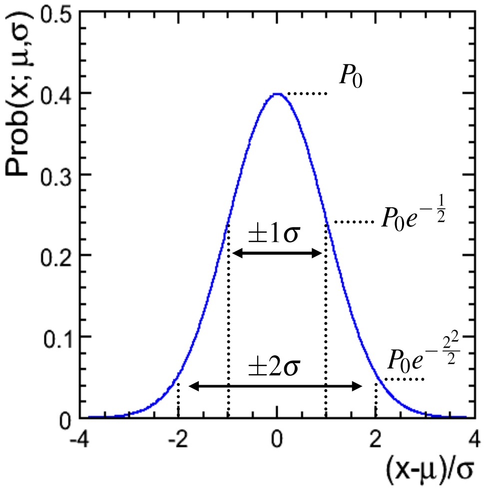
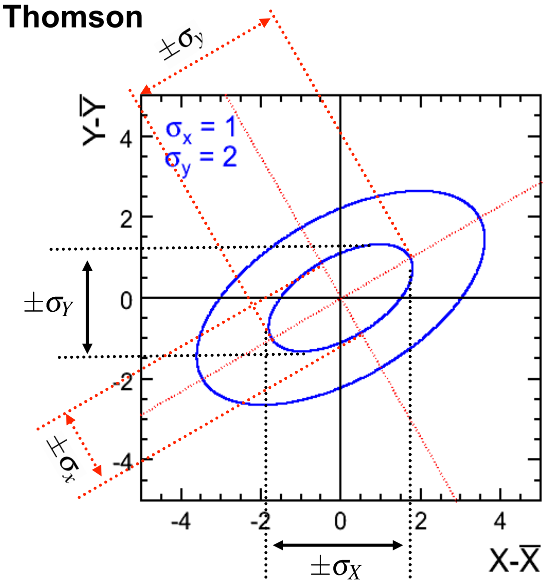
# Statistics

**Lent Term 2015**
**Prof. Mark Thomson**




## Lecture 2 : The Gaussian Limit



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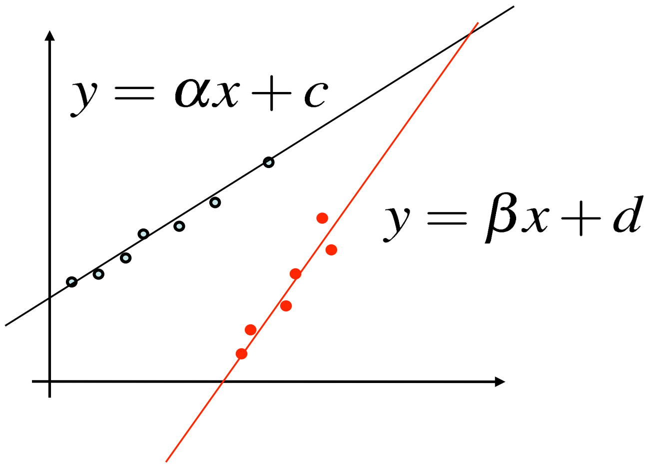
Lecture 1: Back to basics
Introduction, Probability distribution functions, Binomial distributions, Poisson distribution

**Lecture 2: The Gaussian Limit**
**The central limit theorem, Gaussian errors, Error propagation, Combination of measurements, Multi-dimensional Gaussian errors, Error Matrix**

Lecture 3: Fitting and Hypothesis Testing
The $\chi^2$ test, Likelihood functions, Fitting, Binned maximum likelihood, Unbinned maximum likelihood

Lecture 4: Dark Arts
Bayesian statistics, Confidence intervals, systematic errors.

★Problem: given the results of two straight line fits with errors, calculate the uncertainty on the intersection




★Solution: first learn about
- Gaussian errors
- Correlations
- Error propagation



# The Central Limit Theorem

★ We have already shown that for large $\mu$ that a Poisson distribution tends to a Gaussian

★ This is one example of a more general theorem, the "Central Limit Theorem"*

**If $n$ random variables, $x_i$, each distributed according to any PDF, are combined then the sum $y = \Sigma x_i$ will have a PDF which, for large $n$, tends to a Gaussian**

★ For this reason the Gaussian distribution plays an important role in statistics

$$G(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

which by make a suitable coordinate transformation, $x \rightarrow \sigma x + \mu$, gives the Normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}$$

Mean = zero
Rms = 1

* The proof of the central limit theorem is non-trivial and isn't reproduced here

# A useful integral relationship

★ We will often take averages of functions of Gaussian distributed quantities $\langle x^2 \rangle, \langle x^4 \rangle$

★ Hence interested in integrals of the form

$$\langle (x-\mu)^n \rangle = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}(x-\mu)^n e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\sigma^n\int_{-\infty}^{+\infty} y^n e^{-\frac{y^2}{2}}\mathrm{d}y$$

★ Define $\boxed{I_n = \int_{-\infty}^{+\infty} x^n e^{-\frac{x^2}{2}}\mathrm{d}x}$ 

For n odd, $I_n = 0$

For even n:

$$= \int_{-\infty}^{+\infty}\mathrm{d}(-x^{n-1}e^{-\frac{x^2}{2}}) + (n-1)\int_{-\infty}^{+\infty} x^{n-2}e^{-\frac{x^2}{2}}\mathrm{d}x$$

$$= \left[-x^{n-1}e^{-\frac{x^2}{2}}\right]_{-\infty}^{+\infty} + (n-1)I_{n-2}$$

★ Hence $\boxed{\frac{I_n}{I_{n-2}} = (n-1)}$ $n > 1$

★ By writing

$$\langle (x-\mu)^n \rangle = \frac{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}(x-\mu)^n e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x}{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x} \Rightarrow \boxed{\langle (x-\mu)^n \rangle = \frac{I_n}{I_0}\sigma^n}$$

e.g. $\langle (x-\mu)^4 \rangle = \frac{I_4}{I_0}\sigma^4 = \frac{I_4}{I_2}\frac{I_2}{I_0}\sigma^4 = (4-1)(2-1)\frac{I_0}{I_0}\sigma^4 = 3\sigma^4$



# Properties of the Gaussian Distribution

★Normalised to unity (it's a PDF)

$$\boxed{\int_{-\infty}^{+\infty} G(x;\mu,\sigma)\mathrm{d}x = 1}$$

**Proof:**

$$\begin{aligned}\int_{-\infty}^{+\infty} G(x;\mu,\sigma)\mathrm{d}x &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}\sigma}\cdot\sqrt{2}\sigma\cdot\int_{-\infty}^{+\infty} e^{-y^2}\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}\sigma}\cdot\sqrt{2}\sigma\cdot\sqrt{\pi} = 1\end{aligned}$$

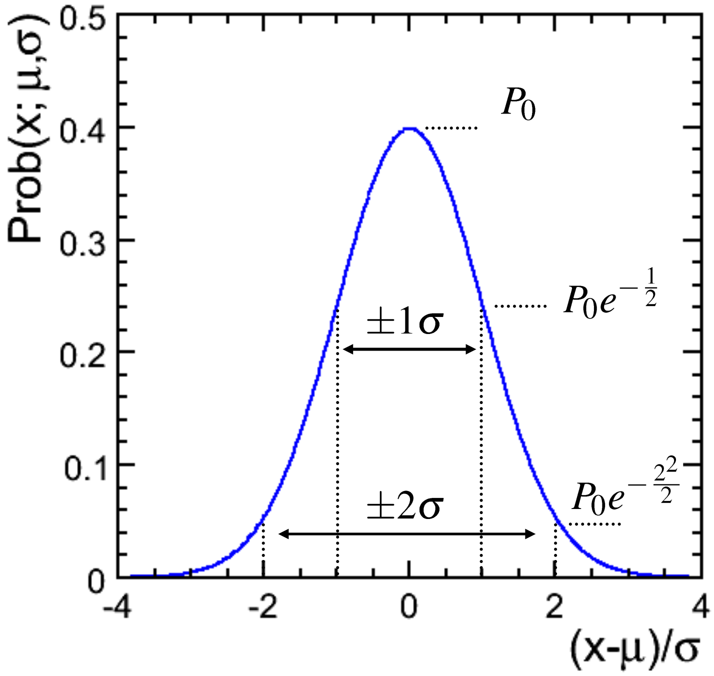
★Variance

$$\boxed{Var(x) = \langle (x-\mu)^2 \rangle = \sigma^2}$$

**Proof:**

$$\begin{aligned}Var(x) = \int_{-\infty}^{+\infty}(x-\mu)^2 G(x;\mu,\sigma)\mathrm{d}x &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x \\ &= \frac{I_2}{I_0}\sigma^2 \\ &= \sigma^2\end{aligned}$$

# Properties of the 1D Gaussian Distribution, cont.




$$\boxed{G(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

★ Natural to introduce $\chi^2(x)$

$$\chi^2 = \frac{(x-\mu)^2}{\sigma^2}$$

"squared deviation from mean in terms of standard error"

$$G(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\chi^2}{2}\right)$$

★ Fractions of events

| | | |
|---|---|---|
| $68.3\,\%$ | $\|x-\mu\| < 1\sigma$ | $(\chi^2 < 1)$ |
| $95.5\,\%$ | $\|x-\mu\| < 2\sigma$ | $(\chi^2 < 4)$ |
| $99.7\,\%$ | $\|x-\mu\| < 3\sigma$ | $(\chi^2 < 9)$ |
| $6\times 10^{-7}$ | $\|x-\mu\| > 5\sigma$ | $(\chi^2 > 25)$ |



# Averaging Gaussian Measurements

★ Suppose we have two **independent** measurements of a quantity, e.g. the W boson mass:

$$x_1 \pm \sigma_1 \quad \text{and} \quad x_2 \pm \sigma_2$$

there are two questions we can ask:

- Are the measurements compatible? [Hypothesis test – we'll return to this ]
- **What is our best estimate of the parameter $x$?** (i.e. how to average)

★ In principle can take any linear combination as an unbiased estimator of $x$

$$x_{12} = \omega_1 x_1 + \omega_2 x_2 \qquad \text{provided} \qquad \omega_1 + \omega_2 = 1$$

since $\langle x_{12}\rangle = \omega_1\langle x_1\rangle + \omega_2\langle x_2\rangle = \omega_1\mu + \omega_2\mu = \mu$

★ Clearly want to give the highest weight to the more precise measurements…
e.g. two undergraduate measurements of $g\,[\mathrm{m\,s^{-2}}]$

$$10.1 \pm 0.3 \qquad 5 \pm 5$$

★ **Method I:** choose the weights to minimise the uncertainty on

$$\sigma_x^2 = \sum_i \omega_i^2 \sigma_i^2$$

subject to constraint $f(\omega_1, \omega_2, \ldots) = 1 - \sum_i \omega_i = 0$

$$\begin{aligned}\frac{\partial(\sigma_x^2+\lambda f)}{\partial\omega_i} &= 0\\ \Rightarrow \quad 2\omega_i\sigma_i^2-\lambda &= 0\\ \omega_i \propto \frac{1}{\sigma_i^2}\end{aligned}$$

★Therefore, since the weights sum to unity:

$$\boxed{\omega_i = \frac{1/\sigma_i^2}{\sum_j 1/\sigma_j^2}}$$

★Hence for two measurements

$$\boxed{\bar{x} = \frac{\frac{x_1}{\sigma_1^2}+\frac{x_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}}$$

with

$$\boxed{\sigma_{\bar{x}}^2 = \frac{1}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}}$$

Problem: derive this.
(just error propagation as described later)



# Averaging Gaussian Measurements II

★Can obtain the same expression using a natural probability based approach

- We can interpret the first measurement in terms of a probability distribution for the true value of $x$, i.e. a Gaussian centred on $x_1$

$$P(x) = P(x;x_1) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left\{-\frac{(x-x_1)^2}{2\sigma_1^2}\right\}$$

- Bayes' theorem then tells us how to modify this in the light of a new measurement

$$P(x;data) \propto P(data;x)P(x)$$

$$P(x;data) \propto \exp\left\{-\frac{(x-x_2)^2}{2\sigma_2^2}\right\}\exp\left\{-\frac{(x-x_1)^2}{2\sigma_1^2}\right\}$$

- So our new expression for the knowledge of x is:

$$P(x) \propto \exp -\frac{1}{2}\left\{\frac{(x-x_1)^2}{\sigma_1^2}+\frac{(x-x_2)^2}{\sigma_2^2}\right\}$$

- Completing the square gives plus a little algebra gives

$$P(x) \propto \exp\left\{-\frac{(x-\bar{x})^2}{2\sigma^2}\right\} \quad \text{with} \quad \bar{x} = \frac{\frac{x_1}{\sigma_1^2}+\frac{x_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}} \quad \text{and} \quad \sigma^2 = \frac{1}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}$$

- Product of n Gaussians is a Gaussian

# Error Propagation I

★ **Suppose measure a quantity $x$ with a Gaussian uncertainty $\sigma_x$; what is the uncertainty on a derived quantity**

$$y = f(x)$$

- **Expand $f(x)$ about $\overline{x}$**

$$f(x) = f(\overline{x}) + (x-\overline{x})\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{\overline{x}} + \ldots$$

- **Define estimate of y:** $\overline{y} = f(\overline{x})$

**so**

$$y - \overline{y} = f(x) - f(\overline{x}) \approx (x-\overline{x})\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{\overline{x}}$$

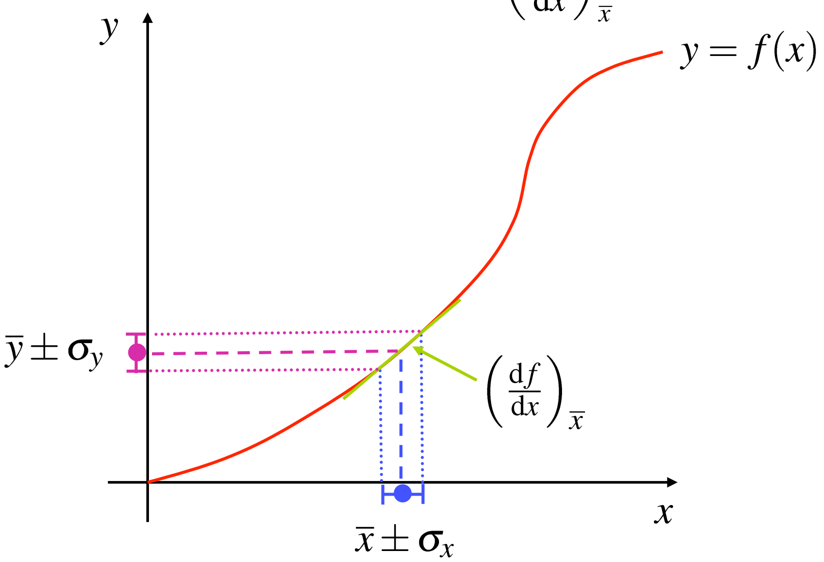
$$\langle (y-\overline{y})^2 \rangle = \langle (x-\overline{x})^2 \rangle \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2_{\overline{x}}$$

$$\sigma_y^2 = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2_{\overline{x}} \sigma_x^2$$

$$\boxed{\sigma_y = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{\overline{x}} \sigma_x}$$



★ **It is easy to understand the origin of** $\sigma_y = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{\overline{x}} \sigma_x$

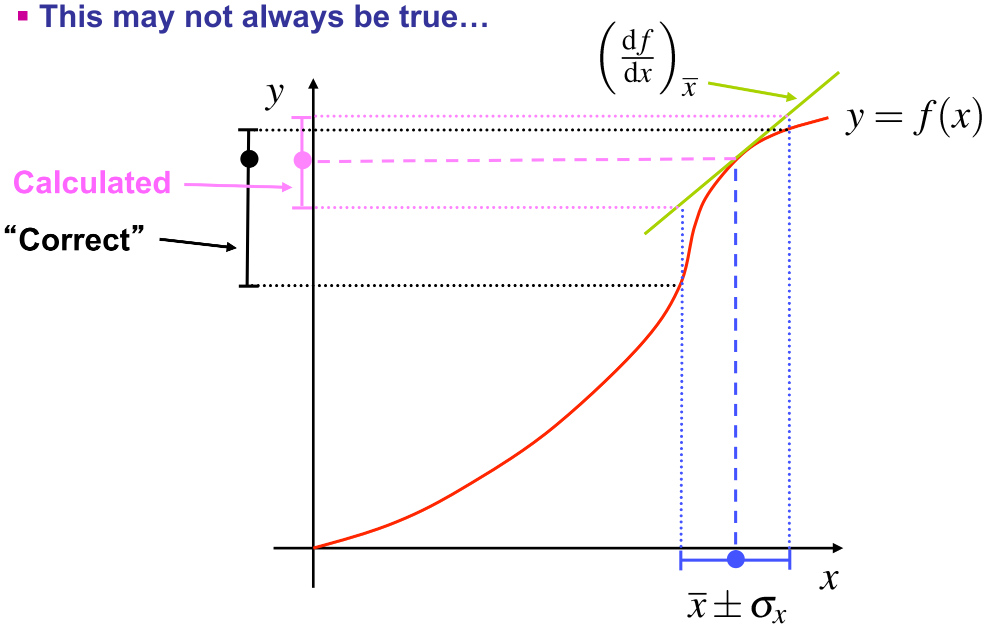



★ **How does a "small" change in $x$, i.e. $\sigma_x$, propagate to a small change in $y$, $\sigma_y$**

$$\frac{\sigma_y}{\sigma_x} \approx \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\overline{x}}$$

★ A word of warning…

- In deriving the error propagation equation $\longrightarrow \quad \sigma_y = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{\overline{x}} \sigma_x$
- Neglected second order terms in the Taylor expansion
- This is equivalent to saying that the derivative is constant in region of interest
- This may not always be true…






# Example

★ Measurement of transverse momentum of a track from a fit

- radius of curvature of track helix, $R$, given by

$$R = 0.3B(\mathrm{T})p_{\mathrm{T}}(\mathrm{GeV})$$

- track fit returns a Gaussian uncertainty in radius of curvature, and hence, the PDF is Gaussian in $1/p_{\mathrm{T}}$

$$\sigma_{1/p_{\mathrm{T}}}$$

- what is the error in $p_{\mathrm{T}}$

let $x = 1/p_{\mathrm{T}}$

$$p_{\mathrm{T}} = 1/x$$

$$\frac{\mathrm{d}p_{\mathrm{T}}}{\mathrm{d}x} = -\frac{1}{x^2} = -p_{\mathrm{T}}^2$$

$$\sigma_{p_{\mathrm{T}}}^2 = \left(\frac{\mathrm{d}p_{\mathrm{T}}}{\mathrm{d}x}\right)^2 \sigma_x^2$$

$$\sigma_{p_{\mathrm{T}}}^2 = \left(p_{\mathrm{T}}^2\right)^2 \sigma_x^2$$

$$\sigma_{p_{\mathrm{T}}} = p_{\mathrm{T}}^2 \sigma_{1/p_{\mathrm{T}}}$$

# Error on Error

★ **Recall question 2:**

Given 5 measurements of a quantity $x$: **10.2, 5.5, 6.7, 3.4, 3.5**

What is the **best estimate** of $x$ and what is the **estimated** uncertainty?

$$\bar{x} = 5.86; \quad s_{n-1} = 2.80; \quad \sigma_{\bar{x}} = \frac{s_{n-1}}{\sqrt{5}} = 1.25$$

So our best estimate of $x$ is: $\boxed{x = 5.9 \pm 1.3}$

★ But how good is our estimate of the error – *i.e.* what is the "**error on the error**" ?

- It can be shown (but not easy):

$$Var(s^2) = \frac{1}{n}\left(\langle (x-\mu)^4 \rangle - \frac{n-3}{n-1}\langle (x-\mu)^2 \rangle^2\right)$$

- For a Gaussian distribution $\langle (x-\mu)^4 \rangle = 3\sigma^4$

so $$Var(s^2) = \frac{\sigma^4}{n}\left(3 - \frac{n-3}{n-1}\right) = \frac{2\sigma^4}{n-1}$$

- Hence (by error propagation – show this) the error on the error estimate is

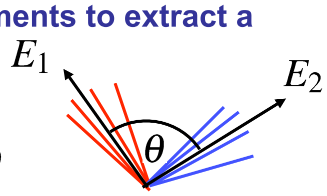
$$\boxed{\sigma_s = \frac{\sigma}{\sqrt{2(n-1)}}}$$

- To obtain a 10% estimate of $\sigma$; need rms of 51 measurements !



# Combining Gaussian Errors

★ There are many cases where we want to combine measurements to extract a single quantity, e.g. di-jet invariant mass




$$m^2 = E_1 E_2 (1 - \cos\theta)$$

- What is the uncertainty on the mass given $\sigma_{E_1}, \sigma_{E_2}, \sigma_\theta$

★ Start by considering a **simple example**

$$a = x + y$$

- Mean of $a$ is $\quad \bar{a} = \bar{x} + \bar{y}$
- Variance of $a$ is given by:

$$\begin{aligned}
\langle (a-\bar{a})^2 \rangle &= \langle (x + y - (\bar{x}+\bar{y}))^2 \rangle \\
\sigma_a^2 &= \langle ([x-\bar{x}] + [y-\bar{y}])^2 \rangle \\
&= \langle (x-\bar{x})^2 \rangle + \langle (y-\bar{y})^2 \rangle + 2\langle (x-\bar{x})(y-\bar{y}) \rangle \\
&= \sigma_x^2 + \sigma_y^2 + 2\langle (x-\bar{x})(y-\bar{y}) \rangle
\end{aligned}$$

★ Two important points:

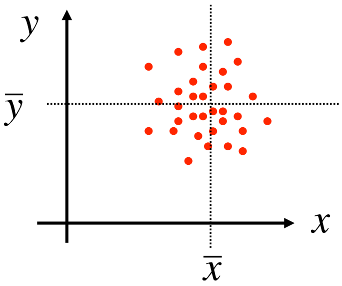
- Errors add **in quadrature** (*i.e.* sum the squares)
- The appearance of a new term, **the covariance** of $x$ and $y$

$$\boxed{\mathrm{cov}(x,y) = \langle (x-\bar{x})(y-\bar{y}) \rangle}$$

# Correlated errors: covariance

★ Consider $\mathrm{cov}(x,y) = \langle (x-\overline{x})(y-\overline{y}) \rangle$

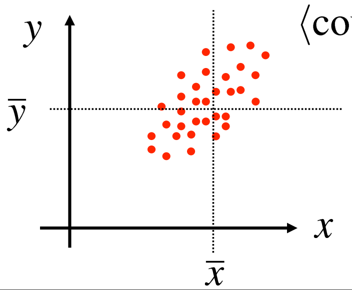
- Suppose in a single experiment measure a value of $x$ and $y$
- Imagine repeating the measurement multiple times ⇨ $\{x_i, y_i\}$
- If the measurements of $x$ and $y$ are uncorrelated, i.e. INDEPENDENT




$$\langle \mathrm{cov}(x,y) \rangle = 0$$

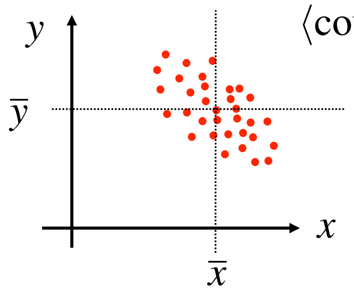
- If $x$ and $y$ are correlated

$$\langle \mathrm{cov}(x,y) \rangle > 0$$




- If $x$ and $y$ are anti-correlated

$$\langle \mathrm{cov}(x,y) \rangle < 0$$

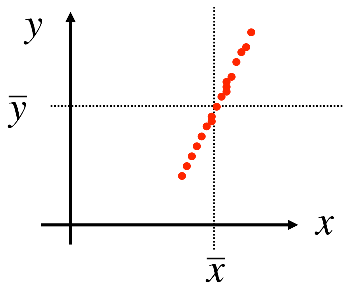





★ Often convenient to express covariance in terms of the correlation coefficient

$$\boxed{\rho = \frac{\mathrm{cov(x,y)}}{\sigma_x \sigma_y}} \qquad \begin{aligned} \mathrm{cov}(x,y) &= \langle (x-\overline{x})(y-\overline{y}) \rangle \\ \sigma_x &= \langle (x-\overline{x})^2 \rangle^{\frac{1}{2}} \end{aligned}$$

- Consider an experiment which returns two values $x$ and $y$; where $y\text{-}\overline{y} = 2(x\text{-}\overline{x})$




$$\begin{aligned} \mathrm{cov}(x,y) &= \langle (x-\overline{x})(2x-2\overline{x}) \rangle \\ &= 2\langle (x-\overline{x})^2 \rangle \\ &= 2\sigma_x^2 = \sigma_x \sigma_y \\ \Rightarrow \quad \rho &= +1 \end{aligned}$$

★ Hence (unsurprisingly) the correlation coefficient expresses the degree of correlation with

$$\boxed{|\rho| \leq 1}$$

★ Going back to $a = x + y$

$$\Rightarrow \quad \sigma_a^2 = \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y$$

# Error Propagation II: the general case

★ **We can now consider the more general case**

$$a = f(x,y)$$

$$\begin{aligned}
a &= f(x,y) = f(\bar{x},\bar{y}) + \frac{\partial f}{\partial x}(x-\bar{x}) + \frac{\partial f}{\partial y}(y-\bar{y}) + \ldots \\
(a-\bar{a})^2 &= (f(x,y) - f(\bar{x},\bar{y}))^2 \\
&\approx \left(\frac{\partial f}{\partial x}\right)^2 (x-\bar{x})^2 + \left(\frac{\partial f}{\partial y}\right)^2 (y-\bar{y})^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}(x-\bar{x})(y-\bar{y}) \\
\langle (a-\bar{a})^2 \rangle &= \left(\frac{\partial f}{\partial x}\right)^2 \langle (x-\bar{x})^2 \rangle + \left(\frac{\partial f}{\partial y}\right)^2 \langle (y-\bar{y})^2 \rangle + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\langle (x-\bar{x})(y-\bar{y}) \rangle \\
\sigma_a^2 &= \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\mathrm{cov}(x,y)
\end{aligned}$$

$$\boxed{\sigma_a^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + 2\rho\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\sigma_x\sigma_y}$$

★ **In order to estimate the error on a derived quantity need to know correlations**



# Example continued

★ **Back to the original problem** $m = \{E_1E_2(1-\cos\theta)\}^{\frac{1}{2}}$

$$\begin{aligned}
\sigma_m^2 = &\left(\frac{\partial m}{\partial E_1}\right)^2 \sigma_{E_1}^2 + \left(\frac{\partial m}{\partial E_2}\right)^2 \sigma_{E_2}^2 + \left(\frac{\partial m}{\partial \theta}\right)^2 \sigma_\theta^2 + \\
&2\rho_{12}\frac{\partial m}{\partial E_1}\frac{\partial m}{\partial E_2}\sigma_{E_1}\sigma_{E_2} + 2\rho_{1\theta}\frac{\partial m}{\partial E_1}\frac{\partial m}{\partial \theta}\sigma_{E_1}\sigma_\theta + 2\rho_{2\theta}\frac{\partial m}{\partial E_2}\frac{\partial m}{\partial \theta}\sigma_{E_2}\sigma_\theta
\end{aligned}$$

★**First assume independent errors on** $E_1, E_2, \theta$ **and for simplicity neglect** $\sigma_\theta$ **term**

$$\frac{\partial m}{\partial E_1} = \frac{1}{2}\left\{\frac{E_2}{E_1}(1-\cos\theta)\right\}^{\frac{1}{2}} \qquad \frac{\partial m}{\partial E_1} = \frac{1}{2}\left\{\frac{E_2}{E_1}(1-\cos\theta)\right\}^{\frac{1}{2}}$$

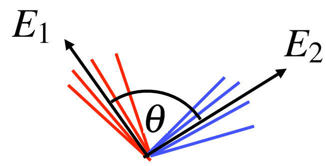
**giving:**

$$\begin{aligned}
\sigma_m^2 &= \frac{1}{4}\left\{\frac{E_2}{E_1}(1-\cos\theta)\right\}\sigma_{E_1}^2 + \frac{1}{4}\left\{\frac{E_1}{E_2}(1-\cos\theta)\right\}\sigma_{E_2}^2 \\
\frac{\sigma_m}{m} &= \frac{1}{2}\left\{\frac{\sigma_{E_1}^2}{E_1^2} + \frac{\sigma_{E_2}^2}{E_2^2}\right\}^{\frac{1}{2}}
\end{aligned}$$

★**EXERCISE: by first considering** $\sigma_{m^2}$**, calculate** $\frac{\sigma_m}{m}$**, including the** $\sigma_\theta$ **term**

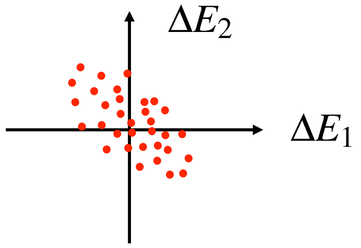
**ANS:** $$\frac{\sigma_m}{m} = \frac{1}{2}\left\{\frac{\sigma_{E_1}^2}{E_1^2} + \frac{\sigma_{E_2}^2}{E_2^2} + \cot^2\left(\frac{\theta}{2}\right)\sigma_\theta^2\right\}^{\frac{1}{2}}$$

# Estimating the Correlation Coefficient




- Correlations can arise from physical effects, e.g.
  - Would expect $E_1$ and $E_2$ to be (slightly) anti-correlated why ?
  - Can always check (in MC) by plotting

$$\Delta E_1 = E_1 - E_1^{\mathrm{MC}} \quad \text{against} \quad \Delta E_2 = E_2 - E_2^{\mathrm{MC}}$$




$$\boxed{\rho = \frac{\mathrm{cov}(\mathrm{x},\mathrm{y})}{\sigma_x \sigma_y}}$$

$$\mathrm{cov}(x,y) = \langle (x-\bar{x})(y-\bar{y}) \rangle$$
$$\sigma_x = \langle (x-\bar{x})^2 \rangle^{\frac{1}{2}}$$

NOTE: uncertainty on correlation coefficient $s_\rho \approx \dfrac{(1-\rho^2)}{\sqrt{n-2}}$

- Correlations also arise when calculating derived quantities from uncorrelated measurements
  - e.g. $x = a + b \qquad y = a - b$
  - this type of correlation can be handled mathematically (see later)



# Properties of the 2D Gaussian Distribution

- For two independent variables $(x,y)$ the joint probability distribution $P(x,y)$ is simply the product of the two distributions

$$P(x,y) = P(x)P(y) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(x-\bar{x})^2}{2\sigma_x^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(y-\bar{y})^2}{2\sigma_y^2}\right\}$$

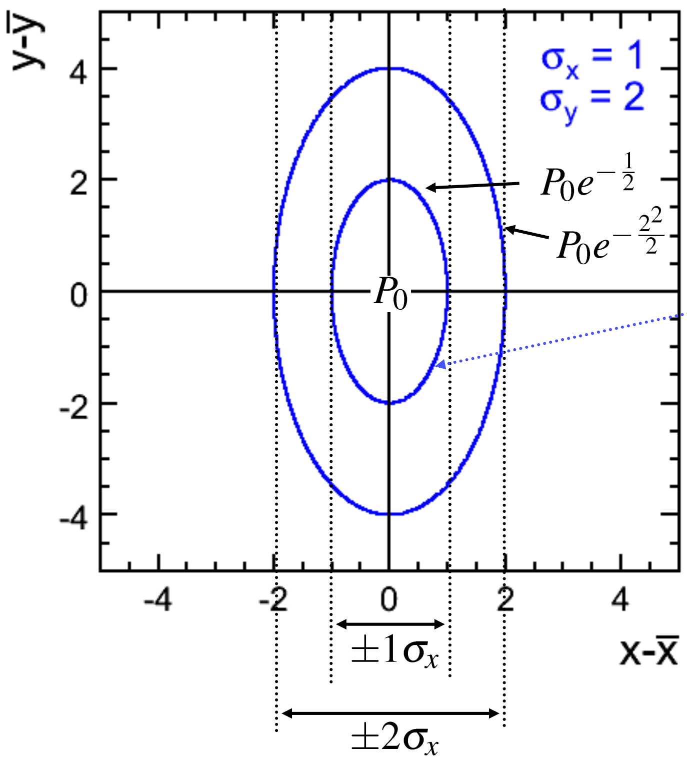
$$P(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left[\frac{(x-\bar{x})^2}{\sigma_x^2} + \frac{(y-\bar{y})^2}{\sigma_y^2}\right]\right\}$$

<u>NOTE:</u>

$$\int_{-\infty}^{+\infty} P(x,y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(x-\bar{x})^2}{2\sigma_x^2}\right\} = P(x)$$

- Can write in terms of $\chi^2$ with

$$P(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{\chi^2}{2}\right\} \qquad \chi^2 = \chi_x^2 + \chi_y^2 = \frac{(x-\bar{x})^2}{\sigma_x^2} + \frac{(y-\bar{y})^2}{\sigma_y^2}$$




- **68 %** of events within **±1** $\sigma_x$
- **68 %** of events within **±1** $\sigma_y$
- Now consider contours of

$$\chi^2 = \frac{(x-\overline{x})^2}{\sigma_x^2} + \frac{(y-\overline{y})^2}{\sigma_y^2}$$

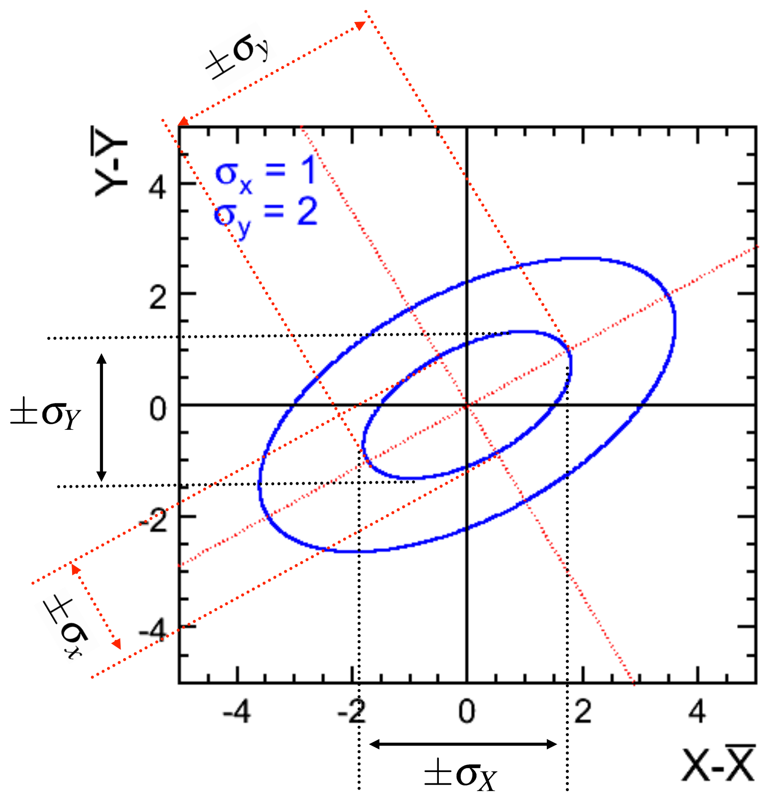
- $\chi^2 = 1$ corresponds to contour where PDF falls to $e^{-\frac{1}{2}}$ of peak
- Only **39%** of events within $\chi^2 < 1$
- Only **86%** of events within $\chi^2 < 4$

Now to introduce correlations… rotate the ellipse

$$\begin{pmatrix} X-\overline{X} \\ Y-\overline{Y} \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x-\overline{x} \\ y-\overline{y} \end{pmatrix}$$

$$s = \sin\theta; \quad c = \cos\theta$$






- Same PDF, but now w.r.t. different axes
- Simple to derive the general error ellipse with correlations…

**Let** $\quad X = +cx+sy$

$\quad\quad\; Y = -sx+cy$

**To find the equivalent correlation coefficient, evaluate**

$$\langle XY\rangle = \langle scy^2 - scx^2 + (c^2-s^2)xy\rangle = sc(\sigma_y^2-\sigma_x^2)$$

**hence** $$\boxed{\rho_{XY}\sigma_X\sigma_Y = sc(\sigma_y^2-\sigma_x^2)}$$

**To eliminate the rotation angle, write**

$$\sigma_X^2 = \langle X^2\rangle = \langle c^2x^2+s^2y^2+2csxy\rangle = c^2\sigma_x^2+s^2\sigma_y^2$$
$$\sigma_Y^2 = \langle Y^2\rangle = \langle c^2y^2+s^2x^2-2csxy\rangle = c^2\sigma_y^2+s^2\sigma_x^2$$

**giving** $$\sigma_X^2\sigma_Y^2 = s^2c^2(\sigma_x^4+\sigma_y^4)+(c^4+s^4)\sigma_x^2\sigma_y^2$$

**Compare to:** $$\rho^2\sigma_X^2\sigma_Y^2 = s^2c^2(\sigma_y^4+\sigma_x^4-2\sigma_x^2\sigma_y^2)$$

**gives** $$\sigma_X^2\sigma_Y^2 = \rho^2\sigma_X\sigma_Y+(c^4+2s^2c^2+s^4)\sigma_x^2\sigma_y^2$$

**hence** $$\boxed{(1-\rho^2)\sigma_X^2\sigma_Y^2 = \sigma_x^2\sigma_y^2}$$



# Properties of the 2D Gaussian Distribution

★ **Start from uncorrelated 2D Gaussian:**

$$P(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right\}$$

★ **Make the coordinate transformation**

$$x = cX-sY;\quad y = sX+cY \qquad P(x,y)\mathrm{d}x\mathrm{d}y = P(X,Y)\mathrm{d}X\mathrm{d}Y$$

$$\begin{aligned}
P(X,Y) &= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{(cX-sY)^2}{2\sigma_x^2}-\frac{(cY+sX)^2}{2\sigma_y^2}\right\}\\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{X^2}{2}\left[\frac{c^2}{\sigma_x^2}+\frac{s^2}{\sigma_y^2}\right]-\frac{Y^2}{2}\left[\frac{c^2}{\sigma_y^2}+\frac{s^2}{\sigma_x^2}\right]+\frac{2XY}{2}sc\left[\frac{1}{\sigma_x^2}-\frac{1}{\sigma_y^2}\right]\right\}\\
&= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{X^2}{2}\left[\frac{c^2\sigma_y^2+s^2\sigma_x^2}{\sigma_x^2\sigma_y^2}\right]-\frac{Y^2}{2}\left[\frac{c^2\sigma_x^2+s^2\sigma_y^2}{\sigma_x^2\sigma_y^2}\right]+\frac{2XY}{2}sc\left[\frac{\sigma_y^2-\sigma_x^2}{\sigma_x^2\sigma_y^2}\right]\right\}
\end{aligned}$$

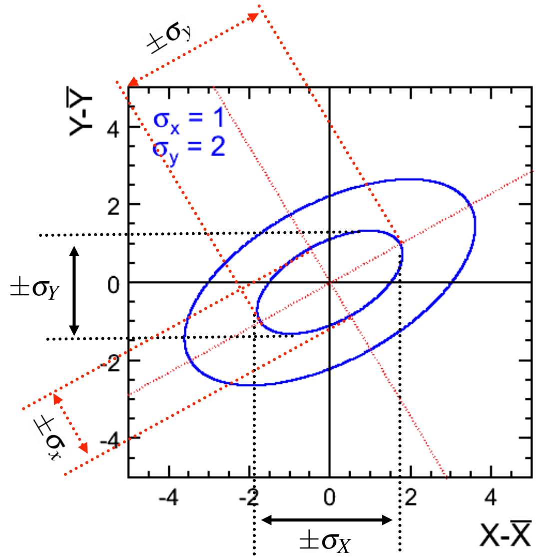
★ **From previous page identify**

$$\langle X^2\rangle = \sigma_X^2 = c^2\sigma_x^2+s^2\sigma_y^2 \qquad \langle Y^2\rangle = \sigma_Y^2 = c^2\sigma_y^2+s^2\sigma_x^2 \qquad (1-\rho^2)\sigma_X^2\sigma_Y^2 = \sigma_x^2\sigma_y^2$$

$$
\begin{aligned}
P(X,Y) &= \frac{1}{2\pi\sqrt{(1-\rho^2)}\sigma_X\sigma_Y}\exp\left\{-\frac{X^2}{2}\left[\frac{\sigma_Y^2}{(1-\rho^2)\sigma_X^2\sigma_Y^2}\right]-\frac{Y^2}{2}\left[\frac{\sigma_X^2}{(1-\rho^2)\sigma_X^2\sigma_Y^2}\right]+\frac{2\rho XY}{2(1-\rho^2)\sigma_X\sigma_Y}\right\} \\
&= \frac{1}{2\pi\sqrt{(1-\rho^2)}\sigma_X\sigma_Y}\exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}\left[\frac{X^2}{\sigma_X^2}+\frac{Y^2}{\sigma_Y^2}-\frac{2\rho XY}{\sigma_X\sigma_Y}\right]\right\}
\end{aligned}
$$

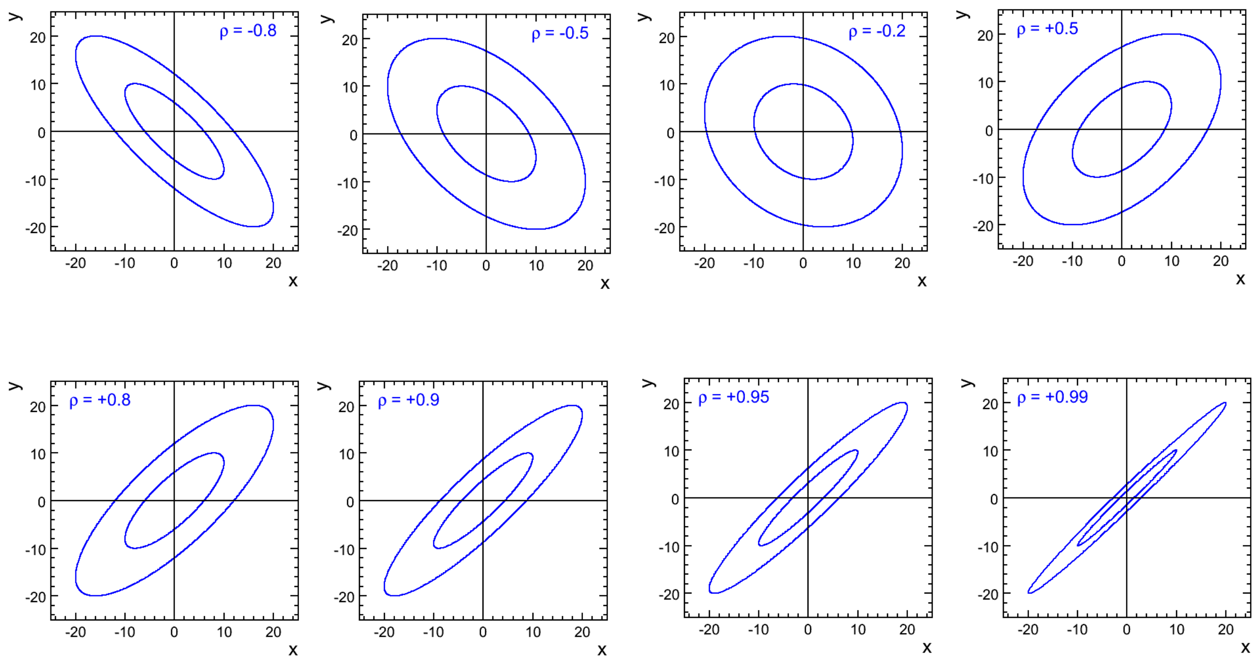
★ Note we have now expressed the same ellipse in terms of the new coordinates, where the errors are now correlated.

★ If dealing with correlated errors can always find a linear combination of variables which are uncorrelated






★ Example 2D error ellipses with different correlation coefficients



# The Error Ellipse and Error Matrix

★ Now we have the general equation for two correlated Gaussian distributed quantities

$$\boxed{P(x,y) = \frac{1}{2\pi\sqrt{(1-\rho^2)}\sigma_x\sigma_y}\exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}\left[\frac{(x-\bar{x})^2}{\sigma_x^2}+\frac{(y-\bar{y})^2}{\sigma_y^2}-2\frac{\rho(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y}\right]\right\}}$$

★ Defines the error ellipse
★ Ultimately want to generalise this to an N variable hyper-ellipsoid
★ Sounds hard… but is actually rather simple in matrix form
★ Define the ERROR MATRIX

$$\mathbf{M} = \begin{pmatrix} \langle x^2\rangle & \langle xy\rangle \\ \langle xy\rangle & \langle y^2\rangle \end{pmatrix} \quad \text{i.e.} \quad \boxed{\mathbf{M} = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}}$$

★and define the DISCREPANCY VECTOR $\boxed{\mathbf{x} = \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}}$

$\det\mathbf{M}$

using $\mathbf{M}^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_x^2} & -\frac{\rho}{\sigma_x\sigma_y} \\ -\frac{\rho}{\sigma_x\sigma_y} & \frac{1}{\sigma_y^2} \end{pmatrix}$ and $|\mathbf{M}| = (1-\rho^2)\sigma_x^2\sigma_y^2$

we can write $\boxed{P(x,y) = \frac{1}{2\pi|\mathbf{M}|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{M}^{-1}\mathbf{x}\right\}}$



★ The beauty of this formalism is that it can be extended to any number of correlated Gaussian distributed variables

$$P(x_1,x_2,\ldots,x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}|\mathbf{M}|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{M}^{-1}\mathbf{x}\right\}$$

with

$$\mathbf{M} = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \ldots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \ldots & \rho_{2n}\sigma_2\sigma_n \\ \ldots & \ldots & \ldots & \ldots \\ \rho_{1n}\sigma_1\sigma_n & \rho_{2n}\sigma_2\sigma_n & \ldots & \sigma_n^2 \end{pmatrix}$$

★ Can write this in terms of the $\chi^2$ for n-variables (including correlations)

$$P(x_1,x_2,\ldots,x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}|\mathbf{M}|^{\frac{1}{2}}}\exp\left\{-\frac{\chi^2}{2}\right\} = P_0 e^{-\frac{\chi^2}{2}}$$

with $\boxed{\chi^2 = \mathbf{x}^{\mathbf{T}}\mathbf{M}^{-1}\mathbf{x}}$

# General transformation of Errors

★ Suppose we have a set of variables, $x_i$, and the error matrix, **M**, and now wish to transform to a set of variables, $y_i$, defined by

★ Taylor expansion about mean:

$$
\begin{aligned}
y_i &= \bar{y}_i + \sum_k \frac{\partial y_i}{\partial x_k}(x_l - \bar{x}_k) + \mathscr{O}(\Delta x^2) \\
y_i - \bar{y}_i &\approx \sum_k \frac{\partial y_i}{\partial x_k}(x_k - \bar{x}_k) \\
\langle (y_i - \bar{y}_i)(y_j - \bar{y}_j) \rangle &= \left\langle \sum_k \frac{\partial y_i}{\partial x_k}(x_k - \bar{x}_k) \sum_\ell \frac{\partial y_j}{\partial x_\ell}(x_\ell - \bar{x}_\ell) \right\rangle \\
&= \sum_{k\ell} \frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_\ell} \langle (x_k - \bar{x}_k)(x_\ell - \bar{x}_\ell) \rangle \\
\mathbf{M}_{\{\mathbf{y}\}}{}^{ij} &= \sum_{k\ell} \frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_\ell} \mathbf{M}_{\{\mathbf{x}\}}{}^{k\ell} \\
\mathbf{M}_{\{\mathbf{y}\}} &= \mathbf{T}^{\mathbf{T}} \mathbf{M}_{\{\mathbf{x}\}} \mathbf{T}
\end{aligned}
$$



★ **T** is the error transformation matrix

$$
\mathbf{T} = \begin{pmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\
\frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n}
\end{pmatrix}
$$

For Gaussian errors we can now do anything !

★ Can deal with:
- correlated errors
- arbitrary dimensions
- parameter transformations

Examples…

# A simple example

★ **Measure two uncorrelated variables** $a \pm \sigma_a,\ b \pm \sigma_b$

**Error matrix** $\mathbf{M} = \begin{pmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{pmatrix} \qquad \mathbf{M}^{-1} = \begin{pmatrix} 1/\sigma_a^2 & 0 \\ 0 & 1/\sigma_b^2 \end{pmatrix}$

★ **Calculate two derived quantities**

$$x = a + b \qquad y = a - b$$

★ **Transformation matrix**

$$\mathbf{T} = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

★ **Giving**

$$\begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} = \mathbf{T}^{\mathbf{T}}\mathbf{M}\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

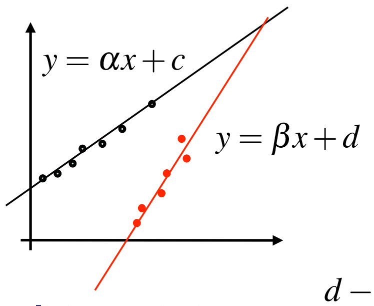
$$= \begin{pmatrix} \sigma_a^2 + \sigma_b^2 & \sigma_a^2 - \sigma_b^2 \\ \sigma_a^2 - \sigma_b^2 & \sigma_a^2 + \sigma_b^2 \end{pmatrix}$$

$$\boxed{\sigma_x^2 = \sigma_y^2 = \sigma_a^2 + \sigma_b^2; \quad \rho = \frac{\sigma_a^2 - \sigma_b^2}{\sigma_a^2 + \sigma_b^2}}$$



# A more involved example

★ **Given the results of two straight line fits, calculate the uncertainty on the intersection**




• **With the error matrix (note the results of the two fits are uncorrelated)**

$$\mathbf{M} = \begin{pmatrix} \sigma_\alpha^2 & \rho_1\sigma_\alpha\sigma_c & 0 & 0 \\ \rho_1\sigma_\alpha\sigma_c & \sigma_c^2 & 0 & 0 \\ 0 & 0 & \sigma_\beta^2 & \rho_2\sigma_\beta\sigma_d \\ 0 & 0 & \rho_2\sigma_\beta\sigma_d & \sigma_d^2 \end{pmatrix}$$

▪**Lines Intersect at:** $x = \dfrac{d-c}{\alpha-\beta} \qquad y = \dfrac{\alpha d - \beta c}{\alpha - \beta}$

▪**To calculate error on intersection need error transformation matrix, i.e. need the partial derivatives, e.g.** $\dfrac{\partial x}{\partial \alpha} = \dfrac{c-d}{(\alpha-\beta)^2}$

▪ **giving**

$$\mathbf{T} = \begin{pmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} \\ \frac{\partial x}{\partial d} & \frac{\partial y}{\partial d} \end{pmatrix} = \frac{1}{\alpha-\beta}\begin{pmatrix} -\kappa & -\beta\kappa \\ -1 & -\beta \\ \kappa & \alpha\kappa \\ +1 & \alpha \end{pmatrix} \quad \text{with } \kappa = \frac{d-c}{\alpha-\beta}$$

- then its just algebra

$$\begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y\\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix} = \mathbf{T}^{\mathbf{T}}\mathbf{M}\mathbf{T}$$

$$= \begin{pmatrix}-\kappa & -1 & \kappa & +1\\ -\beta\kappa & -\beta & \alpha\kappa & \alpha\end{pmatrix}\begin{pmatrix}\sigma_\alpha^2 & \rho_1\sigma_\alpha\sigma_c & 0 & 0\\ \rho_1\sigma_\alpha\sigma_c & \sigma_c^2 & 0 & 0\\ 0 & 0 & \sigma_\beta^2 & \rho_2\sigma_\beta\sigma_d\\ 0 & 0 & \rho_2\sigma_\beta\sigma_d & \sigma_d^2\end{pmatrix}\begin{pmatrix}-\kappa & -\beta\kappa\\ -1 & -\beta\\ \kappa & \alpha\kappa\\ +1 & \alpha\end{pmatrix}$$

- giving

$$\sigma_x^2 = \frac{1}{(\alpha-\beta)^2}\left[\kappa^2(\sigma_\alpha^2+\sigma_\beta^2)+2\kappa(\rho_1\sigma_\alpha\sigma_c+\rho_2\sigma_\beta\sigma_d)+\sigma_c^2+\sigma_d^2\right]$$

$$\rho\sigma_x\sigma_y = \frac{1}{(\alpha-\beta)^2}\left[\kappa^2(\beta\sigma_\alpha^2+\alpha\sigma_\beta^2)+2\kappa(\rho_1\beta\sigma_\alpha\sigma_c+\rho_2\alpha\sigma_\beta\sigma_d)+\beta\sigma_c^2+\alpha\sigma_d^2\right]$$

$$\sigma_y^2 = \frac{1}{(\alpha-\beta)^2}\left[\kappa^2(\beta^2\sigma_\alpha^2+\alpha^2\sigma_\beta^2)+2\kappa(\rho_1\beta^2\sigma_\alpha\sigma_c+\rho_2\alpha^2\sigma_\beta\sigma_d)+\beta^2\sigma_c^2+\alpha^2\sigma_d^2\right]$$

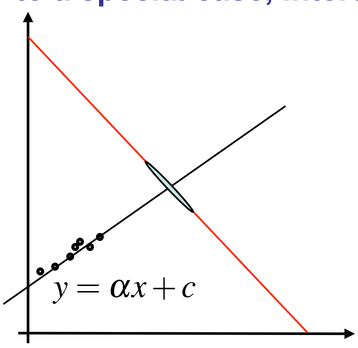
$$\boxed{\kappa = \frac{d-c}{\alpha-\beta}}$$

★ OK, it is not pretty, but we now have an analytic expression
(i.e. once you have done the calculation, computationally very fast)



- Apply to a special case, intersection with a fixed line $y = 1 - x$

$$\boxed{\beta = -1;\ d = +1;\ \sigma_\beta = 0;\ \sigma_d = 0}$$




$$\sigma_x^2 = \frac{1}{(\alpha-\beta)^2}\left[\kappa^2\sigma_\alpha^2+2\kappa\rho_1\sigma_\alpha\sigma_c+\sigma_c^2\right]$$

$$\rho\sigma_x\sigma_y = \frac{1}{(\alpha-\beta)^2}\left[-\kappa^2\sigma_\alpha^2-2\kappa\rho_1\sigma_\alpha\sigma_c-\sigma_c^2\right]$$

$$\sigma_y^2 = \frac{1}{(\alpha-\beta)^2}\left[\kappa^2\sigma_\alpha^2+2\kappa\rho_1\sigma_\alpha\sigma_c+\sigma_c^2\right]$$

Hence $\sigma_x^2 = \sigma_y^2;\ \rho = -1$ which makes perfect sense

★ The treatment of Gaussian errors via the error matrix is an extremely powerful technique – it is also easy to apply (once you understand the basic ideas)

# Summary

★ **Should now understand:**

- **Properties of the Gaussian distribution**
- **How to combine errors**
- **Propagation simple of 1D errors**
- **How to include correlations**
- **How to treat multi-dimensional errors**
- **How to use the error matrix**

★ **Next up, chi-squared, likelihood fits, …**



# Appendix: Error on Error - Justification

- **Assume mean of distribution is zero (can always make this transformation without affecting the variance)**

$$
\begin{aligned}
Var(s^2) &= \langle (s^2-\sigma^2)^2 \rangle \\
&= \langle (\frac{1}{n}\sum x^2 - \sigma^2)^2 \rangle \\
&= \frac{1}{n^2}\langle \sum_i x_i^2 \sum_j x_j^2 \rangle - 2\sigma^2 \langle (\frac{1}{n}\sum x^2) \rangle + \sigma^4 \\
&= \frac{1}{n^2}\left(n\langle x^4\rangle + n(n-1)\langle x_i^2 x_j^2\rangle_{i\neq j}\right) - \sigma^4 \\
&\approx \frac{1}{n}\langle x^4 \rangle + \frac{n-1}{n}\sigma^4 - \sigma^4 \\
&= \frac{1}{n}(\langle x^4\rangle - \langle x^2\rangle^2)
\end{aligned}
$$

**For large n**
$\langle x_i^2 x_j^2 \rangle_{i\neq j} \approx \sigma^4$