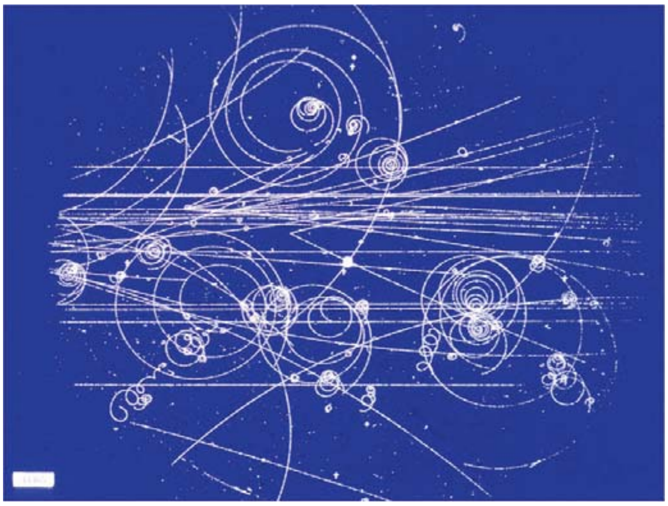
# Particle Physics

**Michaelmas Term 2011**
**Prof Mark Thomson**



# Handout 4 : Electron-Positron Annihilation

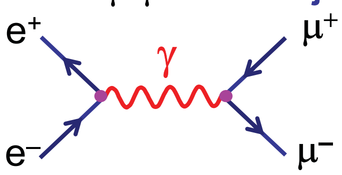


# QED Calculations

- **How to calculate a cross section using QED (e.g. $e^+e^- \rightarrow \mu^+\mu^-$):**

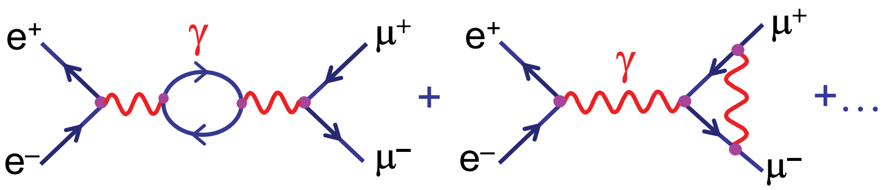
❶ **Draw all possible Feynman Diagrams**

- **For $e^+e^- \rightarrow \mu^+\mu^-$ there is just one lowest order diagram**




$$M \propto e^2 \propto \alpha_{em}$$

**+ many second order diagrams + ...**




$$M \propto e^4 \propto \alpha_{em}^2$$

❷ **For each diagram calculate the matrix element using Feynman rules derived in handout 4.**

❸ **Sum the individual matrix elements (i.e. sum the amplitudes)**

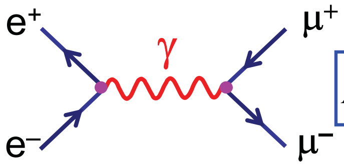
$$M_{fi} = M_1 + M_2 + M_3 + ....$$

- **Note: summing amplitudes therefore different diagrams for the same final state can interfere either positively or negatively!**

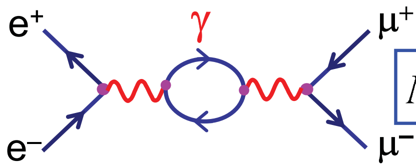
and then square $|M_{fi}|^2 = (M_1 + M_2 + M_3 + ....)(M_1^* + M_2^* + M_3^* + ....)$

➡ this gives the full perturbation expansion in $\alpha_{em}$

- For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to neglect higher order diagrams.




$M^2 \propto \alpha_{em}^2$




$M^2 \propto \alpha_{em}^4$

❹ Calculate decay rate/cross section using formulae from handout 1.

- e.g. for a decay

$$\Gamma = \frac{p^*}{32\pi^2 m_a^2}\int |M_{fi}|^2 \mathrm{d}\Omega$$

- For scattering in the centre-of-mass frame

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|}|M_{fi}|^2 \qquad (1)$$

- For scattering in lab. frame (neglecting mass of scattered particle)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2$$

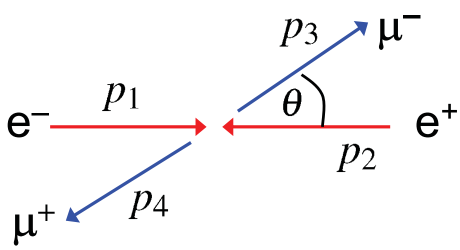


# Electron Positron Annihilation

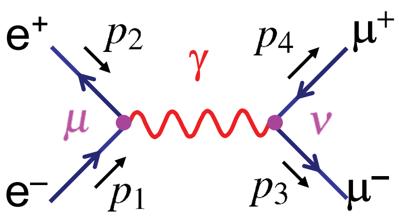
★ Consider the process: $e^+e^- \to \mu^+\mu^-$

- Work in C.o.M. frame (this is appropriate for most $e^+e^-$ colliders).

$$p_1 = (E,0,0,p) \qquad p_2 = (E,0,0,-p)$$
$$p_3 = (E,\vec{p}_f) \qquad p_4 = (E,-\vec{p}_f)$$




- Only consider the lowest order Feynman diagram:




- Feynman rules give:

$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

**NOTE:**
- Incoming anti-particle $\overline{v}$
- Incoming particle $u$
- Adjoint spinor written first

- In the C.o.M. frame have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|}|M_{fi}|^2 \qquad \text{with} \qquad s = (p_1+p_2)^2 = (E+E)^2 = 4E^2$$

# Electron and Muon Currents

- Here $q^2 = (p_1 + p_2)^2 = s$ and matrix element

$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

$$\Rightarrow \quad M = -\frac{e^2}{s}g_{\mu\nu}[\overline{v}(p_2)\gamma^\mu u(p_1)][\overline{u}(p_3)\gamma^\nu v(p_4)]$$

- In handout 2 introduced the four-vector current

$$j^\mu = \overline{\psi}\gamma^\mu\psi$$

which has same form as the two terms in [ ] in the matrix element

- The matrix element can be written in terms of the electron and muon currents

$$(j_e)^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \quad \text{and} \quad (j_\mu)^\nu = \overline{u}(p_3)\gamma^\nu v(p_4)$$

$$\Rightarrow \quad M = -\frac{e^2}{s}g_{\mu\nu}(j_e)^\mu(j_\mu)^\nu$$

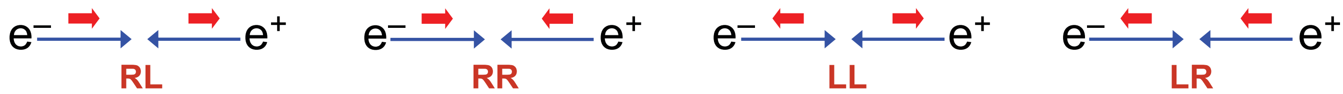
$$\boxed{M = -\frac{e^2}{s}j_e.j_\mu}$$

- Matrix element is a four-vector scalar product – confirming it is Lorentz Invariant



# Spin in e⁺e⁻ Annihilation

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the initial state !




- Similarly there are four possible helicity combinations in the final state
- In total there are **16** combinations e.g. RL→RR, RL→RL, ....
- To account for these states we need to sum over all **16** possible helicity combinations and then average over the number of <u>initial</u> helicity states:

$$\langle |M|^2\rangle = \frac{1}{4}\sum_{\text{spins}}|M_i|^2 = \frac{1}{4}\left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \ldots\right)$$

★ i.e. need to evaluate:

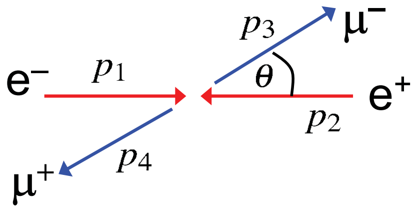
$$M = -\frac{e^2}{s}j_e.j_\mu$$

**for all 16 helicity combinations !**

★ Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements – we will see that this is an important feature of QED/QCD

- In the C.o.M. frame in the limit $E \gg m$

$$p_1 = (E,0,0,E); \quad p_2 = (E,0,0,-E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_4 = (E, -\sin\theta, 0, -E\cos\theta)$$




- Left- and right-handed helicity spinors (handout 3) for particles/anti-particles are:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix} \quad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

where $s = \sin\frac{\theta}{2}$; $c = \cos\frac{\theta}{2}$ and $N = \sqrt{E+m}$

- In the limit $E \gg m$ these become:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \; u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; \; v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; \; v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

- The initial-state electron can either be in a left- or right-handed helicity state

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \; u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix};$$

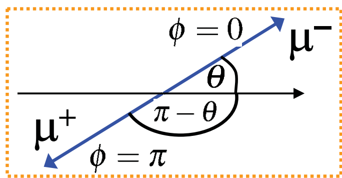


- For the initial state positron $(\theta = \pi)$ can have either:

$$v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}; \; v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

- Similarly for the final state μ⁻ which has polar angle $\theta$ and choosing $\phi = 0$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}; \; u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix};$$



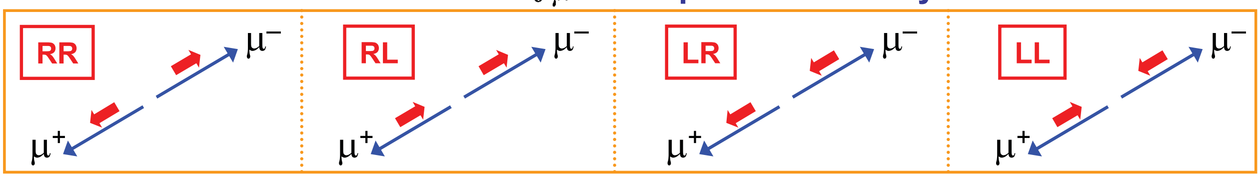

- And for the final state μ⁺ replacing $\theta \to \pi - \theta; \quad \phi \to \pi$ obtain

$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}; \; v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix};$$

using
$$\sin\left(\tfrac{\pi-\theta}{2}\right) = \cos\tfrac{\theta}{2}$$
$$\cos\left(\tfrac{\pi-\theta}{2}\right) = \sin\tfrac{\theta}{2}$$
$$e^{i\pi} = -1$$

- Wish to calculate the matrix element $M = -\frac{e^2}{s} j_e . j_\mu$

★ first consider the muon current $j_\mu$ for 4 possible helicity combinations



# The Muon Current

- Want to evaluate $(j_\mu)^\nu = \overline{u}(p_3)\gamma^\nu v(p_4)$ for all four helicity combinations
- For arbitrary spinors $\psi,\ \phi$ with it is straightforward to show that the components of $\overline{\psi}\gamma^\mu\phi$ are

$$\overline{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1+\psi_2^*\phi_2+\psi_3^*\phi_3+\psi_4^*\phi_4 \quad (3)$$

$$\overline{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4+\psi_2^*\phi_3+\psi_3^*\phi_2+\psi_4^*\phi_1 \quad (4)$$

$$\overline{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4-\psi_2^*\phi_3+\psi_3^*\phi_2-\psi_4^*\phi_1) \quad (5)$$

$$\overline{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3-\psi_2^*\phi_4+\psi_3^*\phi_1-\psi_4^*\phi_2 \quad (6)$$

- Consider the $\mu_R^-\mu_L^+$ combination using $\psi = u_\uparrow \quad \phi = v_\downarrow$

with $v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c\end{pmatrix}$; $u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s\end{pmatrix}$;

$$\overline{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs-sc+cs-sc) = 0$$

$$\Rightarrow \quad \overline{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E(-c^2+s^2-c^2+s^2) = 2E(s^2-c^2) = -2E\cos\theta$$

$$\overline{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE(-c^2-s^2-c^2-s^2) = 2iE$$

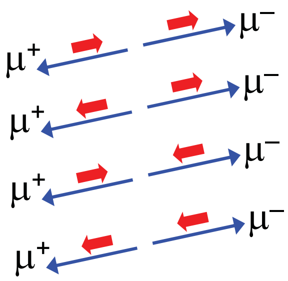
$$\overline{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs+sc+cs+sc) = 4Esc = 2E\sin\theta$$



- Hence the four-vector muon current for the RL combination is

$$\overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta,i,\sin\theta)$$

- The results for the 4 helicity combinations (obtained in the same manner) are:




$$\overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta,i,\sin\theta) \quad \text{RL}$$

$$\overline{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) = (0,0,0,0) \quad \text{RR}$$

$$\overline{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) = (0,0,0,0) \quad \text{LL}$$

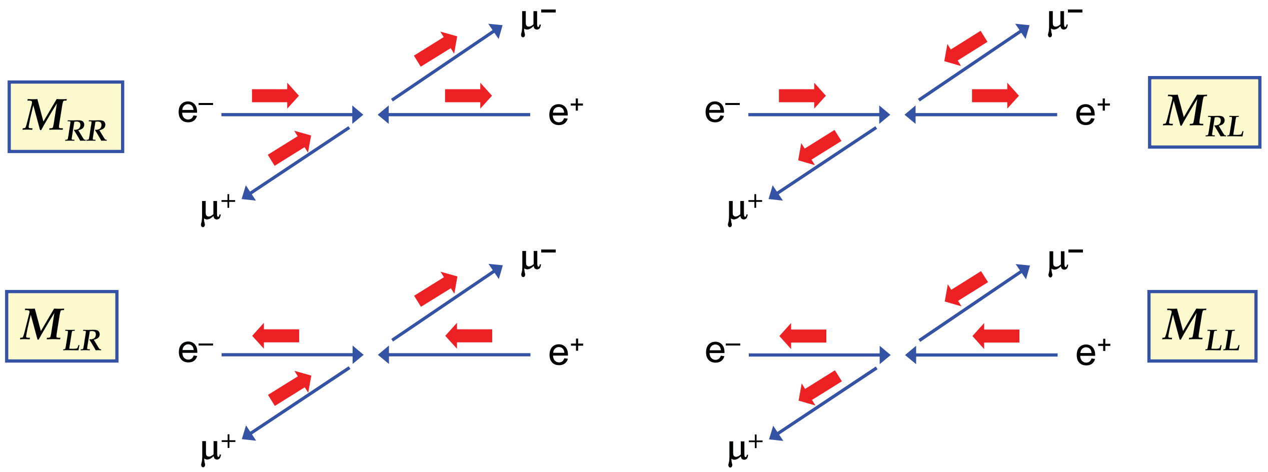
$$\overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0,-\cos\theta,-i,\sin\theta) \quad \text{LR}$$

★ **IN THE LIMIT $E \gg m$ only two helicity combinations are non-zero !**

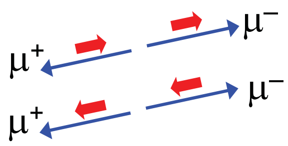
- This is an important feature of QED. It applies equally to QCD.
- In the Weak interaction only one helicity combination contributes.
- The origin of this will be discussed in the last part of this lecture
- But as a consequence of the 16 possible helicity combinations only four given non-zero matrix elements

# Electron Positron Annihilation cont.

★ For $e^+e^- \to \mu^+\mu^-$ now only have to consider the 4 matrix elements:




• Previously we derived the muon currents for the allowed helicities:




$$\begin{array}{lcll} \mu_R^-\mu_L^+ : & \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) & = & 2E(0,-\cos\theta,i,\sin\theta) \\ \mu_L^-\mu_R^+ : & \overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) & = & 2E(0,-\cos\theta,-i,\sin\theta) \end{array}$$

• Now need to consider the electron current



# The Electron Current

• The incoming electron and positron spinors (L and R helicities) are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\; u_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}; \qquad v_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\; v_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

• The electron current can either be obtained from equations (3)-(6) as before or it can be obtained directly from the expressions for the muon current.

$$(j_e)^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \qquad\qquad (j_\mu)^\mu = \overline{u}(p_3)\gamma^\mu v(p_4)$$

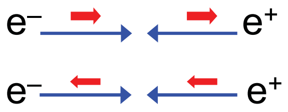
• Taking the Hermitian conjugate of the muon current gives

$$\begin{aligned} [\overline{u}(p_3)\gamma^\mu v(p_4)]^\dagger &= \left[u(p_3)^\dagger\gamma^0\gamma^\mu v(p_4)\right]^\dagger \\ &= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_3) && (AB)^\dagger = B^\dagger A^\dagger \\ &= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^0 u(p_3) && \gamma^{0\dagger} = \gamma^0 \\ &= v(p_4)^\dagger\gamma^0\gamma^\mu u(p_3) && \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu \\ &= \overline{v}(p_4)\gamma^\mu u(p_3) \end{aligned}$$

• Taking the complex conjugate of the muon currents for the two non-zero helicity configurations:

$$\overline{v}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_3) = \left[\overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)\right]^* = 2E(0,-\cos\theta,-i,\sin\theta)$$
$$\overline{v}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_3) = \left[\overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4)\right]^* = 2E(0,-\cos\theta,i,\sin\theta)$$

To obtain the electron currents we simply need to set $\theta = 0$



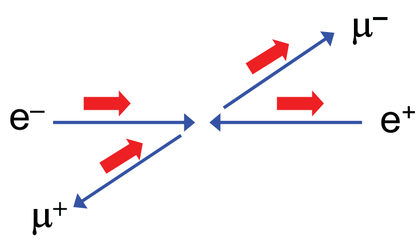

$$e_R^- e_L^+ : \quad \overline{v}_\downarrow(p_2)\gamma^\nu u_\uparrow(p_1) = 2E(0,-1,-i,0)$$
$$e_L^- e_R^+ : \quad \overline{v}_\uparrow(p_2)\gamma^\nu u_\downarrow(p_1) = 2E(0,-1,i,0)$$



# Matrix Element Calculation

• We can now calculate $M = -\frac{e^2}{s} j_e . j_\mu$ for the four possible helicity combinations.

**e.g.** the matrix element for $e_R^- e_L^+ \rightarrow \mu_R^- \mu_L^+$ which will denote $M_{RR}$



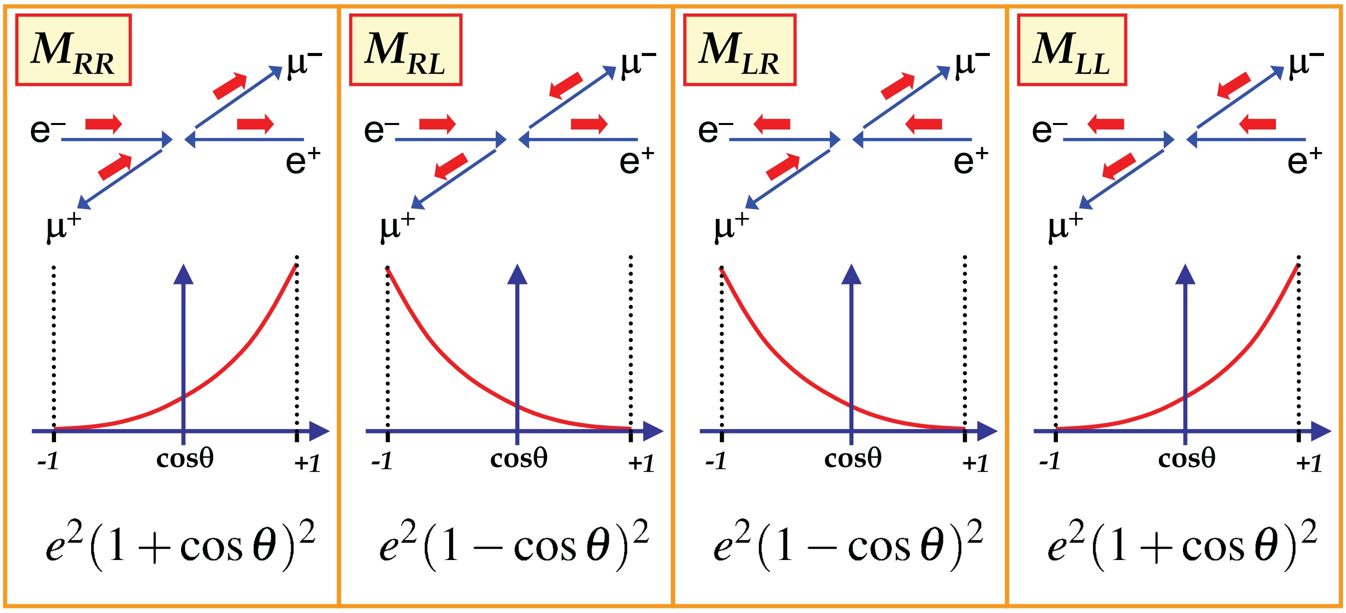

Here the first subscript refers to the helicity of the e⁻ and the second to the helicity of the μ⁻. Don't need to specify other helicities due to "helicity conservation", only certain chiral combinations are non-zero.

★ Using:
$$e_R^- e_L^+ : \quad (j_e)^\mu = \overline{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) = 2E(0,-1,-i,0)$$
$$\mu_R^- \mu_L^+ : \quad (j_\mu)^\nu = \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta,i,\sin\theta)$$

gives
$$M_{RR} = -\frac{e^2}{s}\left[2E(0,-1,-i,0)\right].\left[2E(0,-\cos\theta,i,\sin\theta)\right]$$
$$= -e^2(1+\cos\theta)$$
$$= -4\pi\alpha(1+\cos\theta)$$
where $\alpha = e^2/4\pi \approx 1/137$

**Similarly**

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$
$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$




- Assuming that the incoming electrons and positrons are unpolarized, all 4 possible initial helicity states are equally likely.

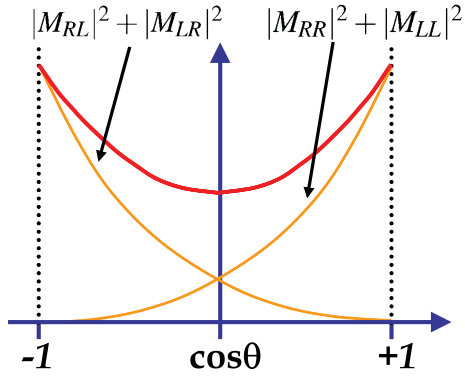


# Differential Cross Section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\times\frac{1}{64\pi^2 s}(|M_{RR}|^2+|M_{RL}|^2+|M_{LR}|^2+|M_{LL}^2|)$$
$$= \frac{(4\pi\alpha)^2}{256\pi^2 s}(2(1+\cos\theta)^2+2(1-\cos\theta)^2)$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

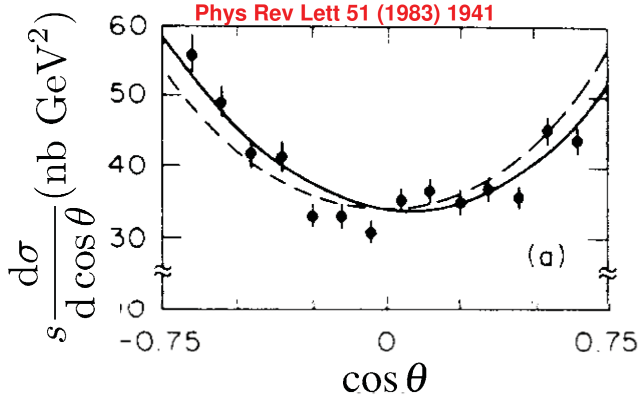



**Example:**

$e^+e^- \rightarrow \mu^+\mu^-$

$\sqrt{s} = 29$ GeV

Mark II Expt., M.E.Levi et al., Phys Rev Lett 51 (1983) 1941




- - - - - pure QED, $O(\alpha^3)$

——— QED plus Z contribution

Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

- The total cross section is obtained by integrating over $\theta,\ \phi$ using

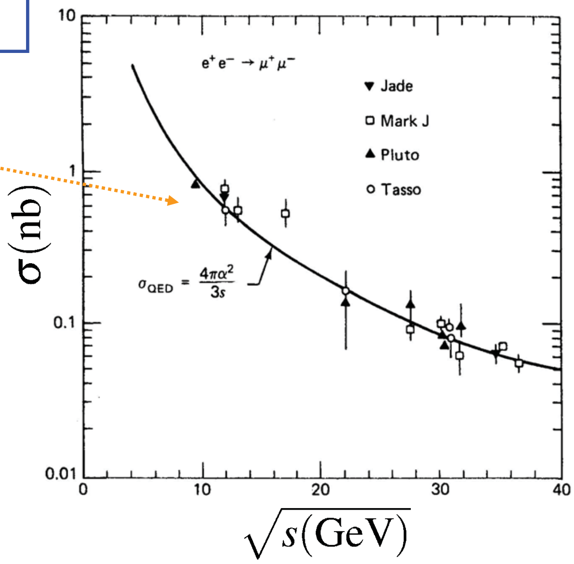
$$\int (1+\cos^2\theta)\mathrm{d}\Omega = 2\pi\int_{-1}^{+1}(1+\cos^2\theta)\mathrm{d}\cos\theta = \frac{16\pi}{3}$$

giving the **QED** total cross-section for the process $e^+e^- \to \mu^+\mu^-$

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

★ Lowest order cross section calculation provides a good description of the data !

This is an impressive result. From first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to 1%



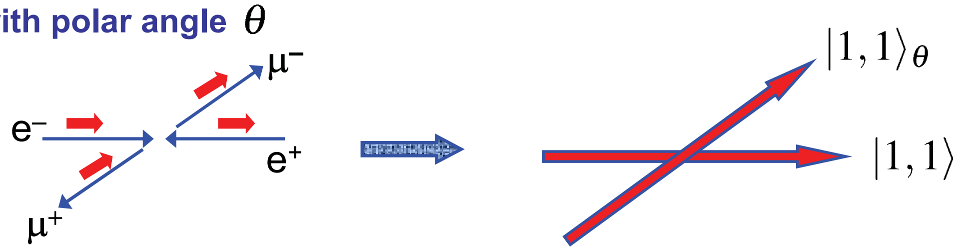



# Spin Considerations $(E \gg m)$

★ The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum

- Because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$, i.e. in a total spin 1 state aligned along the z axis: $|1,+1\rangle$ or $|1,-1\rangle$
- Similarly the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle $\theta$

e.g. $M_{RR}$



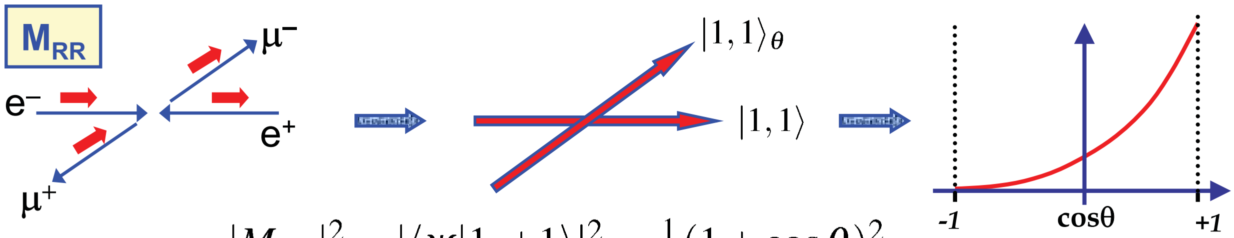

- Hence $M_{RR} \propto \langle\psi|1,1\rangle$ where $\psi$ corresponds to the spin state, $|1,1\rangle_\theta$, of the muon pair.
- To evaluate this need to express $|1,1\rangle_\theta$ in terms of eigenstates of $S_z$
- In the appendix (and also in IB QM) it is shown that:

$$|1,1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$

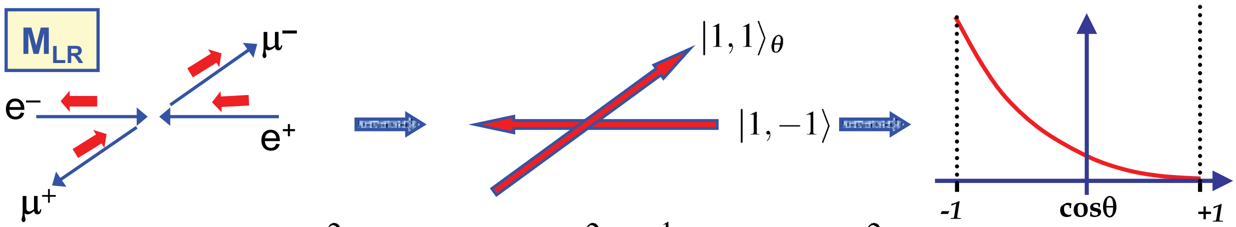
- Using the wave-function for a spin 1 state along an axis at angle $\theta$

$$\psi = |1,1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$

can immediately understand the angular dependence




$$|M_{RR}|^2 \propto |\langle\psi|1,+1\rangle|^2 = \tfrac{1}{4}(1+\cos\theta)^2$$



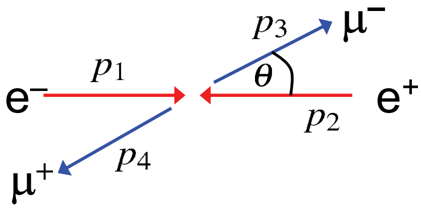

$$|M_{LR}|^2 \propto |\langle\psi|1,-1\rangle|^2 = \tfrac{1}{4}(1-\cos\theta)^2$$



# Lorentz Invariant form of Matrix Element

- Before concluding this discussion, note that the spin-averaged Matrix Element derived above is written in terms of the muon angle in the C.o.M. frame.

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{1}{4}\times(|M_{RR}|^2+|M_{RL}|^2+|M_{LR}|^2+|M_{LL}^2|) \\ &= \frac{1}{4}e^4(2(1+\cos\theta)^2+2(1-\cos\theta)^2) \\ &= e^4(1+\cos^2\theta)\end{aligned}$$




- The matrix element is **Lorentz Invariant** (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz Invariant 4-vector scalar products
- In the C.o.M. $p_1 = (E,0,0,E) \quad p_2 = (E,0,0,-E)$

$$p_3 = (E, E\sin\theta, 0, E\cos\theta) \quad p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

giving: $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

- Hence we can write

$$\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_3)^2+(p_1.p_4)^2}{(p_1.p_2)^2} \qquad \equiv 2e^4\left(\frac{t^2+u^2}{s^2}\right)$$

★Valid in any frame !

# CHIRALITY

- The helicity eigenstates for a particle/anti-particle for $E \gg m$ are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix};\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix};\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

where $s = \sin\frac{\theta}{2};\quad c = \cos\frac{\theta}{2}$

- Define the matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- In the limit $E \gg m$ the helicity states are also eigenstates of $\gamma^5$

$$\gamma^5 u_\uparrow = +u_\uparrow;\ \gamma^5 u_\downarrow = -u_\downarrow;\ \gamma^5 v_\uparrow = -v_\uparrow;\ \gamma^5 v_\downarrow = +v_\downarrow$$

★ In general, define the eigenstates of $\gamma^5$ as LEFT and RIGHT HANDED CHIRAL states $u_R;\quad u_L;\quad v_R;\quad v_L$

i.e. $\gamma^5 u_R = +u_R;\ \gamma^5 u_L = -u_L;\ \gamma^5 v_R = -v_R;\ \gamma^5 v_L = +v_L$

- In the LIMIT $E \gg m$ (and ONLY IN THIS LIMIT):

$$u_R \equiv u_\uparrow;\quad u_L \equiv u_\downarrow;\quad v_R \equiv v_\uparrow;\quad v_L \equiv v_\downarrow$$



★ This is a subtle but important point: in general the HELICITY and CHIRAL eigenstates are not the same. It is only in the ultra-relativistic limit that the chiral eigenstates correspond to the helicity eigenstates.

★ Chirality is an import concept in the structure of QED, and any interaction of the form $\bar{u}\gamma^\nu u$

- In general, the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R;\ \gamma^5 u_L = -u_L;\ \gamma^5 v_R = -v_R;\ \gamma^5 v_L = +v_L$$

- Define the projection operators:

$$P_R = \tfrac{1}{2}(1+\gamma^5);\qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

- The projection operators, project out the chiral eigenstates

$$P_R u_R = u_R;\quad P_R u_L = 0;\quad P_L u_R = 0;\quad P_L u_L = u_L$$
$$P_R v_R = 0;\quad P_R v_L = v_L;\quad P_L v_R = v_R;\quad P_L v_L = 0$$

- Note $P_R$ projects out right-handed particle states and left-handed anti-particle states
- We can then write any spinor in terms of it left and right-handed chiral components:

$$\psi = \psi_R + \psi_L = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi$$

# Chirality in QED

- In QED the basic interaction between a fermion and photon is:

$$ie\overline{\psi}\gamma^{\mu}\phi$$

- Can decompose the spinors in terms of Left and Right-handed chiral components:

$$\begin{aligned} ie\overline{\psi}\gamma^{\mu}\phi &= ie(\overline{\psi}_L+\overline{\psi}_R)\gamma^{\mu}(\phi_R+\phi_L) \\ &= ie(\overline{\psi}_R\gamma^{\mu}\phi_R+\overline{\psi}_R\gamma^{\mu}\phi_L+\overline{\psi}_L\gamma^{\mu}\phi_R+\overline{\psi}_L\gamma^{\mu}\phi_L) \end{aligned}$$

- Using the properties of $\gamma^5$ (Q8 on examples sheet)

$$(\gamma^5)^2=1; \quad \gamma^{5\dagger}=\gamma^5; \quad \gamma^5\gamma^{\mu}=-\gamma^{\mu}\gamma^5$$

it is straightforward to show (Q9 on examples sheet)

$$\overline{\psi}_R\gamma^{\mu}\phi_L=0; \quad \overline{\psi}_L\gamma^{\mu}\phi_R=0$$

★ Hence only certain combinations of chiral eigenstates contribute to the interaction. This statement is **ALWAYS** true.

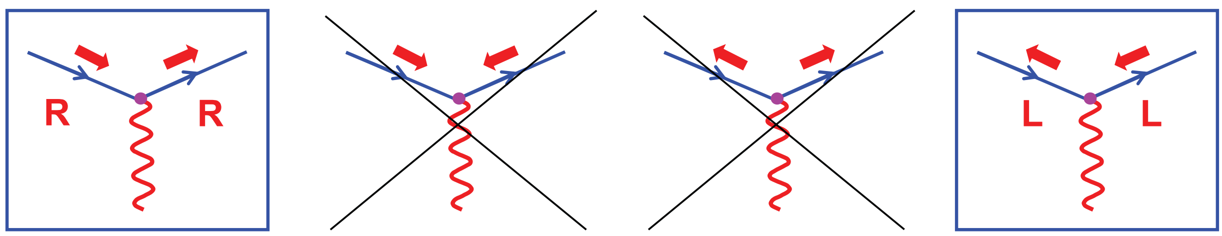
- For $E \gg m$, the chiral and helicity eigenstates are equivalent. This implies that for $E \gg m$ only certain helicity combinations contribute to the QED vertex ! This is why previously we found that for two of the four helicity combinations for the muon current were zero



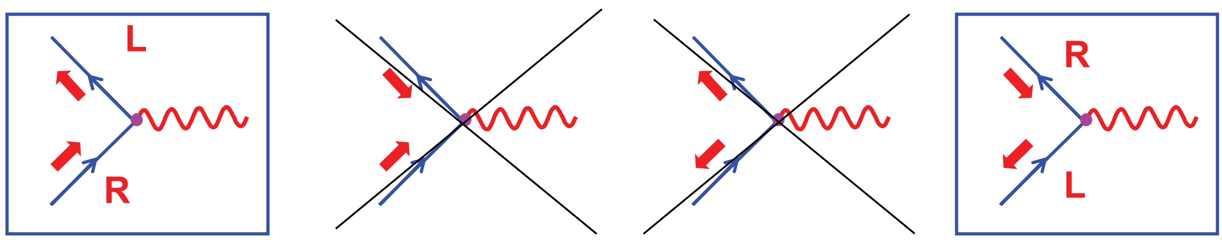
# Allowed QED Helicity Combinations

- In the **ultra-relativistic limit** the **helicity eigenstates ≡ chiral eigenstates**
- In this limit, the only non-zero helicity combinations in QED are:

**Scattering:** "Helicity conservation"




**Annihilation:**



# Summary

★ In the centre-of-mass frame the $e^+e^- \rightarrow \mu^+\mu^-$ differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

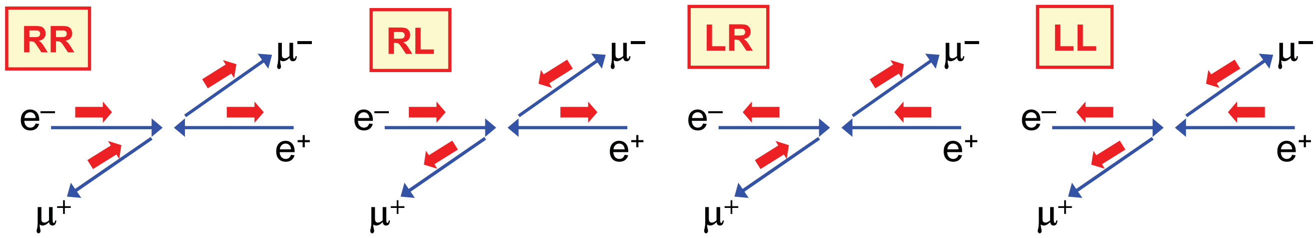
**NOTE:** neglected masses of the muons, i.e. assumed $E \gg m_\mu$

★ In QED only certain combinations of LEFT- and RIGHT-HANDED CHIRAL states give non-zero matrix elements

★ CHIRAL states defined by chiral projection operators

$$P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

★ In limit $E \gg m$ the chiral eigenstates correspond to the HELICITY eigenstates and only certain HELICITY combinations give non-zero matrix elements

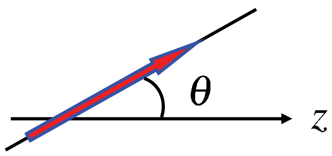





# Appendix : Spin 1 Rotation Matrices

• Consider the spin-1 state with spin +1 along the axis defined by unit vector

$$\vec{n} = (\sin\theta, 0, \cos\theta)$$




• Spin state is an eigenstate of $\vec{n}.\vec{S}$ with eigenvalue +1

$$(\vec{n}.\vec{S})|\psi\rangle = +1|\psi\rangle \qquad \textbf{(A1)}$$

• Express in terms of linear combination of spin 1 states which are eigenstates of $S_z$

$$|\psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

with
$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

• (A1) becomes

$$(\sin\theta S_x + \cos\theta S_z)(\alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle) = \alpha|1,1\rangle + \beta|1,0\rangle\gamma|1,-1\rangle \qquad \textbf{(A2)}$$

• Write $S_x$ in terms of ladder operators $S_x = \frac{1}{2}(S_+ + S_-)$

where
$$S_+|1,1\rangle = 0 \qquad S_+|1,0\rangle = \sqrt{2}|1,1\rangle \qquad S_+|1,-1\rangle = \sqrt{2}|1,0\rangle$$
$$S_-|1,1\rangle = \sqrt{2}|1,0\rangle \qquad S_-|1,0\rangle = \sqrt{2}|1,-1\rangle \qquad S_-|1,-1\rangle = 0$$

- from which we find $S_x|1,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$

$$S_x|1,0\rangle = \tfrac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$

$$S_x|1,-1\rangle = \tfrac{1}{\sqrt{2}}|1,0\rangle$$

- (A2) becomes

$$\sin\theta\left[\frac{\alpha}{\sqrt{2}}|1,0\rangle + \frac{\beta}{\sqrt{2}}|1,-1\rangle + \frac{\beta}{\sqrt{2}}|1,1\rangle + \frac{\gamma}{\sqrt{2}}|1,0\rangle\right] +$$

$$\alpha\cos\theta|1,1\rangle - \gamma\cos\theta|1,-1\rangle = \alpha|1,1\rangle + \beta|1,0\rangle\gamma|1,-1\rangle$$

- which gives

$$\left.\begin{aligned} \beta\frac{\sin\theta}{\sqrt{2}} + \alpha\cos\theta &= \alpha \\ (\alpha+\gamma)\frac{\sin\theta}{\sqrt{2}} &= \beta \\ \beta\frac{\sin\theta}{\sqrt{2}} - \gamma\cos\theta &= \gamma \end{aligned}\right\}$$

- using $\alpha^2+\beta^2+\gamma^2 = 1$ the above equations yield

$$\alpha = \tfrac{1}{\sqrt{2}}(1+\cos\theta) \qquad \beta = \tfrac{1}{\sqrt{2}}\sin\theta \qquad \gamma = \tfrac{1}{\sqrt{2}}(1-\cos\theta)$$

- hence

$$\psi = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$



- The coefficients $\alpha,\beta,\gamma$ are examples of what are known as quantum mechanical rotation matrices. The express how angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction

$$\boxed{d^{j}_{m',m}(\theta)}$$

- For spin-1 $(j=1)$ we have just shown that

$$d^1_{1,1}(\theta) = \tfrac{1}{2}(1+\cos\theta) \qquad d^1_{0,1}(\theta) = \tfrac{1}{\sqrt{2}}\sin\theta \qquad d^1_{-1,1}(\theta) = \tfrac{1}{2}(1-\cos\theta)$$

- For spin-1/2 it is straightforward to show

$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \qquad d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\frac{\theta}{2}$$