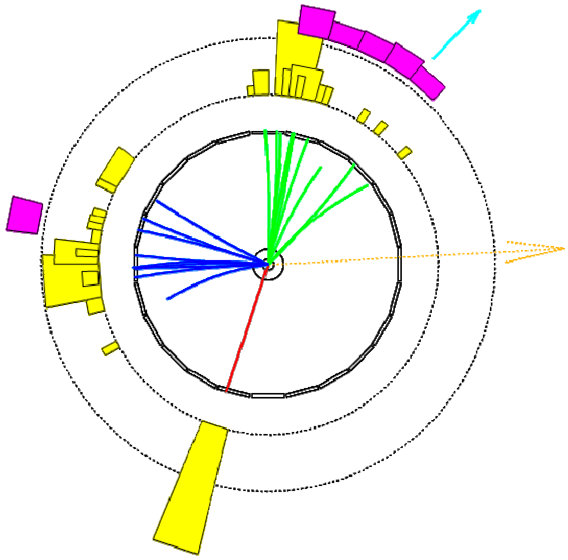
Particle Physics

Michaelmas Term 2009
Prof Mark Thomson



Handout 13 : Electroweak Unification and the W and Z Bosons

Boson Polarization States

★ In this handout we are going to consider the decays of W and Z bosons, for this we will need to consider the polarization. Here simply quote results although the justification is given in Appendices A and B

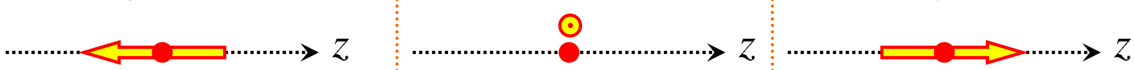
★ A real (i.e. not virtual) massless spin-1 boson can exist in two transverse polarization states, a massive spin-1 boson also can be longitudinally polarized

★ Boson wave-functions are written in terms of the polarization four-vector ε^μ

$$\boxed{B^\mu = \varepsilon^\mu e^{-ip.x} = \varepsilon^\mu e^{i(\vec{p}.\vec{x}-Et)}}$$

★ For a spin-1 boson travelling along the z-axis, the polarization four vectors are:

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = \frac{1}{m}(p_z,0,0,E) \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0,1,i,0)$$



$\boxed{S_z=-1}$ $\boxed{S_z=0}$ $\boxed{S_z=+1}$

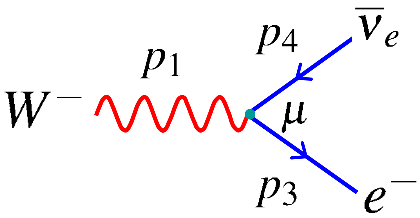
transverse | longitudinal | transverse

Longitudinal polarization isn't present for on-shell massless particles, the photon can exist in two helicity states $h=\pm1$ (LH and RH circularly polarized light)

W-Boson Decay

★ To calculate the W-Boson decay rate first consider $W^- \to e^-\overline{\nu}_e$

★ Want matrix element for :



Incoming W-boson : $\epsilon_\mu(p_1)$
Out-going electron : $\overline{u}(p_3)$
Out-going $\overline{\nu}_e$: $v(p_4)$
Vertex factor : $-i\frac{g_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)$

$$-iM_{fi} = \epsilon_\mu(p_1).\overline{u}(p_3).-i\frac{g_W}{\sqrt{2}}\gamma^\mu\tfrac{1}{2}(1-\gamma^5).v(p_4)$$

Note, no propagator

$$\Rightarrow \quad \boxed{M_{fi} = \frac{g_W}{\sqrt{2}}\epsilon_\mu(p_1)\overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)v(p_4)}$$

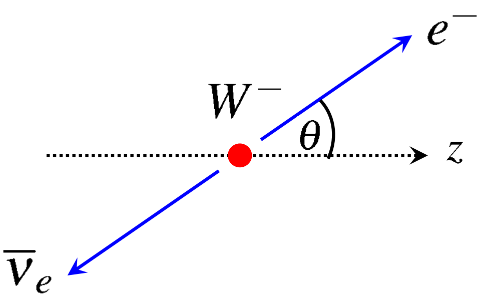
★ This can be written in terms of the four-vector scalar product of the W-boson polarization $\epsilon_\mu(p_1)$ and the weak charged current j^μ

$$\boxed{M_{fi} = \frac{g_W}{\sqrt{2}}\epsilon_\mu(p_1).j^\mu} \quad \text{with} \quad \boxed{j^\mu = \overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)v(p_4)}$$

W-Decay : The Lepton Current

★ First consider the lepton current $j^\mu = \overline{u}(p_3)\gamma^\mu\frac{1}{2}(1-\gamma^5)v(p_4)$

★ Work in Centre-of-Mass frame



$$p_1 = (m_W, 0, 0, 0);$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$
with $E = \dfrac{m_W}{2}$

★ In the ultra-relativistic limit only <u>LH particles</u> and <u>RH anti-particles</u> participate in the weak interaction so

$$j^\mu = \overline{u}(p_3)\gamma^\mu\tfrac{1}{2}(1-\gamma^5)v(p_4) = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4)$$

Note: $\frac{1}{2}(1-\gamma^5)v(p_4) = v_\uparrow(p_4)$ $\qquad$ $\overline{u}(p_3)\gamma^\mu v_\uparrow(p_4) = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4)$

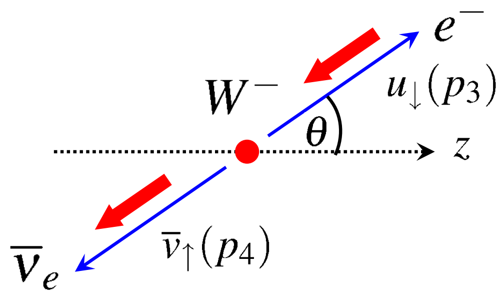
Chiral projection operator, e.g. see p.131 or p.294

"Helicity conservation", e.g. see p.133 or p.295

- We have already calculated the current

$$j^\mu = \bar{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4)$$

when considering $e^+e^- \to \mu^+\mu^-$



- From page 128 we have for $\mu_L^-\mu_R^+$

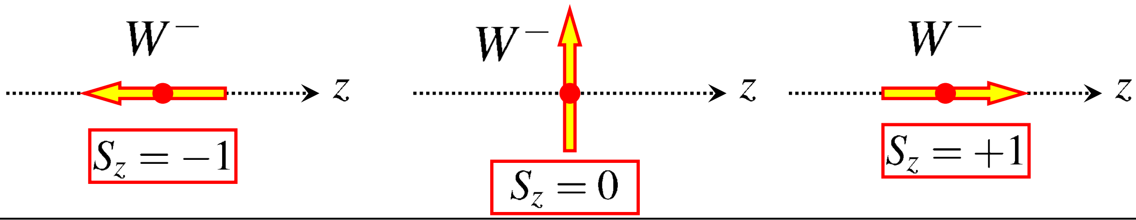
$$\boxed{j^\mu_{\uparrow\downarrow} = 2E(0,-\cos\theta,-i,\sin\theta)}$$

- For the charged current weak Interaction we only have to consider this **single** combination of helicities

$$j^\mu = \bar{u}(p_3)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)v(p_4) = \bar{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4) = 2E(0,-\cos\theta,-i,\sin\theta)$$

and the three possible W-Boson polarization states:

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = \frac{1}{m}(p_z,0,0,E) \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0,1,i,0)$$



★ For a W-boson at rest these become:

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0,1,-i,0); \quad \varepsilon_L = (0,0,0,1) \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

★ Can now calculate the matrix element for the different polarization states

$$M_{fi} = \frac{g_W}{\sqrt{2}}\varepsilon_\mu(p_1)j^\mu \quad \text{with} \quad j^\mu = 2\frac{m_W}{2}(0,-\cos\theta,-i,\sin\theta)$$

Decay at rest : $E_e = E_\nu = m_W/2$

★ giving

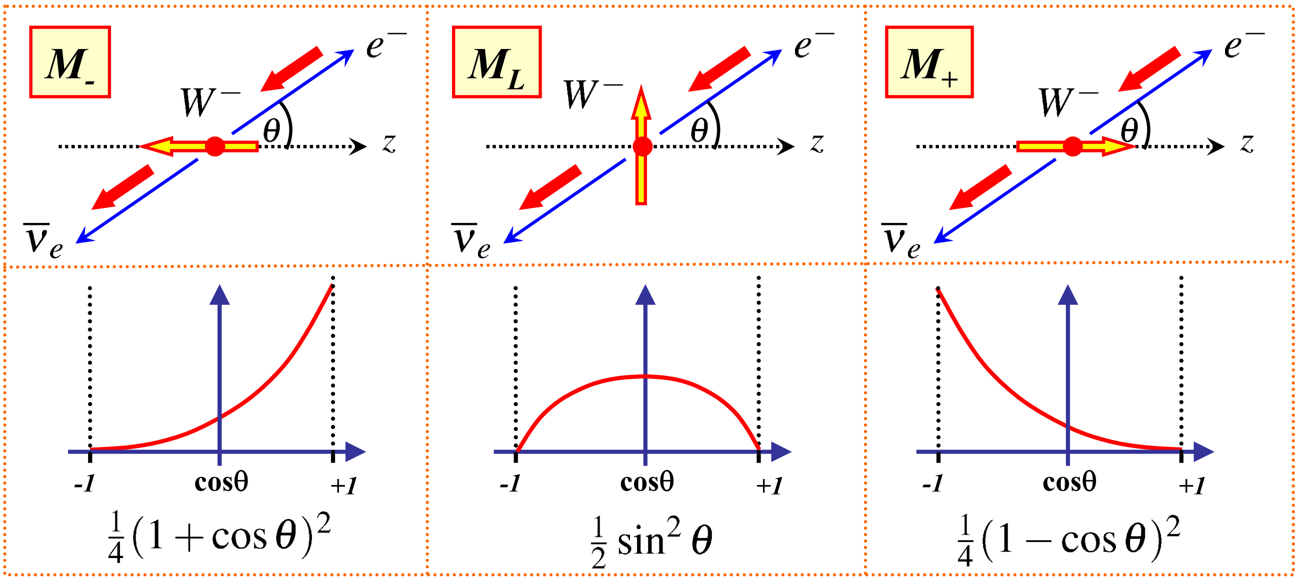
$$\boxed{\varepsilon_-} \quad M_- = \frac{g_W}{\sqrt{2}}\frac{1}{\sqrt{2}}(0,1,-i,0).m_W(0,-\cos\theta,-i,\sin\theta) = \tfrac{1}{2}g_W m_W(1+\cos\theta)$$

$$\boxed{\varepsilon_L} \quad M_L = \frac{g_W}{\sqrt{2}}(0,0,0,1).m_W(0,-\cos\theta,-i,\sin\theta) = -\tfrac{1}{\sqrt{2}}g_W m_W\sin\theta$$

$$\boxed{\varepsilon_+} \quad M_+ = -\frac{g_W}{\sqrt{2}}\frac{1}{\sqrt{2}}(0,1,i,0).m_W(0,-\cos\theta,-i,\sin\theta) = \tfrac{1}{2}g_W m_W(1-\cos\theta)$$

$$\Rightarrow \quad \boxed{\begin{aligned} |M_-|^2 &= g_W^2 m_W^2 \tfrac{1}{4}(1+\cos\theta)^2 \\ |M_L|^2 &= g_W^2 m_W^2 \tfrac{1}{2}\sin^2\theta \\ |M_+|^2 &= g_W^2 m_W^2 \tfrac{1}{4}(1-\cos\theta)^2 \end{aligned}}$$

★ The angular distributions can be understood in terms of the spin of the particles



★ The differential decay rate (see page 26) can be found using:

$$\frac{d\Gamma}{d\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}|M|^2$$

where p* is the C.o.M momentum of the final state particles, here $p^* = \frac{m_W}{2}$

★ Hence for the three different polarisations we obtain:

$$\frac{d\Gamma_-}{d\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1+\cos\theta)^2 \qquad \frac{d\Gamma_L}{d\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{2}\sin^2\theta \qquad \frac{d\Gamma_+}{d\Omega} = \frac{g_W^2 m_w}{64\pi^2}\frac{1}{4}(1-\cos\theta)^2$$

★ Integrating over all angles using

$$\int \tfrac{1}{4}(1\pm\cos\theta)^2 d\phi\, d\cos\theta = \int \tfrac{1}{2}\sin^2\theta\, d\phi\, d\cos\theta = \frac{4\pi}{3}$$

★ Gives

$$\boxed{\Gamma_- = \Gamma_L = \Gamma_+ = \frac{g_W^2 m_W}{48\pi}}$$

★ The total W-decay rate is independent of polarization; this has to be the case as the decay rate cannot depend on the arbitrary definition of the z-axis

★ For a sample of unpolarized W boson each polarization state is equally likely, for the average matrix element sum over all possible matrix elements and average over the three initial polarization states

$$\begin{aligned}\langle |M_{fi}|^2\rangle &= \tfrac{1}{3}(|M_-|^2+|M_L|^2+|M_+|^2)\\ &= \tfrac{1}{3}g_W^2 m_W^2\left[\tfrac{1}{4}(1+\cos\theta)^2+\tfrac{1}{2}\sin^2\theta+\tfrac{1}{4}(1-\cos\theta)^2\right]\\ &= \tfrac{1}{3}g_W^2 m_W^2\end{aligned}$$

★ For a sample of unpolarized W-bosons, the decay is isotropic (as expected)

★For this isotropic decay

$$\frac{d\Gamma}{d\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}\langle |M|^2 \rangle \;\Rightarrow\; \Gamma = \frac{4\pi |p^*|}{32\pi^2 m_W^2}\langle |M|^2 \rangle$$

$$\Rightarrow \boxed{\Gamma(W^- \to e^-\overline{\nu}) = \frac{g_W^2 m_W}{48\pi}}$$

★ The calculation for the other decay modes (neglecting final state particle masses) is same. For quarks need to account for colour and CKM matrix. No decays to top – the top mass (175 GeV) is greater than the W-boson mass (80 GeV)

$$\begin{array}{llllll}
W^- \to e^-\overline{\nu}_e & W^- \to d\overline{u} & \times 3|V_{ud}|^2 & W^- \to d\overline{c} & \times 3|V_{cd}|^2 \\
W^- \to \mu^-\overline{\nu}_\mu & W^- \to s\overline{u} & \times 3|V_{us}|^2 & W^- \to s\overline{c} & \times 3|V_{cs}|^2 \\
W^- \to \tau^-\overline{\nu}_\tau & W^- \to b\overline{u} & \times 3|V_{ub}|^2 & W^- \to b\overline{c} & \times 3|V_{cb}|^2
\end{array}$$

★ Unitarity of CKM matrix gives, e.g. $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$

★ Hence $BR(\mathrm{W} \to qq') = 6BR(\mathrm{W} \to e\nu)$

and thus the total decay rate :

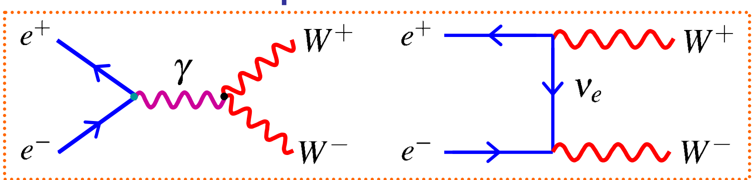
$$\boxed{\Gamma_W = 9\Gamma_{W\to e\nu} = \frac{3g_W^2 m_W}{16\pi} = 2.07\,\mathrm{GeV}}$$

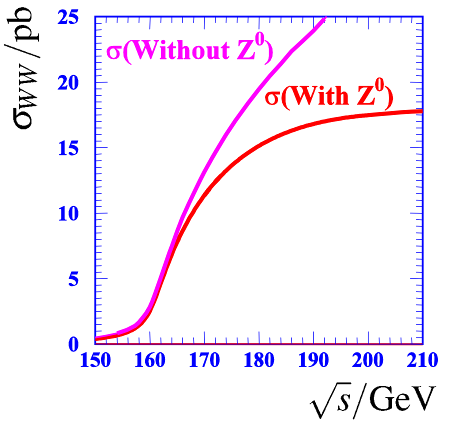
Experiment: 2.14±0.04 GeV
(our calculation neglected a 3% QCD correction to decays to quarks)

From W to Z

★ The W± bosons carry the EM charge - suggestive Weak are EM forces are related.

★ W bosons can be produced in e⁺e⁻ annihilation

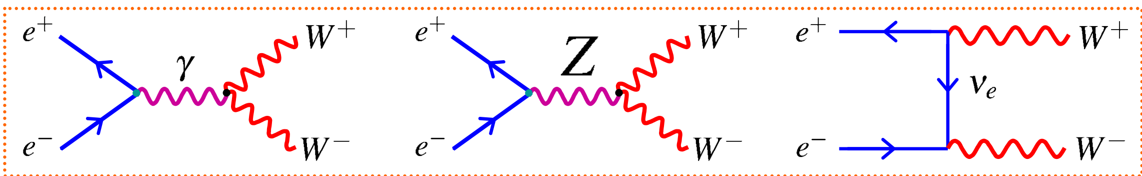




★ With just these two diagrams there is a problem: the cross section increases with C.o.M energy and at some point violates QM unitarity

UNITARITY VIOLATION: when QM calculation gives larger flux of W bosons than incoming flux of electrons/positrons

★ Problem can be "fixed" by introducing a new boson, the Z. The new diagram interferes negatively with the above two diagrams fixing the unitarity problem



$$|M_{\gamma\mathrm{WW}} + M_{\mathrm{ZWW}} + M_{\nu\mathrm{WW}}|^2 < |M_{\gamma\mathrm{WW}} + M_{\nu\mathrm{WW}}|^2$$

★ Only works if Z, γ, W couplings are related: need ELECTROWEAK UNIFICATION

The Local Gauge Principle

(see the Appendices A,C and D for more details)

★ All the interactions between fermions and spin-1 bosons in the SM are specified by the principle of LOCAL GAUGE INVARIANCE

★ To arrive at QED, require physics to be invariant under the local phase transformation of particle wave-functions

$$\psi \to \psi' = \psi e^{iq\chi(x)}$$

★ Note that the change of phase depends on the space-time coordinate: $\chi(t,\vec{x})$

- Under this transformation the Dirac Equation transforms as

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0} \Rightarrow \boxed{i\gamma^\mu(\partial_\mu + iq\partial_\mu\chi)\psi - m\psi = 0}$$

- To make "physics", i.e. the Dirac equation, invariant under this local phase transformation FORCED to introduce a massless gauge boson, A_μ .

\+ The Dirac equation has to be modified to include this new field:

$$\boxed{i\gamma^\mu(\partial_\mu - qA_\mu)\psi - m\psi = 0}$$

- The modified Dirac equation is invariant under local phase transformations if:

$$\boxed{A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi}$$ **Gauge Invariance**

★ For physics to remain unchanged – must have GAUGE INVARIANCE of the new field, i.e. physical predictions unchanged for $A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi$

★ Hence the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (i.e. photon):

$$i\gamma^\mu(\partial_\mu\psi - \boxed{qA_\mu})\psi - m\psi = 0$$

⇒ interaction vertex: $i\gamma^\mu q A_\mu$ (see p.111)

⇒ **QED !**

★ The local phase transformation of QED is a unitary U(1) transformation

$$\psi \to \psi' = \hat{U}\psi \quad \text{i.e.} \quad \psi \to \psi' = \psi e^{iq\chi(x)} \quad \text{with} \quad U^\dagger U = 1$$

Now extend this idea…

From QED to QCD

★ Suppose there is another fundamental symmetry of the universe, say "invariance under SU(3) local phase transformations"

- i.e. require invariance under $\psi \rightarrow \psi' = \psi e^{i\vec{\lambda}.\vec{\theta}(x)}$ where

$\vec{\lambda}$ are the eight 3x3 Gell-Mann matrices introduced in handout 7

$\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time

⇨ 8 spin-1 gauge bosons

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$ wave function is now a vector in COLOUR SPACE

⇨ QCD !

★ QCD is fully specified by require invariance under SU(3) **local** phase transformations

Corresponds to rotating states in colour space about an axis whose direction is different at every space-time point

⇨ interaction vertex: $-\frac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu$

★ Predicts 8 massless gauge bosons – the gluons (one for each λ)

★ Also predicts exact form for interactions between gluons, i.e. the 3 and 4 gluon vertices – the details are beyond the level of this course

SU(2)$_L$: The Weak Interaction

★ The Weak Interaction arises from SU(2) local phase transformations

$$\psi \rightarrow \psi' = \psi e^{i\vec{\alpha}(x).\frac{\vec{\sigma}}{2}}$$

where the $\vec{\sigma}$ are the generators of the SU(2) symmetry, i.e the three Pauli spin matrices

⇨ 3 Gauge Bosons $W_1^\mu,\ W_2^\mu,\ W_3^\mu$

★ The wave-functions have two components which, in analogy with isospin, are represented by "weak isospin"

★ The fermions are placed in isospin doublets and the local phase transformation corresponds to

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \rightarrow \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}' = e^{i\vec{\alpha}(x).\frac{\vec{\sigma}}{2}} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$$

★ Weak Interaction only couples to LH particles/RH anti-particles, hence only place LH particles/RH anti-particles in weak isospin doublets: $I_W = \frac{1}{2}$

RH particles/LH anti-particles placed in weak isospin singlets: $I_W = 0$

Weak Isospin

$I_W = \frac{1}{2}$ $\quad \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \begin{pmatrix} u \\ d' \end{pmatrix}_L, \begin{pmatrix} c \\ s' \end{pmatrix}_L, \begin{pmatrix} t \\ b' \end{pmatrix}_L$ ← $I_W^3 = +\frac{1}{2}$ ← $I_W^3 = -\frac{1}{2}$

$I_W = 0$ $\quad (\nu_e)_R,\ (e^-)_R, \ldots (u)_R,\ (d)_R, \ldots$

Note: RH/LH refer to chiral states

★ For simplicity only consider $\chi_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$

•The gauge symmetry specifies the form of the interaction: one term for each of the 3 generators of SU(2) – [note: here include interaction strength in current]

$$j^1_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_1 \chi_L \qquad j^2_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_2 \chi_L \qquad j^3_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_3 \chi_L$$

★The charged current W^+/W^- interaction enters as a linear combinations of W_1, W_2

$$W^{\pm\mu} = \tfrac{1}{\sqrt{2}}(W_1^\mu \pm W_2^\mu)$$

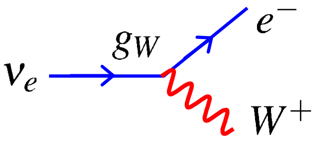
★ The $W^\pm$ interaction terms

$$j^\mu_\pm = \tfrac{g_W}{\sqrt{2}}(j^\mu_1 \pm i j^\mu_2) = \tfrac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)\chi_L$$

★ Express in terms of the weak isospin ladder operators $\sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)$

$$\boxed{j^\mu_\pm = \tfrac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_\pm \chi_L} \quad \Big\} \text{ Origin of } \tfrac{1}{\sqrt{2}} \text{ in Weak CC}$$

W$^+$



corresponds to $\boxed{j^\mu_+ = \tfrac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_+ \chi_L}$

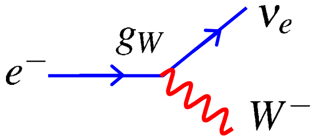
Bars indicates adjoint spinors

which can be understood in terms of the weak isospin doublet

$$j^\mu_+ = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_+ \chi_L = \frac{g_W}{\sqrt{2}} (\overline{\nu}_L, \overline{e}_L)\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix}_L = \frac{g_W}{\sqrt{2}} \overline{\nu}_L \gamma^\mu e_L = \boxed{\frac{g_W}{\sqrt{2}} \overline{\nu} \gamma^\mu \tfrac{1}{2}(1-\gamma^5) e}$$

★ Similarly

W$^-$



corresponds to $\boxed{j^\mu_- = \tfrac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_- \chi_L}$

$$j^\mu_- = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_- \chi_L = \frac{g_W}{\sqrt{2}} (\overline{\nu}_L, \overline{e}_L)\gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix}_L = \frac{g_W}{\sqrt{2}} \overline{e}_L \gamma^\mu \nu_L = \frac{g_W}{\sqrt{2}} \overline{e} \gamma^\mu \tfrac{1}{2}(1-\gamma^5) \nu$$

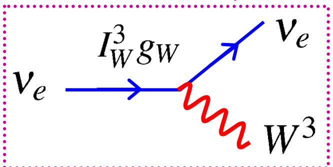
★However have an **additional interaction** due to W^3

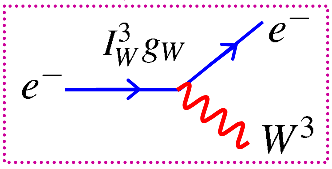
$$\boxed{j^\mu_3 = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_3 \chi_L}$$

expanding this:

$\boxed{I^3_W = \pm\tfrac{1}{2}}$

$$j^\mu_3 = g_W \frac{1}{2} (\overline{\nu}_L, \overline{e}_L)\gamma^\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix}_L = g_W \tfrac{1}{2} \overline{\nu}_L \gamma^\mu \nu_L - g_W \tfrac{1}{2} \overline{e}_L \gamma^\mu e_L$$





⇒ **NEUTRAL CURRENT INTERACTIONS !**

Electroweak Unification

★Tempting to identify the W^3 as the Z

★However this is not the case, have two physical neutral spin-1 gauge bosons, γ, Z and the W^3 is a mixture of the two,

★ Equivalently write the photon and Z in terms of the W^3 and a new neutral spin-1 boson the B

★The physical bosons (the Z and photon field, A) are:

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W$$

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W$$

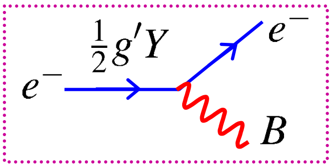
θ_W is the weak mixing angle

★The new boson is associated with a new gauge symmetry similar to that of electromagnetism : $U(1)_Y$

★The charge of this symmetry is called **WEAK HYPERCHARGE** Y

$$\boxed{Y = 2Q - 2I^3_W}$$

Q is the EM charge of a particle
I^3_W is the third comp. of weak isospin



•By convention the coupling to the B_μ is $\frac{1}{2}g'Y$

$$e_L: \; Y = 2(-1) - 2(-\tfrac{1}{2}) = -1 \qquad \nu_L: \; Y = +1$$

$$e_R: \; Y = 2(-1) - 2(0) = -2 \qquad \nu_R: \; Y = 0$$

(this identification of hypercharge in terms of Q and I_3 makes all of the following work out)

★ For this to work the coupling constants of the W^3, B, and photon must be related e.g. consider contributions involving the neutral interactions of electrons:

γ $$j^{em}_\mu = e\overline{\psi}Q_e\gamma_\mu\psi = e\overline{e}_L Q_e \gamma_\mu e_L + e\overline{e}_R Q_e \gamma_\mu e_R$$

W^3 $$j^{W^3}_\mu = -\frac{g_W}{2}\overline{e}_L\gamma_\mu e_L$$

B $$j^Y_\mu = \frac{g'}{2}\overline{\psi}Y_e\gamma_\mu\psi = \frac{g'}{2}\overline{e}_L Y_{e_L}\gamma_\mu e_L + \frac{g'}{2}\overline{e}_R Y_{e_R}\gamma_\mu e_R$$

★ The relation $A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W$ is equivalent to requiring

$$\boxed{j^{em}_\mu = j^Y_\mu \cos\theta_W + j^{W^3}_\mu \sin\theta_W}$$

•Writing this in full:

$$e\overline{e}_L Q_e \gamma_\mu e_L + e\overline{e}_R Q_e \gamma_\mu e_R = \tfrac{1}{2}g'\cos\theta_W[\overline{e}_L Y_{e_L}\gamma_\mu e_L + \overline{e}_R Y_{e_R}\gamma_\mu e_R] - \tfrac{1}{2}g_W \sin\theta_W[\overline{e}_L\gamma_\mu e_L]$$

$$-e\overline{e}_L\gamma_\mu e_L - e\overline{e}_R\gamma_\mu e_R = \tfrac{1}{2}g'\cos\theta_W[-\overline{e}_L\gamma_\mu e_L - 2\overline{e}_R\gamma_\mu e_R] - \tfrac{1}{2}g_W\sin\theta_W[\overline{e}_L\gamma_\mu e_L]$$

which works if: $\boxed{e = g_W \sin\theta_W = g'\cos\theta_W}$ (i.e. equate coefficients of L and R terms)

★ Couplings of electromagnetism, the weak interaction and the interaction of the $U(1)_Y$ symmetry are therefore related.

The Z Boson

★In this model we can now derive the couplings of the Z Boson

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W \qquad \boxed{I^3_W} \qquad \text{for the electron } I^3_W = \tfrac{1}{2}$$

$$j^Z_\mu = -\tfrac{1}{2} g' \sin\theta_W [\bar{e}_L Y_{e_L} \gamma_\mu e_L + \bar{e}_R Y_{e_R} \gamma_\mu e_R] - \tfrac{1}{2} g_W \cos\theta_W [e_L \gamma_\mu e_L]$$

•Writing this in terms of weak isospin and charge:

$$j^Z_\mu = -\tfrac{1}{2} g' \sin\theta_W [\bar{e}_L (2Q - 2I^3_W) \gamma_\mu e_L + \bar{e}_R \underbrace{(2Q)}_{\text{For RH chiral states } I_3=0} \gamma_\mu e_R] + I^3_W g_W \cos\theta_W [e_L \gamma_\mu e_L]$$

•Gathering up the terms for LH and RH chiral states:

$$j^Z_\mu = \left[g' I^3_W \sin\theta_W - g' Q \sin\theta_W + g_W I^3_W \cos\theta_W\right] \bar{e}_L \gamma_\mu e_L - \left[g' Q \sin\theta_W\right] e_R \gamma_\mu e_R$$

•Using: $e = g_W \sin\theta_W = g' \cos\theta_W$ gives

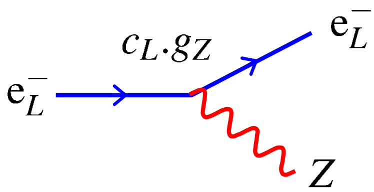
$$j^Z_\mu = \left[g' \frac{(I^3_W - Q\sin^2\theta_W)}{\sin\theta_W}\right] \bar{e}_L \gamma_\mu e_L - \left[g' \frac{Q\sin^2\theta_W}{\sin\theta_W}\right] e_R \gamma_\mu e_R$$

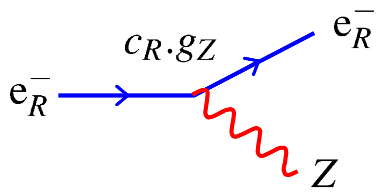
$$\boxed{j^Z_\mu = g_Z (I^3_W - Q\sin^2\theta_W)[\bar{e}_L \gamma_\mu e_L] - g_Z Q \sin^2\theta_W [e_R \gamma_\mu e_R]}$$

with $\boxed{e = g_Z \cos\theta_W \sin\theta_W}$ i.e. $\boxed{g_Z = \dfrac{g_W}{\cos\theta_W}}$

★ Unlike for the Charged Current Weak interaction (W) the Z Boson couples to both LH and RH chiral components, **but not equally…**

$$\begin{aligned} j^Z_\mu &= g_Z (I^3_W - Q\sin^2\theta_W)[\bar{e}_L \gamma_\mu e_L] - g_Z Q \sin^2\theta_W [e_R \gamma_\mu e_R] \\ &= g_Z c_L [\bar{e}_L \gamma_\mu e_L] + g_Z c_R [e_R \gamma_\mu e_R] \end{aligned}$$





$$\boxed{c_L = I^3_W - Q\sin^2\theta_W} \qquad \boxed{c_R = -Q\sin^2\theta_W}$$

W³ part of Z couples only to LH components (like W±)

B_μ part of Z couples equally to LH and RH components

★ Use projection operators to obtain vector and axial vector couplings

$$\bar{u}_L \gamma_\mu u_L = \bar{u}\gamma_\mu \tfrac{1}{2}(1-\gamma_5)u \qquad \bar{u}_R \gamma_\mu u_R = \bar{u}\gamma_\mu \tfrac{1}{2}(1+\gamma_5)u$$

$$j^Z_\mu = g_Z \bar{u}\gamma_\mu \left[c_L \tfrac{1}{2}(1-\gamma_5) + c_R \tfrac{1}{2}(1+\gamma_5)\right] u$$

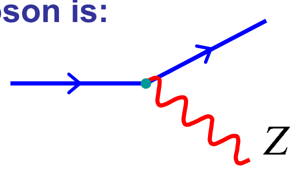
$$j_\mu^Z = \frac{g_Z}{2}\bar{u}\gamma_\mu\left[(c_L + c_R) + (c_R - c_L)\gamma_5\right]u$$

★ Which in terms of **V** and **A** components gives: $$j_\mu^Z = \frac{g_Z}{2}\bar{u}\gamma_\mu\left[c_V - c_A\gamma_5\right]u$$

with $\boxed{c_V = c_L + c_R = I_W^3 - 2Q\sin^2\theta_W}$ $\boxed{c_A = c_L - c_R = I_W^3}$

★ Hence the vertex factor for the Z boson is:

$$\boxed{-ig_Z\tfrac{1}{2}\gamma_\mu\left[c_V - c_A\gamma_5\right]}$$



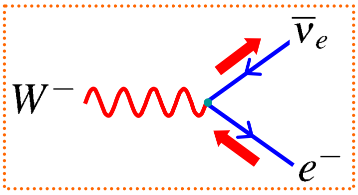
★ Using the experimentally determined value of the weak mixing angle:

$\boxed{\sin^2\theta_W \approx 0.23}$

Fermion	Q	I_W^3	c_L	c_R	c_V	c_A
ν_e, ν_μ, ν_τ	0	$+\frac{1}{2}$	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
e^-, μ^-, τ^-	-1	$-\frac{1}{2}$	-0.27	0.23	-0.04	$-\frac{1}{2}$
u, c, t	$+\frac{2}{3}$	$+\frac{1}{2}$	0.35	-0.15	$+0.19$	$+\frac{1}{2}$
d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	-0.42	0.08	-0.35	$-\frac{1}{2}$

Z Boson Decay : Γ_Z

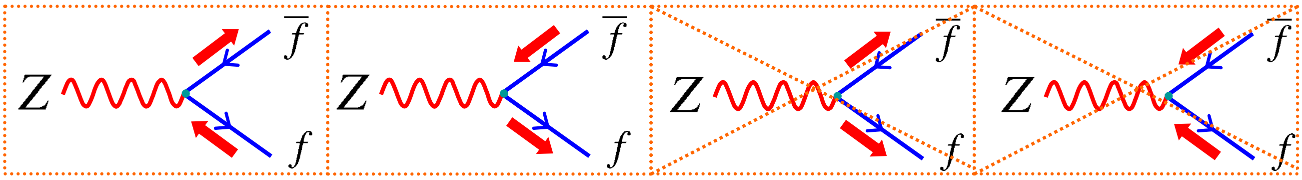
★ In W-boson decay only had to consider one helicity combination of (assuming we can neglect final state masses: helicity states = chiral states)



W-boson couples:
to LH particles
and RH anti-particles

★ But Z-boson couples to LH and RH particles (with different strengths)

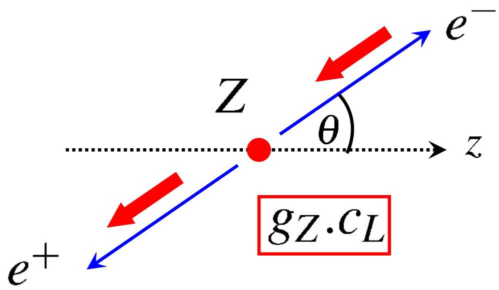
★ Need to consider **only two** helicity (or more correctly chiral) combinations:

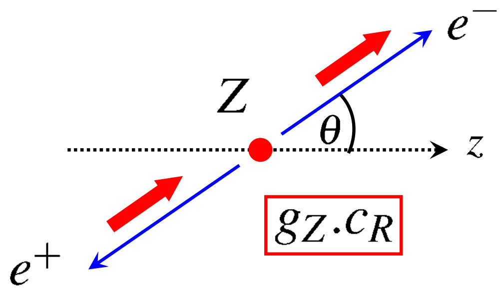


This can be seen by considering either of the combinations which give zero

e.g. $$\bar{u}_R\gamma^\mu(c_V + c_A\gamma_5)v_R = u^\dagger\tfrac{1}{2}(1+\gamma^5)\gamma^0\gamma^\mu(c_V + c_A\gamma^5)\tfrac{1}{2}(1-\gamma^5)v$$
$$= \tfrac{1}{4}u^\dagger\gamma^0(1-\gamma^5)\gamma^\mu(1-\gamma^5)(c_V + c_A\gamma^5)v$$
$$= \tfrac{1}{4}\bar{u}\gamma^\mu(1+\gamma^5)(1-\gamma^5)(c_V + c_A\gamma_5)v = 0$$

★ **In terms of left and right-handed combinations need to calculate:**





★ **For unpolarized Z bosons:** **(Question 26)**

$$\langle |M_{fi}|^2 \rangle = \frac{1}{3}[2c_L^2 g_Z^2 m_Z^2 + 2c_R^2 g_Z^2 m_Z^2] = \frac{2}{3} g_Z^2 m_Z^2 (c_L^2 + c_R^2)$$

average over polarization

★ **Using** $c_V^2 + c_A^2 = 2(c_L^2 + c_R^2)$ **and** $\frac{d\Gamma}{d\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}|M|^2$

$$\Rightarrow \quad \boxed{\Gamma(Z \to e^+e^-) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2)}$$

Z Branching Ratios

(Question 27)

★ **(Neglecting fermion masses) obtain the same expression for the other decays**

$$\Gamma(Z \to f\bar{f}) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2)$$

•**Using values for c_V and c_A on page 471 obtain:**

$$Br(Z \to e^+e^-) = Br(Z \to \mu^+\mu^-) = Br(Z \to \tau^+\tau^-) \boxed{\approx 3.5\%}$$

$$Br(Z \to \nu_1\bar{\nu}_1) = Br(Z \to \nu_2\bar{\nu}_2) = Br(Z \to \nu_3\bar{\nu}_3) \boxed{\approx 6.9\%}$$

$$Br(Z \to d\bar{d}) = Br(Z \to s\bar{s}) = Br(Z \to b\bar{b}) \boxed{\approx 15\%}$$

$$Br(Z \to u\bar{u}) = Br(Z \to c\bar{c}) \boxed{\approx 12\%}$$

•**The Z Boson therefore predominantly decays to hadrons**

$$Br(Z \to \text{hadrons}) \approx 69\%$$

Mainly due to factor 3 from colour

•**Also predict total decay rate (total width)**

$$\Gamma_Z = \sum_i \Gamma_i = 2.5\,\text{GeV}$$

Experiment: $\boxed{\Gamma_Z = 2.4952 \pm 0.0023\,\text{GeV}}$

Summary

★ The Standard Model interactions are mediated by spin-1 **gauge bosons**
★ The form of the interactions are completely specified by the assuming an underlying local phase transformation ➡ GAUGE INVARIANCE

$U(1)_{em}$	⇨	QED
$SU(2)_L$	⇨	Charged Current Weak Interaction + W^3
$SU(3)_{col}$	⇨	QCD

★ In order to "unify" the electromagnetic and weak interactions, introduced a new symmetry gauge symmetry : U(1) hypercharge

$U(1)_Y$ ⇨ B_μ

★ The physical Z boson and the photon are mixtures of the neutral W boson and B determined by the **Weak Mixing angle**

$$\boxed{\sin\theta_W \approx 0.23}$$

★ Have we really unified the EM and Weak interactions ? Well not really…
•Started with two independent theories with coupling constants g_W, e
•Ended up with coupling constants which are related but at the cost of introducing a new parameter in the Standard Model θ_W
•Interactions not unified from any higher theoretical principle… **but it works!**

Appendix A1 : Electromagnetism

(Non-examinable)

★ In Heaviside-Lorentz units $\varepsilon_0 = \mu_0 = c = 1$ Maxwell's equations in the vacuum become

$$\vec{\nabla}.\vec{E} = \rho; \quad \vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}; \quad \vec{\nabla}\cdot\vec{B} = 0; \quad \vec{\nabla}\wedge\vec{B} = \vec{J} + \frac{\partial\vec{E}}{\partial t}$$

★ The electric and magnetic fields can be expressed in terms of scalar and vector potentials

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi; \qquad \vec{B} = \vec{\nabla}\wedge\vec{A} \tag{A1}$$

★ In terms of the the 4-vector potential $A^\mu = (\phi, \vec{A})$ and the 4-vector current $j^\mu = (\rho, \vec{J})$ Maxwell's equations can be expressed in the covariant form:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{A2}$$

where $F^{\mu\nu}$ is the anti-symmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{A3}$$

•Combining (A2) and (A3)

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu \tag{A4}$$

which can be written $\Box^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$ **(A5)**

where the D'Alembertian operator

$$\Box^2 = \partial_\nu \partial^\nu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

- Acting on equation **(A5)** with ∂_ν gives

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial_\nu \partial^\nu A^\mu = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Conservation of Electric Charge

- Conservation laws are associated with symmetries. Here the symmetry is the **GAUGE INVARIANCE** of electro-magnetism

Appendix A2 : Gauge Invariance (Non-examinable)

★ The electric and magnetic fields are unchanged for the **gauge transformation**:

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi; \qquad \phi \rightarrow \phi' = \phi - \frac{\partial \chi}{\partial t}$$

where $\chi = \chi(t, \vec{x})$ is any finite differentiable function of position and time

★ In 4-vector notation the **gauge transformation** can be expressed as:

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi$$

★ Using the fact that the physical fields are gauge invariant, choose χ to be a solution of

$$\Box^2 \chi = -\partial_\mu A^\mu$$

★ In this case we have

$$\partial^\mu A'_\mu = \partial^\mu (A_\mu + \partial_\mu \chi) = \partial^\mu A_\mu + \Box^2 \chi = 0$$

★ Dropping the prime we have a chosen a gauge in which

$$\boxed{\partial_\mu A^\mu = 0}$$

The Lorentz Condition **(A6)**

★ With the Lorentz condition, equation **(A5)** becomes:

$$\boxed{\Box^2 A^\mu = j^\mu} \qquad \textbf{(A7)}$$

★ Having imposed the Lorentz condition we still have freedom to make a further gauge transformation, i.e.

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda$$

where $\Lambda(t, \vec{x})$ is any function that satisfies

$$\Box^2 \Lambda = 0 \qquad \textbf{(A8)}$$

★ Clearly **(A7)** remains unchanged, in addition the Lorentz condition still holds:

$$\partial^\mu A'_\mu = \partial^\mu (A_\mu + \partial_\mu \Lambda) = \partial^\mu A_\mu + \Box^2 \Lambda = \partial^\mu A_\mu = 0$$

Appendix B1 : Photon Polarization

(Non-examinable)

• For a free photon (i.e. $j^\mu = 0$) equation (A7) becomes

$$\boxed{\Box^2 A^\mu = 0} \qquad \text{(B1)}$$

(note have chosen a gauge where the Lorentz condition is satisfied)

★ Equation (A8) has solutions (i.e. the wave-function for a free photon)

$$A^\mu = \varepsilon^\mu(q)e^{-iq.x}$$

where ε^μ is the four-component polarization vector and q is the photon four-momentum

$$0 = \Box^2 A^\mu = -q^2 \varepsilon^\mu e^{-iq.x}$$

$$\Rightarrow \quad q^2 = 0$$

★ Hence equation (B1) describes a massless particle.

★ But the solution has four components – might ask how it can describe a spin-1 particle which has three polarization states?

★ But for (A8) to hold we must satisfy the Lorentz condition:

$$0 = \partial_\mu A^\mu = \partial_\mu(\varepsilon^\mu e^{-iq.x}) = \varepsilon^\mu \partial_\nu(e^{-iq.x}) = -i\varepsilon^\mu q_\mu e^{-iq.x}$$

Hence the Lorentz condition gives $\boxed{q_\mu \varepsilon^\mu = 0}$ (B2)

i.e. only 3 independent components.

★ However, in addition to the Lorentz condition still have the addional gauge freedom of $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ with (A8) $\Box^2 \Lambda = 0$

•Choosing $\Lambda = iae^{-iq.x}$ which has $\Box^2\Lambda = q^2\Lambda = 0$

$$\begin{aligned} A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda &= \varepsilon_\mu e^{-iq.x} + ia\partial_\mu e^{-iq.x} \\ &= \varepsilon_\mu e^{-iq.x} + ia(-iq_\mu)e^{-iq.x} \\ &= (\varepsilon_\mu + aq_\mu)e^{-iq.x} \end{aligned}$$

★ Hence the electromagnetic field is left unchanged by

$$\varepsilon_\mu \to \varepsilon'_\mu = \varepsilon_\mu + aq_\mu$$

★ Hence the two polarization vectors which differ by a mulitple of the photon four-momentum describe the same photon. Choose a such that the time-like component of ε_μ is zero, i.e. $\varepsilon_0 \equiv 0$

★ With this choice of gauge, which is known as the **COULOMB GAUGE**, the Lorentz condition (B2) gives

$$\boxed{\vec{\varepsilon} \cdot \vec{q} = 0} \qquad \text{(B3)}$$

i.e. only 2 independent components, both transverse to the photons momentum

★ A massless photon has two transverse polarisation states. For a photon travelling in the z direction these can be expressed as the transversly polarized states:

$$\varepsilon_1^\mu = (0,1,0,0); \qquad \varepsilon_2^\mu = (0,0,1,0)$$

★ Alternatively take linear combinations to get the circularly polarized states

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \qquad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

★ It can be shown that the ε_+ state corresponds the state in which the photon spin is directed in the **+z** direction, i.e. $S_z = +1$

Appendix B2 : Massive Spin-1 particles

(Non-examinable)

•For a massless photon we had (before imposing the Lorentz condition) we had from equation (A5)

$$\Box^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$$

★The Klein-Gordon equation for a spin-0 particle of mass m is

$$(\Box^2 + m^2)\phi = 0$$

suggestive that the appropriate equations for a massive spin-1 particle can be obtained by replacing $\Box^2 \rightarrow \Box^2 + m^2$

★ This is indeed the case, and from QFT it can be shown that for a massive spin 1 particle equation (A5) becomes

$$(\Box^2 + m^2)B^\mu - \partial^\mu(\partial_\nu B^\nu) = j^\mu$$

★ Therefore a free particle must satisfy

$$(\Box^2 + m^2)B^\mu - \partial^\mu(\partial_\nu B^\nu) = 0 \qquad \text{(B4)}$$

•Acting on equation (B4) with ∂_ν gives

$$\begin{aligned}(\Box^2+m^2)\partial_\mu B^\mu - \partial_\mu\partial^\mu(\partial_\nu B^\nu) &= 0\\ (\Box^2+m^2)\partial_\mu B^\mu - \Box^2(\partial_\nu B^\nu) &= 0\\ m^2\partial_\mu B^\mu &= 0 \qquad \text{(B5)}\end{aligned}$$

★ Hence, for a massive spin-1 particle, unavoidably have $\partial_\mu B^\mu = 0$; note this is not a relation that reflects to choice of gauge.

•Equation (B4) becomes

$$\boxed{(\Box^2+m^2)B^\mu = 0} \qquad \text{(B6)}$$

★ For a free spin-1 particle with 4-momentum, p^μ , equation (B6) admits solutions

$$B_\mu = \varepsilon_\mu e^{-ip.x}$$

★ Substituting into equation (B5) gives

$$\boxed{p_\mu \varepsilon^\mu = 0}$$

★The four degrees of freedom in ε^μ are reduced to three, but for a massive particle, equation (B6) does <u>not</u> allow a choice of gauge and we can not reduce the number of degrees of freedom any further.

★ Hence we need to find three orthogonal polarisation states satisfying

$$\boxed{p_\mu \varepsilon^\mu = 0} \qquad \text{(B7)}$$

★ For a particle travelling in the z direction, can still admit the circularly polarized states.

$$\varepsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0); \qquad \varepsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0)$$

★ Writing the third state as

$$\varepsilon_L^\mu = \frac{1}{\sqrt{\alpha^2+\beta^2}}(\alpha,0,0,\beta)$$

equation (B7) gives $\alpha E - \beta p_z = 0$

$$\Rightarrow \qquad \varepsilon_L^\mu = \frac{1}{m}(p_z,0,0,E)$$

★ This <u>longitudinal</u> polarisation state is only present for massive spin-1 particles, i.e. there is no analogous state for a free photon (although off-mass shell virtual photons can be longitudinally polarized – a fact that was alluded to on page 114).

Appendix C : Local Gauge Invariance

(Non-examinable)

★The Dirac equation for a charged particle in an electro-magnetic field can be obtained from the free particle wave-equation by making the minimal substitution

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi \qquad (q = \text{charge})$$ (see p.112)

In QM: $i\partial_\mu \to i\partial_\mu - qA_\mu$ and the Dirac equation becomes

$$\boxed{\gamma^\mu(i\partial_\mu - qA_\mu)\psi - m\psi = 0} \qquad \text{(C1)}$$

★ In Appendix A2 : saw that the physical EM fields where invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi \qquad \Box^2\chi = 0$$

★ Under this transformation the Dirac equation becomes

$$\gamma^\mu(i\partial_\mu - qA_\mu + q\partial_\mu\chi)\psi - m\psi = 0$$

which is not the same as the original equation. If we require that the Dirac equation is invariant under the Gauge transformation then under the gauge transformation we need to modify the wave-functions

$$\psi \to \psi' = \psi e^{iq\chi}$$

A Local Phase Transformation

★To prove this, applying the gauge transformation :

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi \qquad \psi \to \psi' = \psi e^{iq\chi}$$

to the original Dirac equation gives

$$\gamma^\mu(i\partial_\mu - qA_\mu + q\partial_\mu\chi)\psi e^{iq\chi} - m\psi e^{iq\chi} = 0 \qquad \text{(C2)}$$

★ But $\quad i\partial_\mu(\psi e^{iq\chi}) = ie^{iq\chi}\partial_\mu\psi - q(\partial_\mu\chi)e^{iq\chi}\psi$

★ Equation (C2) becomes

$$\gamma^\mu e^{iq\chi}(i\partial_\mu - qA_\mu + q\partial_\mu\chi - q\partial_\mu\chi)\psi - m\psi e^{iq\chi} = 0$$

$$\Rightarrow \qquad \gamma^\mu e^{iq\chi}(i\partial_\mu - qA_\mu)\psi - m\psi e^{iq\chi} = 0$$

$$\Rightarrow \qquad \gamma^\mu(i\partial_\mu - qA_\mu)\psi - m\psi = 0$$

which is the original form of the Dirac equation

Appendix D : Local Gauge Invariance 2

(Non-examinable)

★ Reverse the argument of Appendix D. Suppose there is a fundamental symmetry of the universe under **local phase transformations**

$$\psi(x) \to \psi'(x) = \psi(x)e^{iq\chi(x)}$$

★ Note that the local nature of these transformations: the phase transformation depends on the space-time coordinate $x = (t, \vec{x})$

★ Under this transformation the free particle Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

becomes

$$i\gamma^\mu \partial_\mu (\psi e^{iq\chi}) - m\psi e^{iq\chi} = 0$$

$$ie^{iq\chi}\gamma^\mu(\partial_\mu \psi + iq\psi\partial_\mu \chi) - m\psi e^{iq\chi} = 0$$

$$i\gamma^\mu(\partial_\mu + iq\partial_\mu \chi)\psi - m\psi = 0$$

Local phase invariance is not possible for a free theory, i.e. one without interactions

★ To restore invariance under local phase transformations have to introduce a massless "gauge boson" A^μ which transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

and make the substitution

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$