Particle Physics

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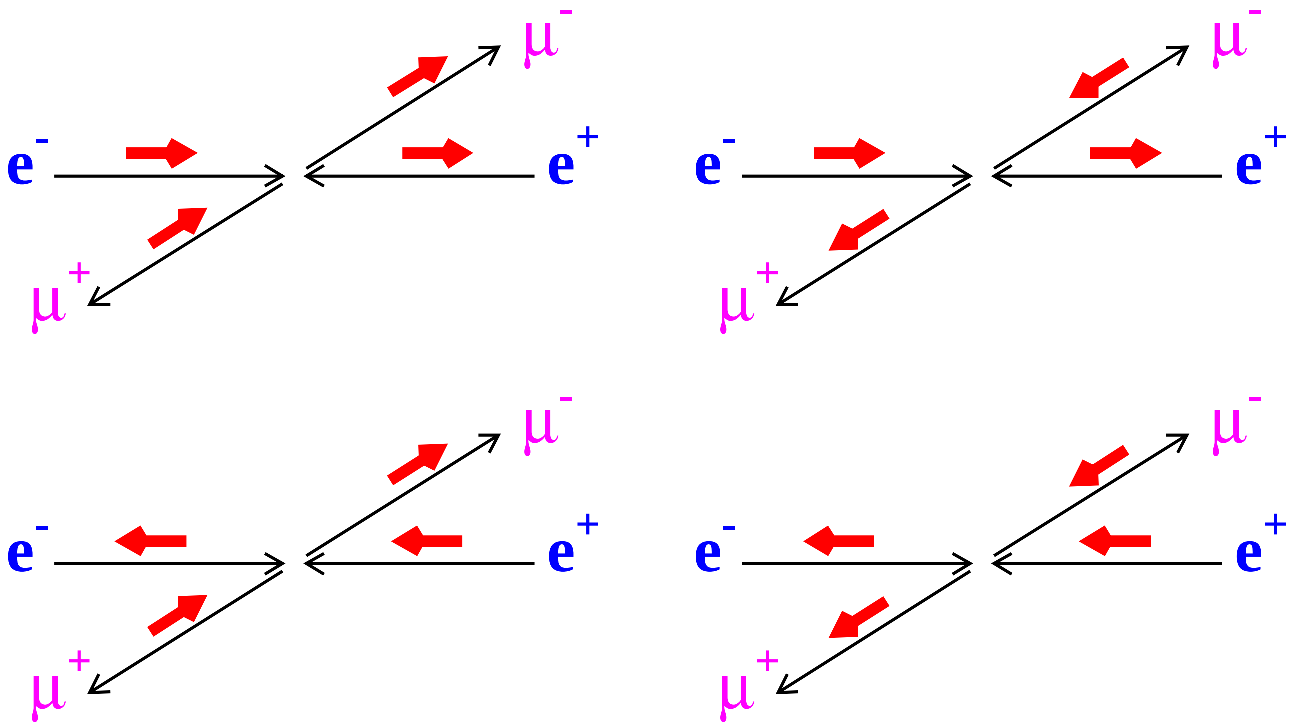
$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$



Part II, Lent Term 2004

HANDOUT V

Spin, Helicity and the Dirac Equation

★ Upto this point we have taken a hands-off approach to “spin”.

★ Scattering cross sections calculated for spin-less particles

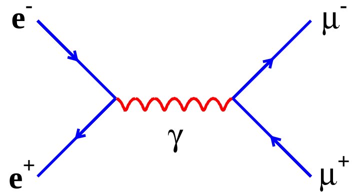
★ To understand the WEAK interaction need to understand SPIN

★ Need a relativistic theory of quantum mechanics that includes spin

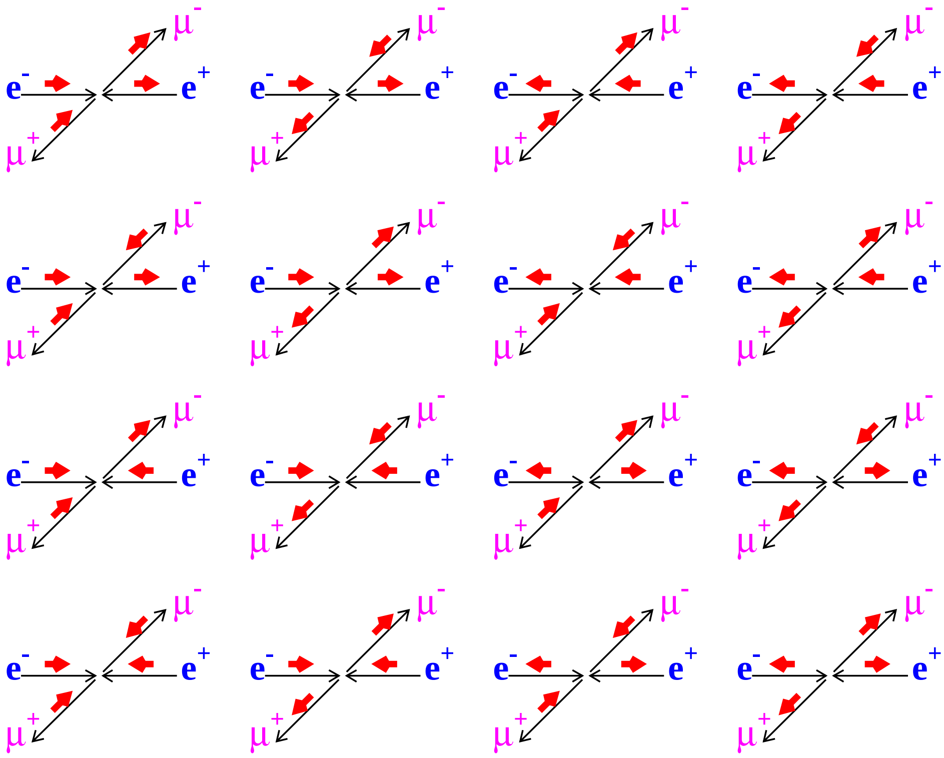
★ $\Rightarrow$ The DIRAC EQUATION

★ SPIN complicates things....

The process



represents the sum over all possible spin states



$$M \rightarrow \sum_i M_i$$

$$|M|^2 \rightarrow |\sum_i M_i|^2 = \sum_i |M_i|^2$$

since ORTHOGONAL SPIN states.

$$\sigma = \frac{1}{4}\left[2\pi \sum_i |M_i|^2 \rho(E_f)\right]$$

Cross-section : sum over all spin assignments,

averaged over initial spin states.

The Klein-Gordon Equation Revisited

Schrödinger Equation for a free particle can be written as

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2\mathrm{m}}\nabla^2\psi$$

Derivatives : 1st order in time and 2nd order in space coordinates $\Rightarrow$ not Lorentz invariant

From Special Relativity:

$$E^2 = p^2 + m^2$$

from Quantum Mechanics:

$$\hat{\mathrm{E}} = i\,\frac{\partial}{\partial \mathrm{t}} \quad , \quad \hat{\mathrm{p}} = -i\nabla$$

Combine to give the Klein-Gordon Equation:

$$\frac{\partial^2\psi}{\partial t^2} = (\nabla^2 - m^2)\psi$$

Second order in both space and time derivatives - by construction Lorentz invariant.

- ★ Negative energy solutions
  $\Rightarrow$ anti-particles
- ★ BUT negative energy solutions also give negative particle densities !?
  $\psi^*\psi < 0$

Try another approach.....

Weyl Equations

(The massless version of the Dirac Equation)

Klein-Gordon Eqn. for massless particles:

$$(\frac{\partial^2\psi}{\partial t^2} - \nabla^2)\psi = 0$$
$$i.e. \; (\hat{E}^2 - \hat{p}^2)\psi = 0$$

Try to factorize 2nd Order KG equation → equation linear in ∇ AND $\frac{\partial}{\partial t}$:

$$(\frac{\partial}{\partial t} - \tilde{\sigma}.\nabla)(\frac{\partial}{\partial t} + \tilde{\sigma}.\nabla)\psi = 0$$

with as yet undetermined constants $\tilde{\sigma}$

Gives the two decoupled WEYL Equations

$$(\sigma_x\frac{\partial}{\partial x} + \sigma_y\frac{\partial}{\partial y} + \sigma_z\frac{\partial}{\partial z})\psi = +\frac{\partial\psi}{\partial t}$$
$$(\sigma_x\frac{\partial}{\partial x} + \sigma_y\frac{\partial}{\partial y} + \sigma_z\frac{\partial}{\partial z})\psi = -\frac{\partial\psi}{\partial t}$$

both linear in space and time derivatives.

BUT must satisfy the Klein-Gordon Equation

***i.e.* in operator form**

$$(\hat{E} - \tilde{\sigma}.\hat{p})(\hat{E} + \tilde{\sigma}.\hat{p})\psi = 0$$

must satisfy

$$(\hat{E}^2 - \hat{p}^2)\psi = 0$$

Weyl equations give:

$$\begin{aligned}(\hat{E}^2 &- \sigma_x\hat{p}_x\sigma_x\hat{p}_x - \sigma_y\hat{p}_y\sigma_y\hat{p}_y - \sigma_z\hat{p}_z\sigma_z\hat{p}_z\\ &- \sigma_x\hat{p}_x\sigma_y\hat{p}_y - \sigma_y\hat{p}_y\sigma_x\hat{p}_x\\ &- \sigma_y\hat{p}_y\sigma_z\hat{p}_z - \sigma_z\hat{p}_z\sigma_y\hat{p}_y\\ &- \sigma_z\hat{p}_z\sigma_x\hat{p}_x - \sigma_x\hat{p}_x\sigma_z\hat{p}_z)\psi = 0\end{aligned}$$

Therefore in order to recover the KG equation:

$$(\hat{E}^2 - \hat{p}_x\hat{p}_x - \hat{p}_y\hat{p}_y - \hat{p}_z\hat{p}_z) = 0,$$

require:

$$\begin{aligned}{\sigma_x}^2 = {\sigma_y}^2 = {\sigma_z}^2 &= 1\\ (\sigma_x\sigma_y + \sigma_y\sigma_x) &= 0 \ etc.\end{aligned}$$

$\therefore \sigma_x, \sigma_y, \sigma_z$ **ANTI-COMMUTE**

The simplest choice for σ are the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence solutions to the Klein-Gordon equation

$$(\hat{E} - \tilde{\sigma}.\hat{p})(\hat{E} + \tilde{\sigma}.\hat{p})\psi = 0$$

are given by the Weyl Equations:

$$\begin{aligned}(\hat{E} - \tilde{\sigma}.\hat{p})\phi = 0\\ (\hat{E} + \tilde{\sigma}.\hat{p})\chi = 0\end{aligned}$$

Since σ_i are 2×2 matrices, need 2 component wave-functions - WEYL SPINORS.

$$\phi = N\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} e^{-i(Et - \tilde{p}.\tilde{r})}$$

The wave-function is forced to have a new degree of freedom - the spin of the fermion.

Consider the FIRST Weyl Equation

$$
\begin{aligned}
(\hat{\mathrm{E}} - \tilde{\sigma}.\hat{\mathrm{p}})\phi &= 0 \\
\left(\frac{\partial}{\partial \mathrm{t}} + \sigma_x \frac{\partial}{\partial \mathrm{x}} + \sigma_y \frac{\partial}{\partial \mathrm{y}} + \sigma_z \frac{\partial}{\partial \mathrm{z}}\right)\phi &= 0
\end{aligned}
$$

For a plane wave solution:

$$
\phi = N \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} e^{-i(\mathrm{Et} - \tilde{\mathrm{p}}.\tilde{\mathrm{r}})}
$$

the first Weyl Equation gives

$$
\begin{aligned}
(E - \sigma_x \mathrm{p}_x - \sigma_y \mathrm{p}_y - \sigma_z \mathrm{p}_z)\phi &= 0 \\
(E - \tilde{\sigma}.\tilde{\mathrm{p}})\phi &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\sigma}.\tilde{\mathrm{p}} &= \sigma_x \mathrm{p}_x + \sigma_y \mathrm{p}_y + \sigma_z \mathrm{p}_z \\
&= \begin{pmatrix} \mathrm{p}_z & \mathrm{p}_x - i\mathrm{p}_y \\ \mathrm{p}_x + i\mathrm{p}_y & -\mathrm{p}_z \end{pmatrix}
\end{aligned}
$$

Hence for the first WEYL equation, the SPINOR solutions of $(E - \tilde{\sigma}.\tilde{\mathrm{p}})\phi = 0$ are given the coupled equations:

$$
\Rightarrow \left.\begin{aligned}
\mathrm{p}_z\phi_1 + (\mathrm{p}_x - i\mathrm{p}_y)\phi_2 &= E\phi_1 \\
(\mathrm{p}_x + i\mathrm{p}_y)\phi_1 - \mathrm{p}_z\phi_2 &= E\phi_2
\end{aligned}\right\}
$$

★ Choose $\tilde{\mathrm{p}}$ along z axis, i.e. $\mathrm{p}_z = |p|$:

$$(E - |p|)\phi_1 = 0$$
$$(E + |p|)\phi_2 = 0$$

There are two solutions $\phi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ and $\phi_- = \begin{pmatrix}0\\1\end{pmatrix}$.

★ The first solution, $\phi_+ = \begin{pmatrix}1\\0\end{pmatrix}$, requires $E = +|p|$, i.e. a positive energy (particle) solution. Similarly, the second $\phi_- = \begin{pmatrix}0\\1\end{pmatrix}$, requires $E = -|p|$, i.e. a negative energy (anti-particle) solution.

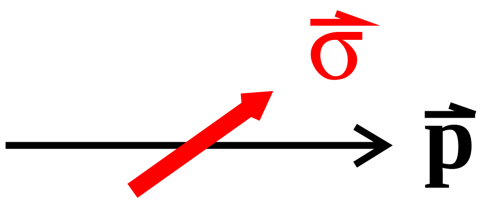
★ Back to the FIRST WEYL equation

$$\begin{aligned} (E - \tilde{\sigma}.\hat{\mathrm{p}})\phi &= 0 \\ \tilde{\sigma}.\hat{\mathrm{p}}\phi &= E\phi \\ \frac{\tilde{\sigma}.\hat{\mathrm{p}}}{|p|}\phi &= \frac{E}{|p|}\phi = \begin{cases} +1 & E > 0 \\ -1 & E < 0 \end{cases} \end{aligned}$$

★ The solutions of the WEYL equations are Eigenstates of the HELICITY operator.

$$\hat{H} = \frac{\tilde{\sigma}.\hat{\mathrm{p}}}{|p|}$$

with Eigenvalues $+1$ and -1 respectively.



HELICITY is the projection of a particle's SPIN onto its flight direction.

(Recall WEYL equations applicable for massless particles)

★ Interpret the two solutions of the FIRST WEYL equation as a RIGHT-HANDED $H = +1$ particle and a LEFT-HANDED anti-particle $H = -1$.

E > 0, H = +1 | E < 0, H = -1

RH particle | LH anti-particle

★ The SECOND WEYL equation:

$$(\hat{\mathrm{E}} + \tilde{\sigma}.\hat{\mathrm{p}})\chi = 0$$

has LEFT-HANDED particle and RIGHT-HANDED anti-particle solutions.

E < 0, H = +1 | E > 0, H = -1

RH anti-particle | LH particle

<u>SUMMARY:</u>

★ By factorizing the Klein-Gordon equation into a form linear in the derivatives $\Rightarrow$ force particles to have a non-commuting degree of freedom, SPIN !

★ Still obtain anti-particle solutions

★ Probability densities always positive

★ ‘Natural’ states are the Helicity Eigenstates

★ Weyl Equations are the ultra-relativistic (massless) limit of the Dirac Equation

Spin in the Fundamental Interactions

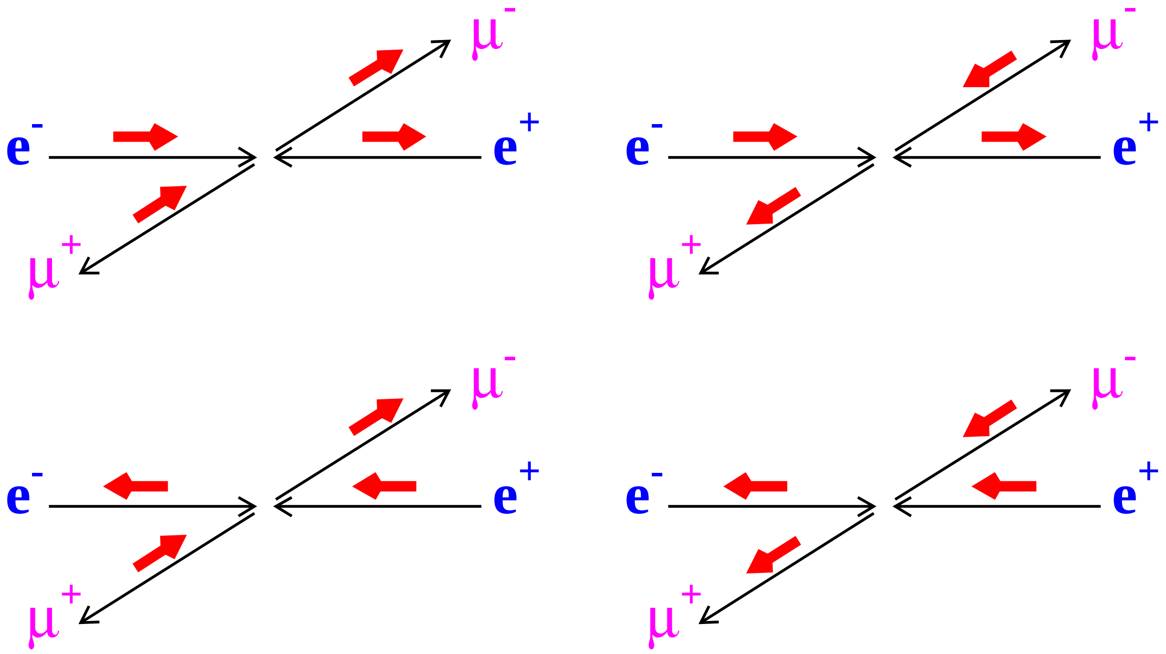
★ The ELECTROMAGNETIC, STRONG, and WEAK interactions are all mediated by VECTOR (spin-1) fields. In the massless limit, the fundamental fermion states are eigenstates of the helicity operator. HANDEDNESS

★ (CHIRALITY) plays a central rôle in the interactions between the field bosons and the fermions; the only allowed couplings are:

LH particle	to	LH particle
RH particle	to	RH particle
LH anti-particle	to	LH anti-particle
RH anti-particle	to	RH anti-particle
LH particle	to	RH anti-particle
RH particle	to	LH anti-particle

<u>EXAMPLE</u> $e^+e^- \rightarrow \mu^+\mu^-$

Of the 16 possibilities ONLY the following SPIN combinations contribute to the cross-section



All other SPIN combinations give zero $|M_i|^2$

Solutions of the Weyl Equations

Consider the general case of a particle with travelling at an angle θ with respect to the z-axis

$$p_z = |p| \cos\theta$$
$$p_x = |p| \sin\theta$$

WEYL 1

$$(\hat{E} - \hat{\sigma}.\hat{p})\phi = 0$$
$$(\hat{\sigma}.\hat{p})\phi = \hat{E}\phi$$

$$\Rightarrow \left.\begin{aligned} p_z\phi_1 + p_x\phi_2 &= E\phi_1 \\ p_x\phi_1 - p_z\phi_2 &= E\phi_2 \end{aligned}\right\}$$

For the positive energy solution $E = +|p|$:

$$\Rightarrow \left.\begin{aligned} \phi_1\cos\theta + \phi_2\sin\theta &= \phi_1 \\ \phi_1\sin\theta - \phi_2\cos\theta &= \phi_2 \end{aligned}\right\}$$

$$\Rightarrow \quad \phi_1 = \phi_2 \frac{\sin\theta}{(1-\cos\theta)}$$

Therefore

$$\frac{\phi_1}{\phi_2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{(1-\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2})}$$

$$\frac{\phi_1}{\phi_2} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

Normalizing such that ${\phi_1}^2 + {\phi_2}^2 = 1$ gives:

$$\phi_1 = \cos\frac{\theta}{2}$$
$$\phi_2 = \sin\frac{\theta}{2}$$

★ So the positive energy solution to the first Weyl equation (RH particle) gives

$$\phi_{\mathrm{RH}} \quad = \quad N\begin{pmatrix} +\cos\frac{\theta}{2} \\ +\sin\frac{\theta}{2} \end{pmatrix} e^{-i(\mathbf{Et}-\tilde{\mathbf{p}}.\tilde{\mathbf{r}})} \quad \text{RH fermion}$$

This is still a Eigenvalue of the helicity operator with $(H = +1)$ *i.e.* a RH particle but now referred to an external axis.

The positive energy solution to the SECOND Weyl equation (LH particle) gives

$$\chi_{\mathrm{LH}} \quad = \quad N\begin{pmatrix} -\sin\frac{\theta}{2} \\ +\cos\frac{\theta}{2} \end{pmatrix} e^{-i(\mathbf{Et}-\tilde{\mathbf{p}}.\tilde{\mathbf{r}})} \quad \text{LH fermion}$$

Not much more than the Quantum Mechanical rotation properties of spin-$\frac{1}{2}$.

★ The spin part of a RH particle/anti-particle wave-function can be written

$$\psi_R(\theta) = \begin{pmatrix} +\cos\frac{\theta}{2} \\ +\sin\frac{\theta}{2} \end{pmatrix} \quad = \quad \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

★ Similarly for a LH particle/anti-particle

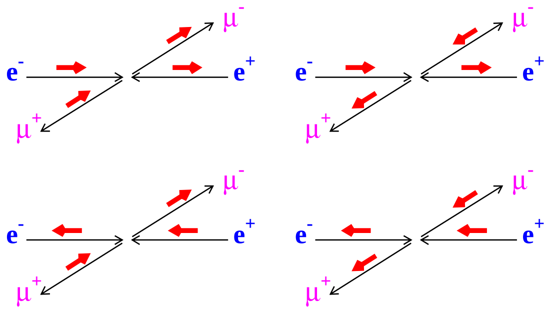
$$\psi_L(\theta) = \begin{pmatrix} -\sin\frac{\theta}{2} \\ +\cos\frac{\theta}{2} \end{pmatrix} \quad = \quad -\sin\frac{\theta}{2}\uparrow + \cos\frac{\theta}{2}\downarrow$$

For particles/anti-particles with momentum $-\tilde{\mathrm{p}}(\theta)$, i.e. an angle $\theta + \pi$ to the z-axis:

$$\begin{aligned} \psi_R(\theta+\pi) &= -\sin\frac{\theta}{2}\uparrow + \cos\frac{\theta}{2}\downarrow \\ \psi_L(\theta+\pi) &= -\cos\frac{\theta}{2}\uparrow - \sin\frac{\theta}{2}\downarrow \end{aligned}$$

Application to $e^+e^- \to \mu^+\mu^-$

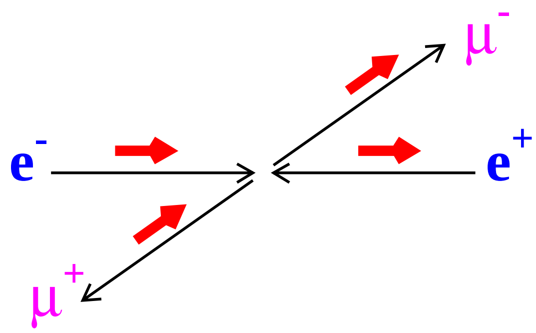
★ Four helicity combinations contribute to the cross-section.



★ Consider the first diagram

$$e^-_R e^+_L \quad \to \quad \mu^-_R \mu^+_L$$

With the e^- direction defining the z axis:



★ The spin parts of the wave-functions :

$$\psi_{e^-_R e^+_L} = \psi_R(0)\psi_L(\pi) = \uparrow\uparrow$$

$$\psi_{\mu^-_R \mu^+_L} = \psi_R(\theta)\psi_L(\theta+\pi)$$

$$= -\cos^2\frac{\theta}{2}\uparrow\uparrow - \cos\frac{\theta}{2}\sin\frac{\theta}{2}(\uparrow\downarrow+\downarrow\uparrow) - \sin^2\frac{\theta}{2}\downarrow\downarrow$$

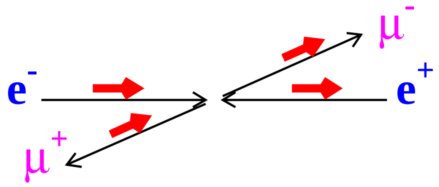
★ Giving the contribution the matrix element:

$$|M_1|^2 = |\langle\psi_{\mu^-_R\mu^+_L}|\frac{e^2}{q^2}|\psi_{e^-_R e^+_L}\rangle|^2$$

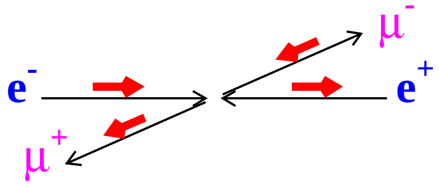
$$= \frac{e^4}{q^4}|\cos^2\frac{\theta}{2}|^2$$

$$= \frac{e^4}{q^4}\left(\frac{1}{2}\right)^2(1+\cos\theta)^2$$

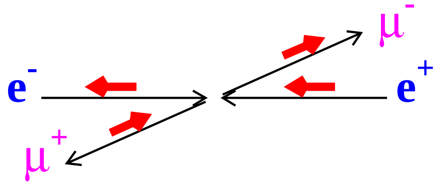
★ **Perform same calculation for the four allowed helicity combinations**



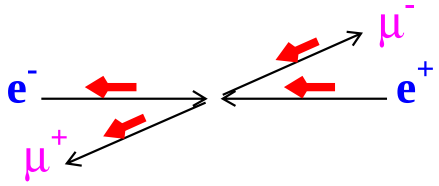
$$\mathbf{M_1\ \alpha\ \cos^2(\theta/2)}$$
$$\rightarrow \frac{1}{2}(1+\cos\theta)$$



$$\mathbf{M_2\ \alpha\ \sin^2(\theta/2)}$$
$$\rightarrow \frac{1}{2}(1-\cos\theta)$$



$$\mathbf{M_3\ \alpha\ \sin^2(\theta/2)}$$
$$\rightarrow \frac{1}{2}(1-\cos\theta)$$



$$\mathbf{M_4\ \alpha\ \cos^2(\theta/2)}$$
$$\rightarrow \frac{1}{2}(1+\cos\theta)$$

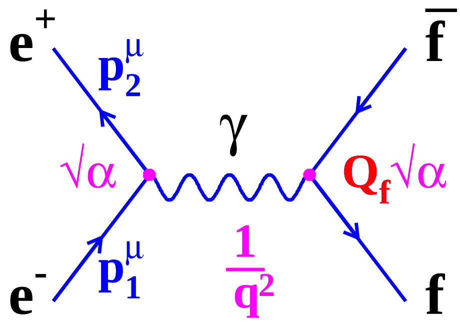
★ **For unpolarized electron/positron beams : each of the above process contributes equally. Therefore SUM over all matrix elements and AVERAGE over initial spin states. Giving the total Matrix Element (remember spin-states are orthogonal so sum squared matrix elements) :**

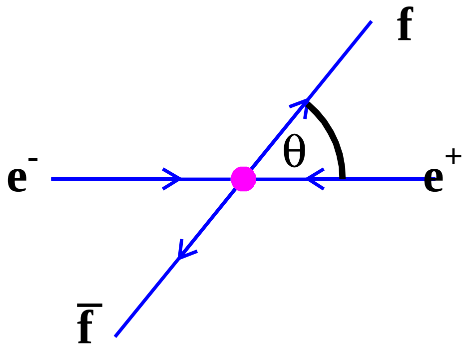
$$|M|^2 = \frac{1}{4}\{|M_1|^2 + |M_2|^2 + |M_3|^2 + |M_4|^2\}$$

$$|M|^2 = \frac{1}{4}\frac{e^4}{q^4}\{\tfrac{1}{4}(1+\cos\theta)^2 + \tfrac{1}{4}(1-\cos\theta)^2 + \tfrac{1}{4}(1-\cos\theta)^2 + \tfrac{1}{4}(1+\cos\theta)^2\}$$

$$|M|^2 = \frac{e^4}{4q^4}(1+\cos^2\theta)$$

★ **Nothing more than the QM properties of a SPIN-1 particle decaying to two SPIN-$\frac{1}{2}$ particles**

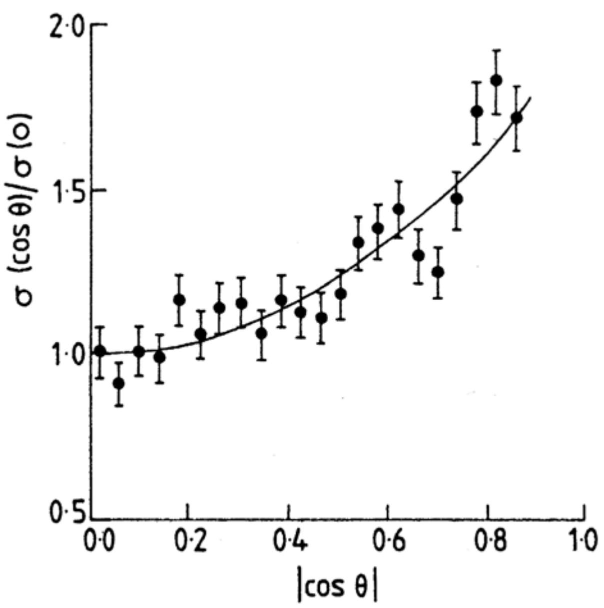




Electron/Positron beams along z-axis $(q^2 = 4E^2 = s)$

Using the spin-averaged matrix element

$$
\begin{aligned}
|M|^2 &= \frac{e^4}{4q^4}(1+\cos^2\theta) \\
|M|^2 &= \frac{(4\pi\alpha)^2}{4s^2}(1+\cos^2\theta) \\
\frac{d\sigma}{d\Omega} &= 2\pi|M|^2\frac{E^2}{(2\pi)^3} \\
&= 2\pi\frac{(4\pi\alpha)^2}{4s^2}(1+\cos^2\theta)\frac{s}{4}\frac{1}{(2\pi)^3} \\
&= \frac{\alpha^2 Q_f^2}{4s}(1+\cos^2\theta)
\end{aligned}
$$



$\frac{d\sigma}{d|\cos\theta|}$ **for $e^+e^- \rightarrow q\bar{q}$. The angle θ is determined from the measured directions of the jets. $|\cos\theta|$ is plotted since it is not possible to uniquely identify which jet corresponds to the quark and which corresponds to the anti-quark. The curve shows the expected $(1+\cos^2\theta)$ distribution.**

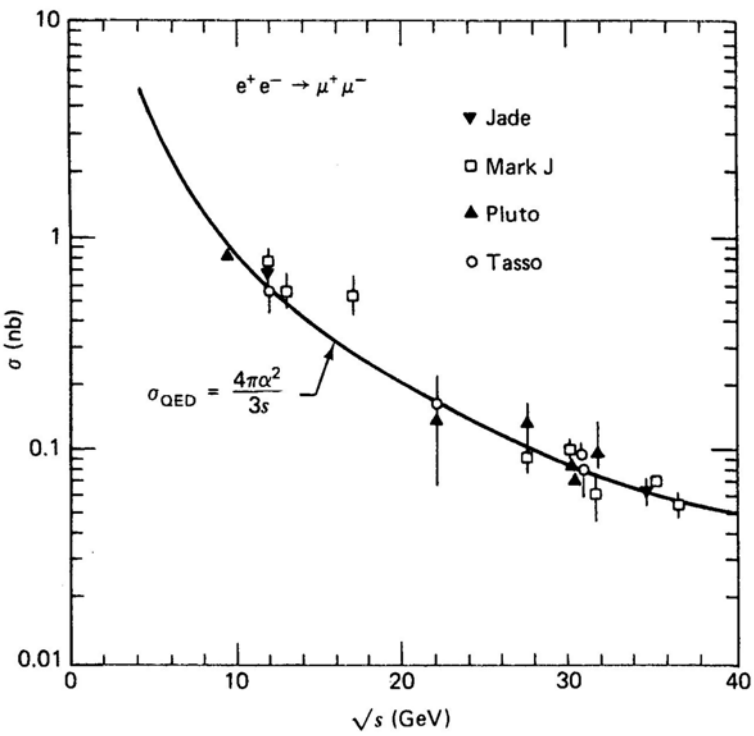
QUARKS are SPIN-$\frac{1}{2}$

(yet again)

Total cross section for $e^+e^- \to f\bar{f}$

$$\begin{aligned}\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \\ &= \int_0^{2\pi}\int_0^{\pi} \frac{\alpha^2 Q_f^2}{4s}(1+\cos^2\theta)\sin\theta\, d\theta\, d\phi \\ &= \frac{\pi\alpha^2 Q_f^2}{2s}\int_{-1}^{+1}(1+y^2)dy \qquad (y=\cos\theta) \\ &= \frac{4\pi\alpha^2 Q_f^2}{3s}\end{aligned}$$

$$\boxed{\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}}$$



$\sigma(e^+e^- \to \mu^+\mu^-)$ **for e^+e^- collider data at centre-of-mass energies 8-36 GeV**

★ **This is the complete lowest order calculation of the $e^+e^- \to \mu^+\mu^-$ cross-section (in the limit of massless fermions).**

The Dirac Equation

NON-EXAMINABLE

WEYL Equations describe massless SPIN-$\frac{1}{2}$ particles. But all known fermions are MASSIVE. Again start from the KG equation.

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial t^2} &= (\nabla^2 - m^2)\psi \\
\hat{\mathrm{H}}^2\psi &= (\hat{\mathrm{p}}^2 + m^2)\psi
\end{aligned}
$$

Write down equation LINEAR in space and time derivatives

$$\hat{\mathrm{H}}\psi = (\vec{\alpha}.\hat{\mathrm{p}} + \beta m)\psi$$

and require it to be a solution of the KG equation:

$$
\begin{aligned}
\hat{\mathrm{H}}\psi &= (\alpha_x.\hat{\mathrm{p}}_x + \alpha_y.\hat{\mathrm{p}}_y + \alpha_z.\hat{\mathrm{p}}_z + \beta.m)\psi \\
\hat{\mathrm{H}}^2\psi &= \alpha_i{}^2.\hat{\mathrm{p}}_x{}^2 + \ldots \\
&\quad +(\alpha_x\alpha_y + \alpha_y\alpha_x)\hat{\mathrm{p}}_x\hat{\mathrm{p}}_y + \ldots \\
&\quad +(\alpha_x\beta + \beta\alpha_x)\hat{\mathrm{p}}_x m + \ldots \\
&\quad +\beta^2 m^2
\end{aligned}
$$

For this to satisfy Klein-Gordon equation:

$$\hat{\mathrm{H}}^2\psi = (\hat{\mathrm{p}}_x{}^2 + \hat{\mathrm{p}}_y{}^2 + \hat{\mathrm{p}}_z{}^2 + m^2)\psi$$

require

$$
\begin{aligned}
\alpha_i{}^2 = \beta^2 &= 1 \quad (i = x, y, z) \\
\alpha_i\alpha_j + \alpha_j\alpha_i &= 0 \quad (i \neq j) \\
\alpha_i\beta + \beta\alpha_i &= 0
\end{aligned}
$$

Now require 4 anti-commuting matrices.

The Dirac Equation:

$$\hat{H}\psi = (\vec{\alpha}.\hat{p} + \beta m)\psi$$

Can be written in a slightly different form

$$\begin{aligned} i\,\frac{\partial\psi}{\partial t} &= (-i\vec{\alpha}.\nabla + \beta m)\psi \\ i\beta\,\frac{\partial\psi}{\partial t} &= (-i\beta\vec{\alpha}.\nabla + m)\psi \end{aligned}$$

$$\begin{aligned} (i\beta\,\frac{\partial}{\partial t} + i\beta\vec{\alpha}.\nabla - \beta^2 m)\psi &= 0 \\ (i\gamma^0\,\frac{\partial}{\partial t} + i\gamma^1\,\frac{\partial}{\partial x} + i\gamma^2\,\frac{\partial}{\partial y} + i\gamma^3\,\frac{\partial}{\partial z} - \beta^2 m)\psi &= 0 \end{aligned}$$

with $\gamma^\mu = (\beta, \beta\vec{\alpha})$

Giving

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$$

with

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 &= -1 \\ (\gamma^i\gamma^j - \gamma^j\gamma^i) &= 0 \quad (i \neq j) \end{aligned}$$

Identify the γ^μ as matrices which must satisfy the anti-commutation relations above. The Pauli spin matrices provide only 3 anti-commuting matrices and the lowest dimension matrices satisfying these requirements are 4×4. The γ-matrices are closely related to the 2×2 Pauli spin matrices.

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}$$

also define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solutions to the Dirac Equation are written as four-component Dirac SPINORs

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

NOTE: this is not the only possible representation of the γ matrices - just the most commonly used

Rest Frame Solutions of the Dirac Equation

Dirac Equation: $$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$(i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial x} + i\gamma^2 \frac{\partial}{\partial y} + i\gamma^3 \frac{\partial}{\partial z} - m)\psi = 0$$

Consider a particle at REST: $p_x = i\frac{\partial}{\partial x}\psi = 0$, **etc.**

Dirac Equation becomes:

$$(i\gamma^0 \frac{\partial}{\partial t} - m)\psi = 0$$

$$i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \partial\psi_1/\partial t \\ \partial\psi_2/\partial t \\ \partial\psi_3/\partial t \\ \partial\psi_4/\partial t \end{pmatrix} = m\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$i\frac{\partial\psi_1}{\partial t} = m\psi_1, \quad i\frac{\partial\psi_2}{\partial t} = m\psi_2$$

Giving two orthogonal $\underline{E = +m}$ solutions:

$$u_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \quad u_2(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}$$

i.e. positive energy spin-up and spin-down PARTICLES

The two other equations

$$i\frac{\partial\psi_3}{\partial t} = -m\psi_3, \quad i\frac{\partial\psi_4}{\partial t} = -m\psi_4$$

give two orthogonal $\underline{E = -m}$ solutions:

$$u_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}, \quad u_4(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}$$

i.e. $-$ve energy spin-up and spin-down ANTI-PARTICLES

The DIRAC equation

- ★ gives PARTICLE/ANTI-PARTICLE solutions
- ★ requires the particles/anti-particles to have an additional degree of freedom (SPIN) !
- ★ in the massless limit, the DIRAC equation reduces to the two uncoupled WEYL equations
- ★ In general the Dirac Equation gives FOUR simultaneous equations for the components of the SPINOR.

e.g. more general solutions

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{(E+m)} \\ \frac{(p_x+ip_y)}{(E+m)} \end{pmatrix}, \quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{(p_x-ip_y)}{(E+m)} \\ \frac{-p_z}{(E+m)} \end{pmatrix}$$

For (u_1, u_2) $\quad E = \sqrt{p^2+m^2}$

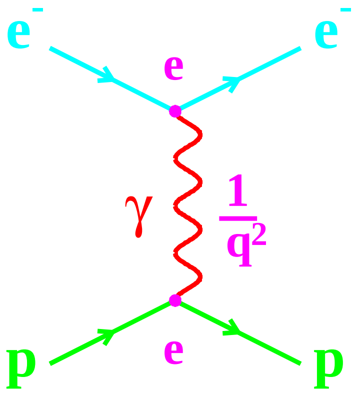
$$u_3 = N\begin{pmatrix} \frac{p_z}{(E-m)} \\ \frac{(p_x+ip_y)}{(E-m)} \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = N\begin{pmatrix} \frac{(p_x-ip_y)}{(E-m)} \\ -\frac{p_z}{(E-m)} \\ 0 \\ 1 \end{pmatrix}$$

For (u_3, u_4) $\quad E = -\sqrt{p^2+m^2}$

The DIRAC equation lives in the realm of PART III Particle Physics.

Lorentz Structure of Interactions

NON-EXAMINABLE



Electron Current

Propagator

Proton Current

Matrix element M factorises into 3 terms :

$$-iM = \langle \overline{u}_e | ie\gamma^\mu | u_e \rangle \quad \text{Electron Current}$$
$$\times \frac{-ig^{\mu\nu}}{q^2} \quad \text{Photon Propagator}$$
$$\times \langle \overline{u}_p | ie\gamma^\nu | u_p \rangle \quad \text{Proton Current}$$

- ★ Fermions are 4-component SPINORS.
- ★ $\therefore$ interaction enters as 4×4 matrices.
- ★ Lorentz invariance allows only five possible forms for the interaction: SCALAR $\overline{u}u$, PSEUDO-SCALAR $\overline{u}\gamma^5 u$, VECTOR $\overline{u}\gamma^\mu u$, AXIAL-VECTOR $\overline{u}\gamma^\mu\gamma^5 u$, TENSOR $\overline{u}\sigma^{\mu\nu}u$
- ★ Electro-magnetic and Strong forces are VECTOR interactions - which determines the HELICITY structure. Treats helicity states symmetrically $\Rightarrow$ PARITY CONSERVATION
- ★ The WEAK interaction has a different form: (V-A) *i.e.* $\gamma^\mu(1-\gamma^5)$. Projects out a single helicity combination : $\Rightarrow$ PARITY VIOLATION

The WEAK interaction is the subject of next lecture