alpha^A\right)^2$$

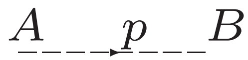
defines *covariant gauges* with gauge parameter λ. Inverse gluon propagator is then

$$\Gamma^{(2)}_{\{AB,\,\alpha\beta\}}(p) = i\delta_{AB}\,\left[p^2 g_{\alpha\beta} - (1-\frac{1}{\lambda})p_\alpha p_\beta\right].$$

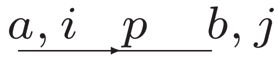
(Check that without gauge-fixing term this function would have no inverse.) Resulting propagator is in Table 1. $\lambda = 1$ (0) is *Feynman* (*Landau*) gauge.



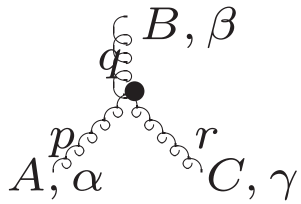
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^{\alpha}p^{\beta}}{p^2+i\varepsilon}\right]\frac{i}{p^2+i\varepsilon}$$



$$\delta^{AB}\frac{i}{p^2+i\varepsilon}$$

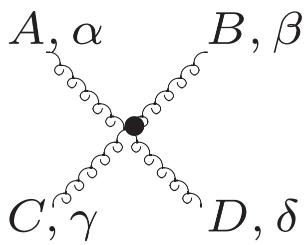


$$\delta^{ab}\frac{i}{(\not p-m+i\varepsilon)_{ji}}$$

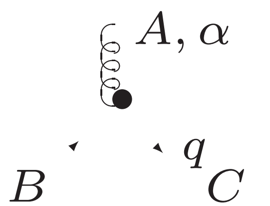


$$-gf^{ABC}\left[g^{\alpha\beta}(p-q)^{\gamma}+g^{\beta\gamma}(q-r)^{\alpha}+g^{\gamma\alpha}(r-p)^{\beta}\right]$$

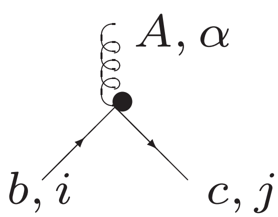
(all momenta incoming)



$$\begin{aligned}&-ig^2f^{XAC}f^{XBD}\left(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\beta\gamma}\right)\\&-ig^2f^{XAD}f^{XBC}\left(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta}\right)\\&-ig^2f^{XAB}f^{XCD}\left(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}\right)\end{aligned}$$



$$gf^{ABC}q^{\alpha}$$



$$-ig\left(t^{A}\right)_{cb}\left(\gamma^{\alpha}\right)_{ji}$$

Table 1: Feynman rules for QCD in a covariant gauge.

- Gauge fixing explicitly breaks gauge invariance. However, in the end physical results will be independent of gauge. For convenience we usually use Feynman gauge.
- In non-Abelian theories like QCD, covariant gauge-fixing term must be supplemented by a *ghost term* which we do not discuss here. Ghost field, shown by dashed lines in Table 1, cancels unphysical degrees of freedom of gluon which would otherwise propagate in covariant gauges.

Running Coupling

- Consider dimensionless physical observable R which depends on a single large energy scale, $Q \gg m$ where m is any mass. Then we can set $m \to 0$ (assuming this limit exists), and dimensional analysis suggests that R should be independent of Q.
- This is not true in quantum field theory. Calculation of R as a perturbation series in the coupling $\alpha_S = g^2/4\pi$ requires renormalization to remove ultraviolet divergences. This introduces a second mass scale μ — point at which subtractions which remove divergences are performed. Then R depends on the ratio Q/μ and is not constant. The renormalized coupling α_S also depends on μ.
- But μ is arbitrary! Therefore, if we hold bare coupling fixed, R cannot depend on μ. Since R is dimensionless, it can only depend on Q^2/μ^2 and the renormalized coupling α_S. Hence

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_S\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S}\right] R = 0 \ .$$

- Introducing

$$\tau = \ln\left(\frac{Q^2}{\mu^2}\right) \ , \qquad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \ ,$$

we have

$$\left[-\frac{\partial}{\partial \tau} + \beta(\alpha_S)\frac{\partial}{\partial \alpha_S}\right] R = 0.$$

This renormalization group equation is solved by defining running coupling $\alpha_S(Q)$:

$$\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)} \ , \quad \alpha_S(\mu) \equiv \alpha_S \ .$$

Then

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)) \ , \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)} \ .$$

and hence $R(Q^2/\mu^2, \alpha_S) = R(1, \alpha_S(Q))$. Thus all scale dependence in R comes from running of $\alpha_S(Q)$.

- We shall see QCD is asymptotically free: $\alpha_S(Q) \to 0$ as $Q \to \infty$. Thus for large Q we can safely use perturbation theory. Then knowledge of $R(1, \alpha_S)$ to fixed order allows us to predict variation of R with Q.

Beta Function

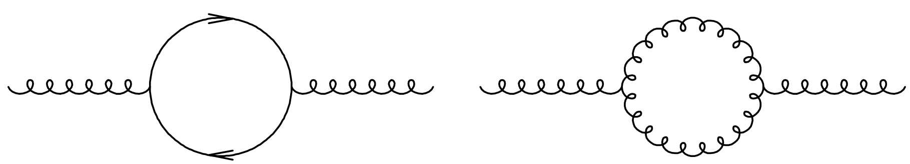
- Running of of the QCD coupling α_S is determined by the β function, which has the expansion

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S) + \mathcal{O}(\alpha_S^4)$$

$$b = \frac{(11C_A - 2N_f)}{12\pi}, \quad b' = \frac{(17C_A^2 - 5C_AN_f - 3C_FN_f)}{2\pi(11C_A - 2N_f)},$$

where N_f is number of "active" light flavours. Terms up to $\mathcal{O}(\alpha_S^5)$ are known.

- Roughly speaking, quark loop "vacuum polarisation" diagram (a) contributes negative N_f term in b, while gluon loop (b) gives positive C_A contribution, which makes β function negative overall.



a) b)

- QED β function is

$$\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \ldots$$

Thus b coefficients in QED and QCD have opposite signs.

- From previous section,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q)\left[1 + b'\alpha_S(Q)\right] + \mathcal{O}(\alpha_S^4).$$

Neglecting b' and higher coefficients gives

$$\alpha_S(Q) = \frac{\alpha_S(\mu)}{1 + \alpha_S(\mu) b\tau}\,, \quad \tau = \ln\left(\frac{Q^2}{\mu^2}\right).$$

- As Q becomes large, $\alpha_S(Q)$ decreases to zero: this is asymptotic freedom. Notice that sign of b is crucial. In QED, $b < 0$ and coupling $increases$ at large Q.
Including next coefficient b' gives implicit equation for $\alpha_S(Q)$:

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b' \ln\left(\frac{\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) - b' \ln\left(\frac{\alpha_S(\mu)}{1 + b'\alpha_S(\mu)}\right)$$

- What type of terms does the solution of the renormalization group equation take into account in the dimensionless physical quantity $R(Q^2/\mu^2, \alpha_S)$?
  Assume that R has perturbative expansion

$$R(1, \alpha_S) = R_1 \alpha_S + R_2 \alpha_S^2 + \mathcal{O}(\alpha_S^3)$$

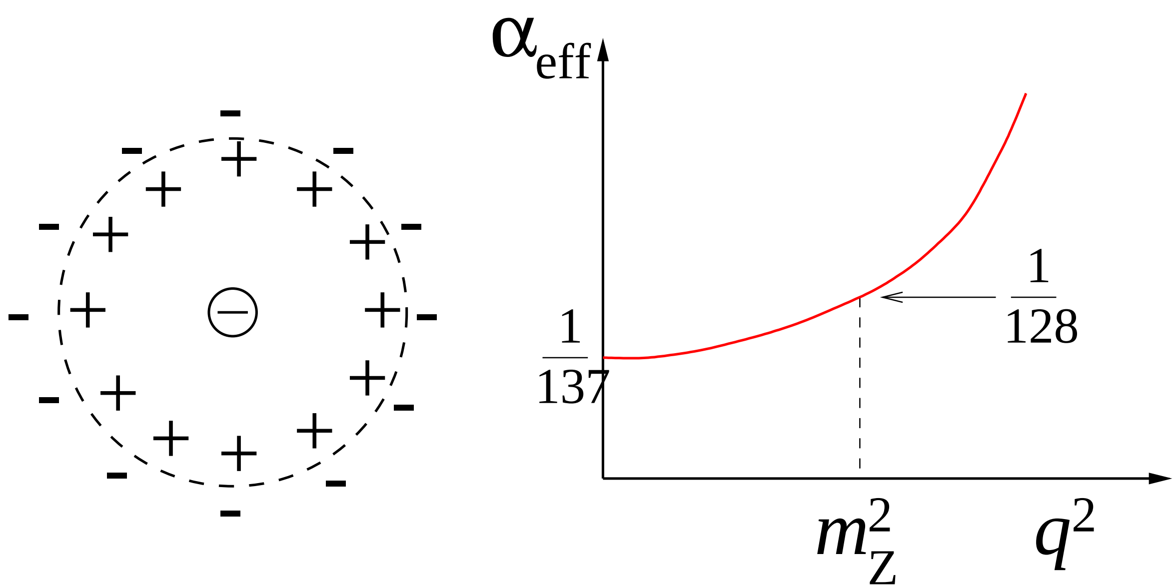
  RGE solution $R(1, \alpha_S(Q))$ can be re-expressed in terms of $\alpha_S(\mu)$:

$$\begin{aligned} \alpha_S(Q) &= \alpha_S(\mu) - b\tau[\alpha_S(\mu)]^2 + \mathcal{O}(\alpha_S^3) \\ R(1, \alpha_S(Q)) &= R_1 \alpha_S(\mu) + (R_2 - b\tau)\alpha_S(\mu)^2 + \mathcal{O}(\alpha_S^3) \end{aligned}$$

  Thus there are powers of $\tau = \log(Q^2/\mu^2)$ that are automatically resummed by using the running coupling.
- Notice that a *leading order* (LO) evaluation of R (i.e. the coefficient R_1) is not very useful since $\alpha_S(\mu)$ can be given any value by varying the scale μ.
  - We need the *next-to-leading order* (NLO) coefficient $(R_2 - b\tau)$ to gain some control of scale dependence: the μ dependence of τ starts to compensate that of $\alpha_S(\mu)$.

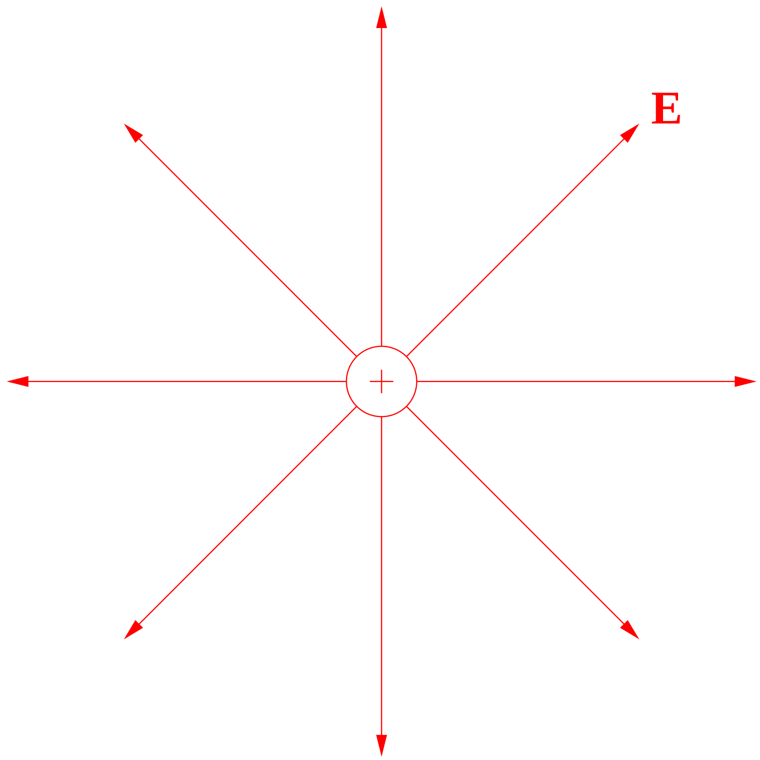
Charge Screening

- In QED the observed electron charge is distance-dependent ($\Rightarrow$ momentum transfer dependent) due to charge screening by the vacuum polarisation:



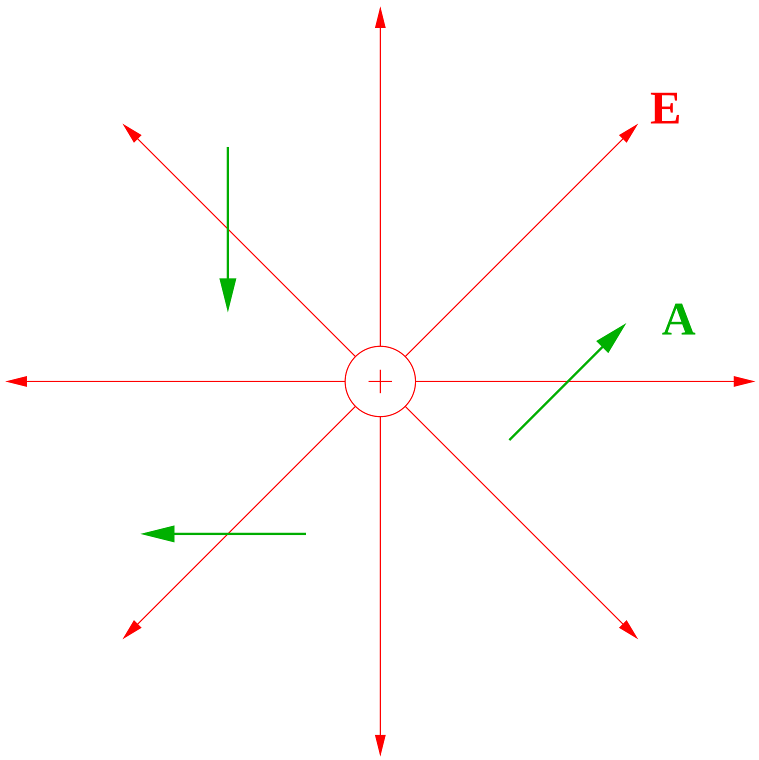
- At short distances (high momentum scales) we see more of the "bare" charge $\Rightarrow$ effective charge (coupling) increases.
- In contrast, the vacuum polarisation of a non-Abelian gauge field gives anti-screening.
  - ❖ Consider for simplicity an SU(2) gauge field: this has 3 "colours". . .

Non-Abelian Vacuum Polarization



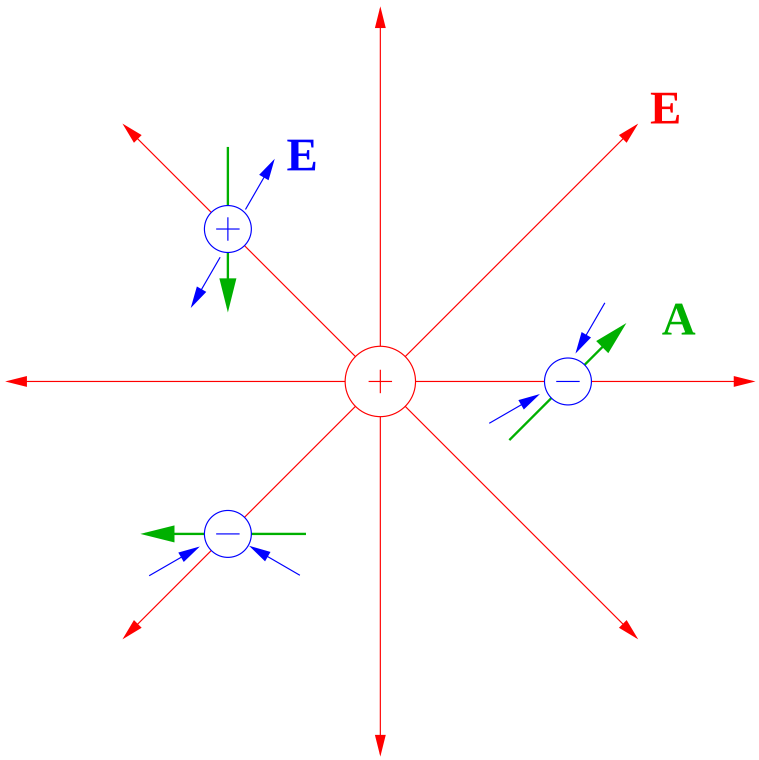
$$\nabla \cdot \boldsymbol{E} = g\,\delta^3(\boldsymbol{r}) + g\,(\boldsymbol{A} \cdot \boldsymbol{E} - \boldsymbol{A} \cdot \boldsymbol{E})$$

Non-Abelian Vacuum Polarization



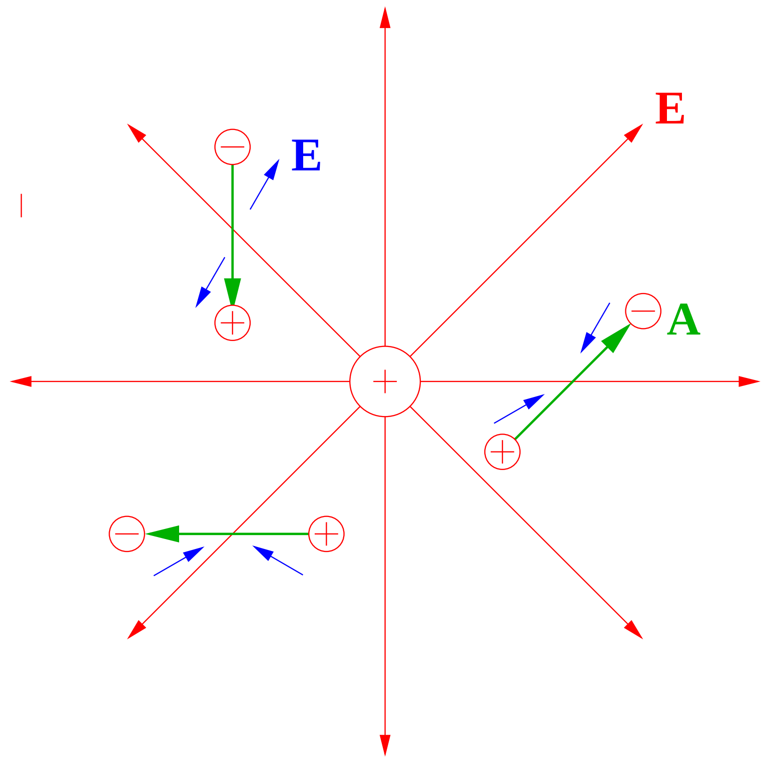
$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = g\,(\boldsymbol{A} \cdot \boldsymbol{E} - \boldsymbol{A} \cdot \boldsymbol{E})$$

Non-Abelian Vacuum Polarization



$$\nabla \cdot \boldsymbol{E} = g\,(\boldsymbol{A} \cdot \boldsymbol{E} - \boldsymbol{A} \cdot \boldsymbol{E})$$

Non-Abelian Vacuum Polarization



$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = g\,\delta^3(\boldsymbol{r}) + g\,(\boldsymbol{A} \cdot \boldsymbol{E} - \boldsymbol{A} \cdot \boldsymbol{E})$$

Lambda Parameter

- Perturbative QCD tells us how $\alpha_S(Q)$ varies with Q, but its absolute value has to be obtained from experiment. Nowadays we usually choose as the fundamental parameter the value of the coupling at $Q = M_Z$, which is simply a convenient reference scale large enough to be in the perturbative domain.
- Also useful to express $\alpha_S(Q)$ directly in terms of a dimensionful parameter (constant of integration) Λ:

$$\ln\frac{Q^2}{\Lambda^2} = -\int_{\alpha_S(Q)}^{\infty}\frac{dx}{\beta(x)} = \int_{\alpha_S(Q)}^{\infty}\frac{dx}{bx^2(1+b'x+\ldots)}.$$

  Then (if perturbation theory were the whole story) $\alpha_S(Q) \to \infty$ as $Q \to \Lambda$. More generally, Λ sets the scale at which $\alpha_S(Q)$ becomes large.
- In leading order (LO) keep only first β-function coefficient b:

$$\alpha_S(Q) = \frac{1}{b\ln(Q^2/\Lambda^2)} \qquad \text{(LO)}.$$

- In next-to-leading order (NLO) include also b':

$$\frac{1}{\alpha_S(Q)} + b' \ln\left(\frac{b'\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) = b \ln\left(\frac{Q^2}{\Lambda^2}\right).$$

This can be solved numerically, or we can obtain an approximate solution to second order in $1/\log(Q^2/\Lambda^2)$:

$$\alpha_S(Q) = \frac{1}{b\ln(Q^2/\Lambda^2)}\left[1 - \frac{b'}{b}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right] \qquad \text{(NLO)}.$$

This is Particle Data Group (PDG) definition.

- Note that Λ depends on number of active flavours N_f. 'Active' means $m_q < Q$. Thus for $5 < Q < 175$ GeV we should use $N_f = 5$. See ESW for relation between Λ's for different values of N_f.

Renormalization Schemes

- Λ also depends on renormalization scheme. Consider two calculations of the renormalized coupling which start from the same bare coupling α_S^0:

$$\alpha_S^A = Z^A \alpha_S^0 \,, \qquad \alpha_S^B = Z^B \alpha_S^0$$

Infinite parts of renormalization constants Z^A and Z^B must be same in all orders of perturbation theory. Therefore two renormalized couplings must be related by a finite renormalization:

$$\alpha_S^B = \alpha_S^A (1 + c_1 \alpha_S^A + \ldots).$$

- Note that first two β-function, coefficients, b and b', are unchanged by such a transformation: they are therefore renormalization-scheme independent.
- Two values of Λ are related by

$$\log \frac{\Lambda^B}{\Lambda^A} = -\frac{1}{2} \int_{\alpha_S^A(Q)}^{\alpha_S^B(Q)} \frac{dx}{\beta(x)}$$

This must be true for all Q, so take $Q \to \infty$, to obtain

$$\Lambda^B = \Lambda^A \exp \frac{c_1}{2b}$$

Thus relations between different definitions of Λ are given by the one-loop calculation that fixes c_1.

- Nowadays, most calculations are performed in *modified minimal subtraction* ($\overline{\text{MS}}$) renormalization scheme. Ultraviolet divergences are 'dimensionally regularized' by reducing number of space-time dimensions to $D < 4$:

$$\frac{\mathrm{d}^4 k}{(2\pi)^4} \longrightarrow (\mu)^{2\epsilon} \frac{\mathrm{d}^{4-2\epsilon} k}{(2\pi)^{4-2\epsilon}}$$

where $\epsilon = 2 - \frac{D}{2}$. Note that renormalization scale μ still has to be introduced to preserve dimensions of couplings and fields.

- Loop integrals of form

$$\int \frac{\mathrm{d}^D k}{(k^2 + m^2)^2}$$

lead to poles at $\epsilon = 0$. The *minimal subtraction* prescription is to subtract poles and replace bare coupling by renormalized coupling $\alpha_S(\mu)$. In practice poles always appear in combination

$$\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E,$$

(Euler's constant $\gamma_E = 0.5772\ldots$). In *modified* minimal subtraction scheme $\ln(4\pi) - \gamma_E$ is subtracted as well. From argument above, it follows that

$$\Lambda_{\overline{\text{MS}}} = \Lambda_{\text{MS}} e^{[\ln(4\pi) - \gamma_E]/2} = 2.66\,\Lambda_{\text{MS}}$$

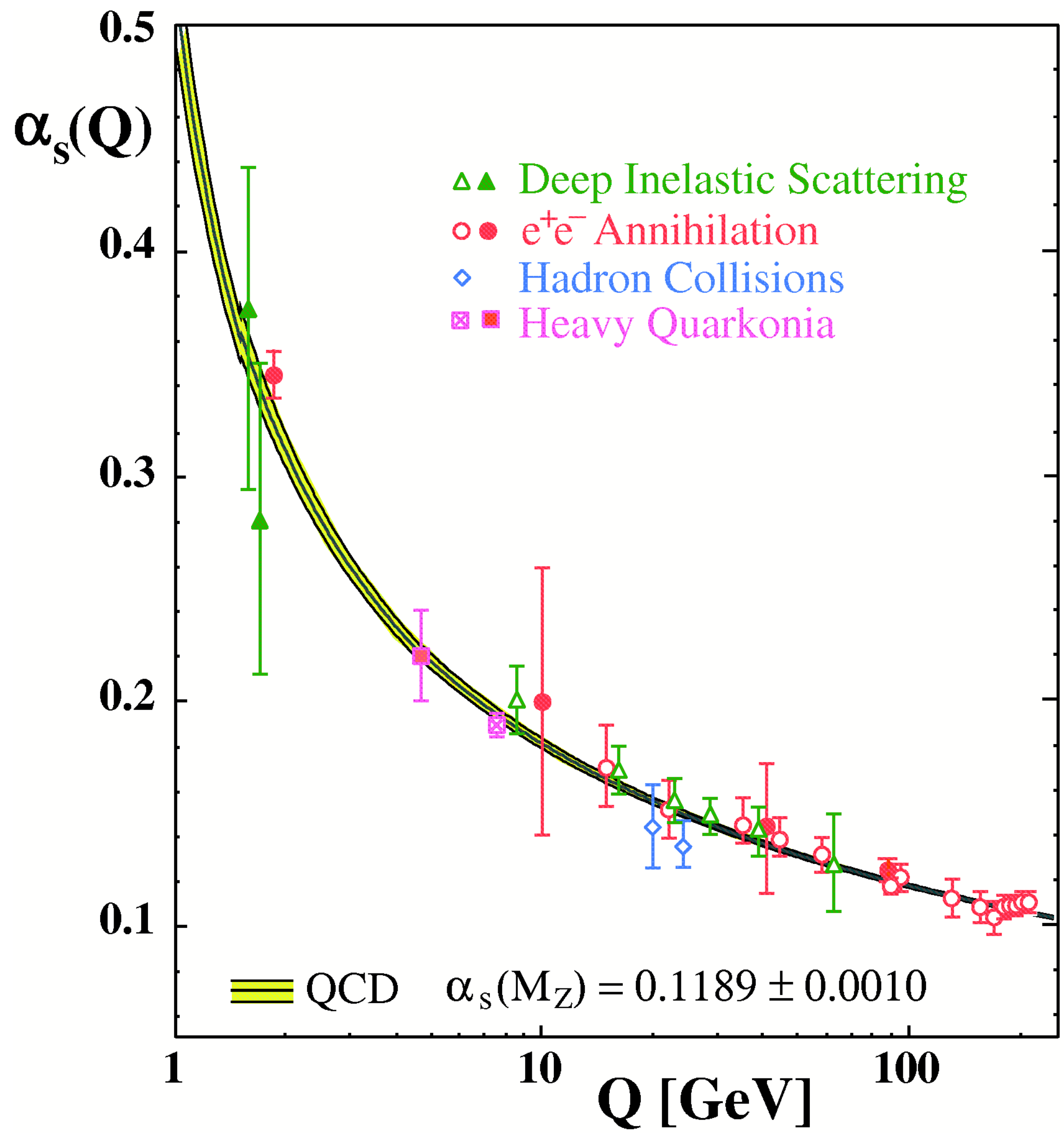
- Current best fit value of α_S at mass of Z is [Bethke, hep-ex/0606035]

$$\alpha_S(M_Z) = 0.1189 \pm 0.0010$$

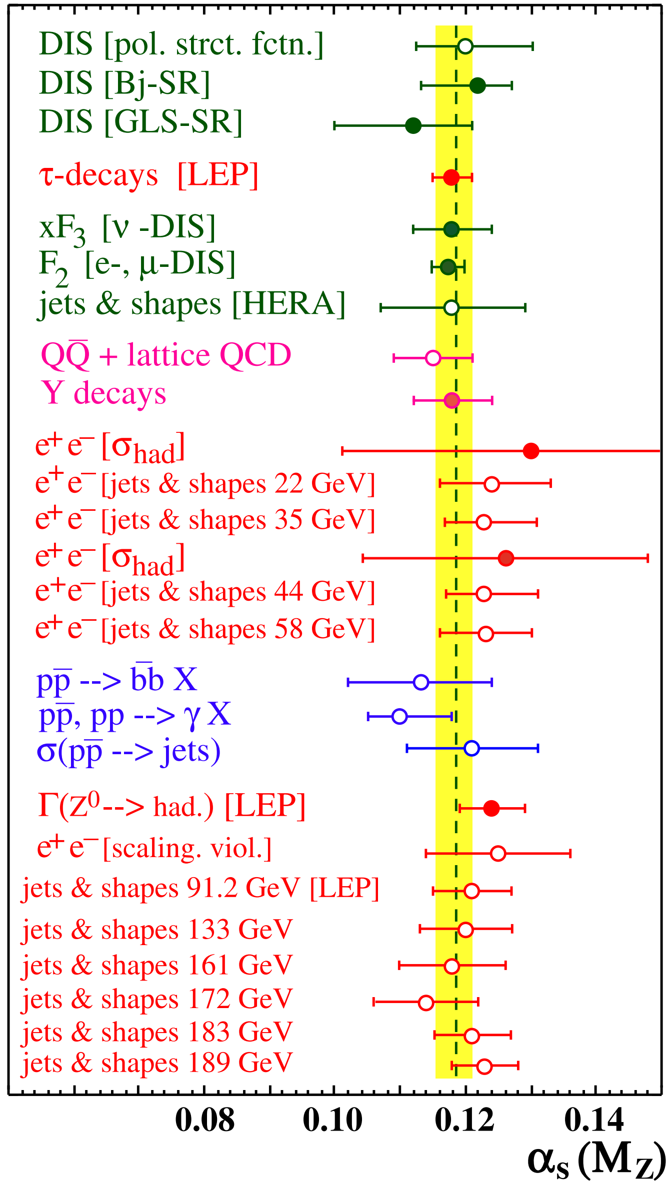
corresponding to a preferred value of $\Lambda_{\overline{MS}}$ (for $N_f = 5$) in the range

$$206 \text{ MeV} < \Lambda_{\overline{MS}}(5) < 231 \text{ MeV}.$$

- Uncertainty in α_S propagates directly into QCD cross sections. Thus we expect errors at the percent level (at least) in prediction of cross sections which begin in order α_S.
- As in the case of scale dependence, we need at least an NLO calculation to start to control the renormalization scheme dependence of any quantity.



- Measurements of α_S are reviewed in ESW. The most recent (2006) compilation of Bethke is shown above. Evidence that $\alpha_S(Q)$ has a logarithmic fall-off with Q is persuasive.



- Using the formula for running $\alpha_S(Q)$ to rescale all measurements to $Q = M_Z$ gives excellent agreement.

History of Asymptotic Freedom

1954 Yang & Mills study vector field theory with non-Abelian gauge invariance.

1965 Vanyashin & Terentyev compute vacuum polarization due to a massive charged vector field. In our notation, they found

$$b = \frac{1}{12\pi}\left(\frac{21}{2} = 11 - \frac{1}{2}\right)$$

The $\frac{1}{2}$ comes from longitudinal polarization states (absent for massless gluons)

- They concluded that this result "... seems extremely undesirable"

1969 Khriplovich correctly computes the one-loop β-function in SU(2) Yang-Mills theory using the Coulomb ($\nabla \cdot A = 0$) gauge

$$b = \frac{C_A}{12\pi}(12 - 1 = 11)$$

In Coulomb gauge the anti-screening (12) is due to an instantaneous Coulomb interaction

- He did not make a connection with strong interactions

1971 't Hooft computes the one-loop β-function for SU(3) gauge theory but does not publish it.

- He wrote it on the blackboard at a conference
- His supervisor (Veltman) told him it wasn't interesting
- 't Hooft & Veltman received the 1999 Nobel Prize for proving the *renormalizability* of QCD (and the whole Standard Model).

1972 Fritzsch & Gell-Mann propose that the strong interaction is an SU(3) gauge theory, later named QCD by Gell-Mann

1973 Gross & Wilczek, and independently Politzer, compute and publish the 1-loop β-function for QCD:

$$b = \frac{1}{12\pi}(11C_A - 2N_f)$$

$\Rightarrow$2004 Nobel Prize (now that 't Hooft has one anyway . . .)

1974 Caswell and Jones compute the 2-loop β-function for QCD.

1980 Tarasov, Vladimirov & Zharkov compute the 3-loop β-function for QCD.

1997 van Ritbergen, Vermaseren & Larin compute the 4-loop β-function for QCD ($\sim 50,000$ Feynman diagrams)

"... We obtained in this way the following result for the 4-loop beta function in the $\overline{\mathrm{MS}}$-scheme:

$$q^2 \frac{\partial a_s}{\partial q^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6)$$

where $a_s = \alpha_S/4\pi$ and . . .

$$
\begin{aligned}
\beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F n_f \ , \quad \beta_1 = \frac{34}{3}C_A^2 - 4C_F T_F n_f - \frac{20}{3}C_A T_F n_f \\
\beta_2 &= \frac{2857}{54}C_A^3 + 2C_F^2 T_F n_f - \frac{205}{9}C_F C_A T_F n_f \\
&\quad - \frac{1415}{27}C_A^2 T_F n_f + \frac{44}{9}C_F T_F^2 n_f^2 + \frac{158}{27}C_A T_F^2 n_f^2 \\
\beta_3 &= C_A^4\left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + C_A^3 T_F n_f\left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right) \\
&\quad + C_A^2 C_F T_F n_f\left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right) + C_A C_F^2 T_F n_f\left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right) \\
&\quad + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2\left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right) + C_F^2 T_F^2 n_f^2\left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right) \\
&\quad + C_A C_F T_F^2 n_f^2\left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right) + \frac{424}{243}C_A T_F^3 n_f^3 + \frac{1232}{243}C_F T_F^3 n_f^3 \\
&\quad + \frac{d_A^{abcd} d_A^{abcd}}{N_A}\left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right) + n_f \frac{d_F^{abcd} d_A^{abcd}}{N_A}\left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right) \\
&\quad + n_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A}\left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right)
\end{aligned}
$$

Here ζ is the Riemann zeta-function ($\zeta_3 = 1.202\cdots$) and the colour factors for SU(N) are

$$T_F = \frac{1}{2}, \quad C_A = N, \quad C_F = \frac{N^2-1}{2N}, \quad \frac{d_A^{abcd}d_A^{abcd}}{N_A} = \frac{N^2(N^2+36)}{24},$$
$$\frac{d_F^{abcd}d_A^{abcd}}{N_A} = \frac{N(N^2+6)}{48}, \quad \frac{d_F^{abcd}d_F^{abcd}}{N_A} = \frac{N^4-6N^2+18}{96N^2}$$

Substitution of these colour factors for $N = 3$ yields the following numerical results for QCD:

$$\begin{aligned}
\beta_0 &\approx 11 - 0.66667n_f \\
\beta_1 &\approx 102 - 12.6667n_f \\
\beta_2 &\approx 1428.50 - 279.611n_f + 6.01852n_f^2 \\
\beta_3 &\approx 29243.0 - 6946.30n_f + 405.089n_f^2 + 1.49931n_f^3
\end{aligned}$$

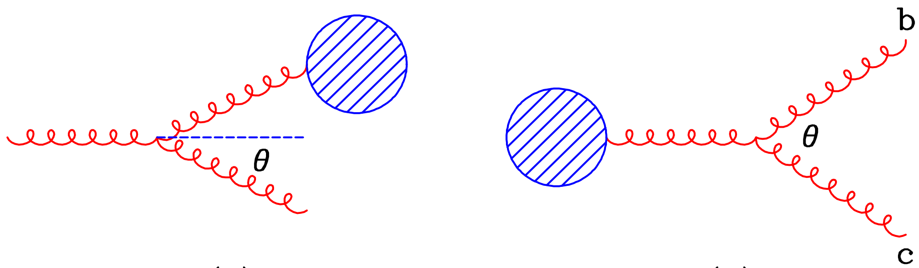

$\beta_g = \frac{-g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$
$\frac{-g^5}{(16\pi^2)^2}\left(\frac{34}{3}N_c^2 + \cdots\right)$
$\frac{-g^7}{(16\pi^2)^3}\left(\frac{2857}{54}N_c^3 + \cdots\right)$
$\frac{-g^9}{(16\pi^2)^4}\left(\cdots\right)$

Nonperturbative QCD

- Corresponding to asymptotic freedom at high momentum scales (short distances), we have infrared slavery: $\alpha_S(Q)$ becomes large at low momenta (long distances). Perturbation theory (PT) not reliable for large α_S, so nonperturbative methods (e.g. lattice) must be used.
- Important low momentum-scale phenomena:
  - Confinement: partons (quarks and gluons) found only in colour-singlet bound states (hadrons), size ~ 1 fm. If we try to separate them, it becomes energetically favourable to create extra partons, forming additional hadrons.
  - Hadronization: partons produced in short-distance interactions reorganize themselves (and multiply) to make observed hadrons.
- Note that confinement is a static (long-distance) property of QCD, treatable by lattice techniques whereas hadronization is a dynamical (long timescale) phenomenon: only models are available at present (see later).

Infrared Divergences

- Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored. Soft or collinear gluon emission gives infrared divergences in PT. Light quarks ($m_q \ll \Lambda$) also lead to divergences in the limit $m_q \to 0$ (mass singularities).



(a) (b)

❖ Spacelike branching: gluon splitting on incoming line (a)

$$p_b^2 = -E_a E_c (1 - \cos\theta) \leq 0 \ .$$

Propagator factor $1/p_b^2$ diverges as $E_c \to 0$ (soft singularity) or $\theta \to 0$ (collinear or mass singularity). If a and b are quarks, inverse propagator factor is

$$p_b^2 - m_q^2 = -E_a E_c (1 - v_a \cos\theta) \leq 0 \ ,$$

Hence $E_c \to 0$ soft divergence remains; collinear enhancement becomes a divergence as $v_a \to 1$, i.e. when quark mass is negligible. If emitted parton c is a quark, vertex factor cancels $E_c \to 0$ divergence.

- Timelike branching: gluon splitting on outgoing line (b)

$$p_a^2 = E_b E_c (1 - \cos\theta) \geq 0 \ .$$

  Diverges when either emitted gluon is soft (E_b or $E_c \to 0$) or when opening angle $\theta \to 0$. If b and/or c are quarks, collinear/mass singularity in $m_q \to 0$ limit. Again, soft quark divergences cancelled by vertex factor.

- Similar infrared divergences in loop diagrams, associated with soft and/or collinear configurations of virtual partons within region of integration of loop momenta.
- Infrared divergences indicate dependence on long-distance aspects of QCD not correctly described by PT. Divergent (or enhanced) propagators imply propagation of partons over long distances. When distance becomes comparable with hadron size ~ 1 fm, quasi-free partons of perturbative calculation are confined/hadronized non-perturbatively, and apparent divergences disappear.
- Can still use PT to perform calculations, provided we limit ourselves to two classes of observables:
  - Infrared safe quantities, i.e. those insensitive to soft or collinear branching. Infrared divergences in PT calculation either cancel between real and virtual contributions or are removed by kinematic factors. Such quantities are determined primarily by hard, short-distance physics; long-distance effects give power corrections, suppressed by inverse powers of a large momentum scale.
  - Factorizable quantities, i.e. those in which infrared sensitivity can be absorbed into an overall non-perturbative factor, to be determined experimentally.

- In either case, infrared divergences must be *regularized* during PT calculation, even though they cancel or factorize in the end.
  - Gluon mass regularization: introduce finite gluon mass, set to zero at end of calculation. However, as we saw, gluon mass breaks gauge invariance.
  - Dimensional regularization: analogous to that used for ultraviolet divergences, except we must *increase* dimension of space-time, $\epsilon = 2 - \frac{D}{2} < 0$. Divergences are replaced by powers of $1/\epsilon$.
- We'll see how this works in the next lecture!

Summary of Lecture 1

- QCD is a non-Abelian gauge theory.
  - Gauge quanta (gluons) are massless and self-interacting.
  - Gauge group SU(3) $\Rightarrow$ 3 colours, 8 gluons.
- Scale invariance is broken by renormalization
  - Physical scale dependence is absorbed in running coupling $\alpha_{\rm S}(Q)$.
- Renormalization scale dependence cancels order-by order.
  - Need at least next-to-leading order (NLO) for meaningful predictions.
- Asymtotic freedom: $\alpha_{\rm S}(Q) \to 0$ as $Q \to \infty$.
  - Due to charge anti-screening.
- All data consistent with $\alpha_{\rm S}(M_Z) \simeq 0.119$.
- Growth of $\alpha_{\rm S}(Q)$ at low scales (infrared slavery) $\Rightarrow$ confinement, hadronization.
  - Must use non-perturbative techniques or models for these.
- Infrared divergences $\Rightarrow$ can only treat infrared safe or factorizable quantities perturbatively.